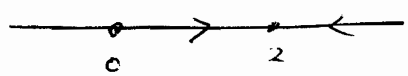
Name A.K.

Math 5260, Dynamical Systems
MIDTERM 2: Fall 2007
B. Peckham

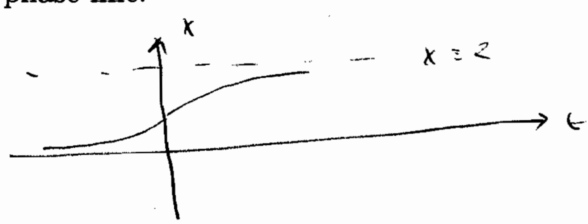
Total: 100 points. Directions: Do problems 2 (10 pts) and 6 (24 pts). Do any 11 of the other 13 problems (6 pts each). Write OUT by the two problems you choose to not count.

1. Consider the differential equation: $\dot{x} = 2x - x^2$.

   (a) Sketch the corresponding phase line.



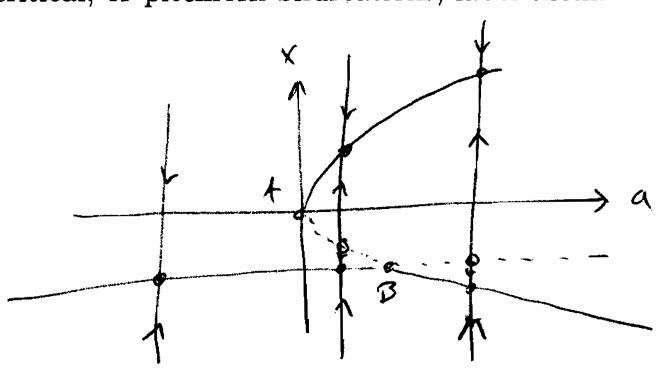
   (b) Sketch a solution curve (x vs t) corresponding to initial conditions $x(0) = 1$. The solution need not be exact, but must be consistent with the phase line.



2. (10 pts) Construct a bifurcation diagram in the parameter $\times$ phase space for the family of differential equations:

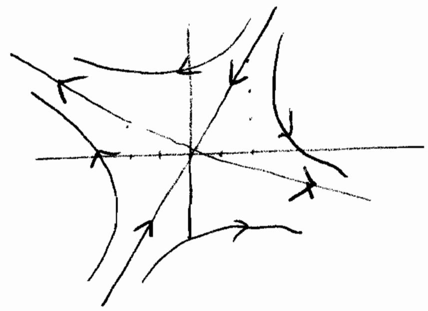
$$\dot{x} = (1 + x)(a - x^2)$$

Include all equilibria (solid lines for attracting and dashed for repelling) and some representative phase lines on the bifurcation diagram. Locate on the diagram any bifurcations. If any of them are saddle-node, transcritical, or pitchfork bifurcations, label them.



A - sn
B - tc

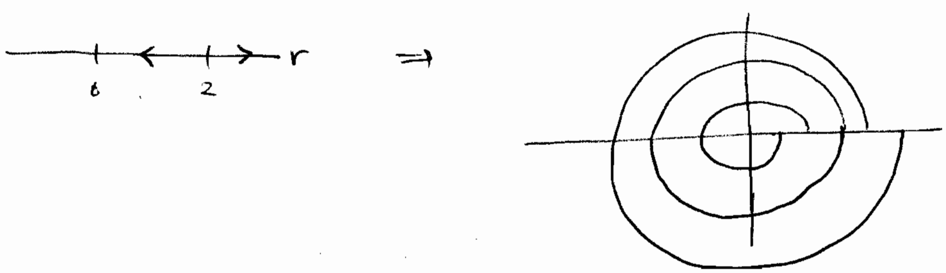
3. Sketch the phase portrait of $\dot{\mathbf{x}} = A\mathbf{x}$ if the 2×2 matrix A has an eigenvector of $(2, 3)$ corresponding to eigenvalue -2 and another eigenvector $(-2, 1)$ corresponding to eigenvalue 1. Include the orbit (in the phase space) corresponding to the solution having initial conditions: $\mathbf{x}(0) = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$.



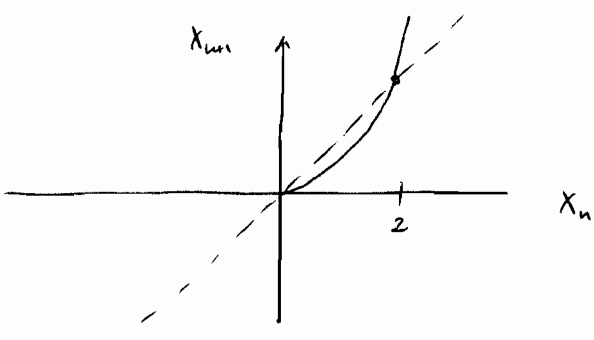
4. Sketch the phase portrait in the "(x, y) space" for the differential equation which is given in polar coordinates as

$$\dot{r} = r(r-2), \quad \dot{\theta} = 2$$

Assume $r \geq 0$. Describe the forward fate of the orbit which starts at $(x, y) = (1, 0)$. Include this orbit on your phase portrait.



5. Sketch a graph of the Poincare map for the above problem. Use the positive x axis as the domain of the map.



6. (15 pts) Consider the following system of differential equations:

$$\dot{x} = -2x + 4y, \quad \dot{y} = y - y^2$$

(a) Locate all equilibrium points.

(b) Find the linear differential equation which approximates the full system near each equilibrium point. (Linearize.)

(c) Classify each linearization (sink, saddle, source, or "other").

(d) Are the axes invariant? Justify.

(e) Sketch the nullclines and the direction of the flow across (or along) the nullclines.

(f) Sketch a plausable phase portrait.

(g) Extra credit: Locate and label any branches of the stable (W^s) and unstable (W^u) manifold of any saddle. Describe the fate of all orbits.

a.) $-2x+4y=0$

$y-y^2=0 \Rightarrow y(1-y)=0 \Rightarrow y=0,1$

$y=0 \Rightarrow -2x=0 \Rightarrow x=0 \Rightarrow \boxed{(0,0)}$

$y=1 \Rightarrow -2x+4\cdot 1=0 \Rightarrow 2x=4$ or $x=2 \Rightarrow \boxed{(2,1)}$

b.) $Df(x) = \begin{pmatrix} -2 & 4 \\ 0 & 1-2y \end{pmatrix} \Rightarrow Df(0,0) = \begin{pmatrix} -2 & 4 \\ 0 & 1 \end{pmatrix}, \quad Df(2,1) = \begin{pmatrix} -2 & 4 \\ 0 & -1 \end{pmatrix}$

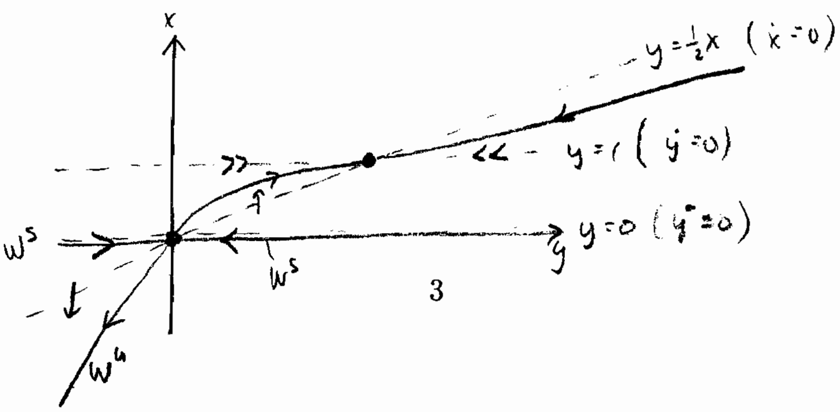
$\Rightarrow$ D.E. is $\dot{x} = Ax$ where $A = Df(0,0)$ or $Df(2,1)$.

c.) $(0,0)$ is a saddle; $(2,1)$ is a sink.

d.) x-axis: $y=0 \Rightarrow \dot{y} = 0-0^2 = 0 \therefore$ x axis is invariant

y axis: $x=0 \Rightarrow \dot{x} = -2\cdot 0+4y \neq 0 \therefore$ y axis is not invariant.

e, f, g)



All orbits w/ $y_0<0$ go to ∞ (approaching lower branch of W^u)

All orbits w/ $y_0>0$ approach equilib. @ $(2,1)$.

All orbits w/ $y_0=0$ approach $(0,0)$.

7. Explain why for a 2×2 matrix A the origin will be an attracting equilibrium for $\dot{x} = Ax$ whenever $Tr(A) < 0$ and $Det(A) > 0$.

Evals $\lambda^2 - TrA\,\lambda + det A = 0$

$$\lambda = \frac{TrA \pm \sqrt{(TrA)^2 - 4\,det A}}{2}$$

Case 1: $(TrA)^2 - 4\,det A < 0 \Rightarrow$ cx evals w/ real part $\frac{TrA}{2} < 0 \Rightarrow$ attr.

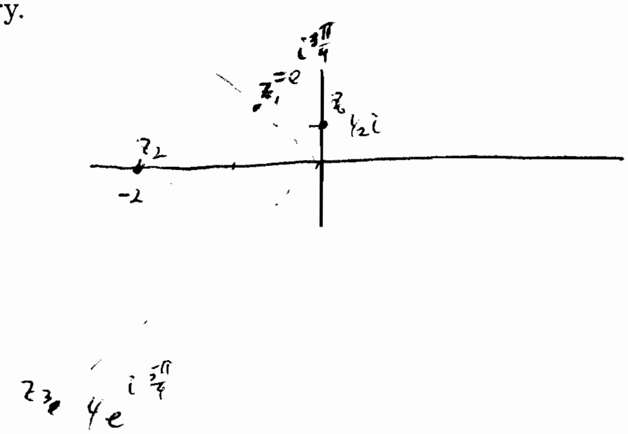
Case 2: " $> 0 \Rightarrow |TrA| > \sqrt{TrA^2 - 4\,det A} \Rightarrow$ both evals real, neg, $\Rightarrow$ attr.

8. Given the differential equation $\dot{x} = 4x - x^2$, determine an appropriate value of a so that the rescaling $y = ax$ leads to the differential equation $\dot{y} = 4y - 4y^2$.

$y = ax \Rightarrow \dot{y} = a\dot{x} = a(4x - x^2) = a\left(4 \cdot \frac{y}{a} - \left(\frac{y}{a}\right)^2\right) = 4y - \frac{y^2}{a}$

$= 4y - 4y^2$ if $\boxed{a = \frac{1}{4}}$

9. In the complex plane, sketch the first four points: $z_0, z_1, z_2 z_3$ on the orbit starting at $z_0 = 0 + 0.5i$ for the iteration function given by $f(z) = 2e^{\frac{\pi i}{4}} z$. As usual, assume $z_{n+1} = f(z_n)$. Exact coordinates are not necessary.



10. Consider the map $Q_i(z) = z^2 + i$. Show that $z_0 = -i$ lies on a period-2 orbit for Q_i. Determine whether the period-2 orbit is attracting or repelling. Justify your answer.

$$Q_i(-i) = (-i)^2 + i = -1+i$$

$$Q_i(-1+i) = (-1+i)^2 + i = 1 - 1 - 2i + i = -i \checkmark$$

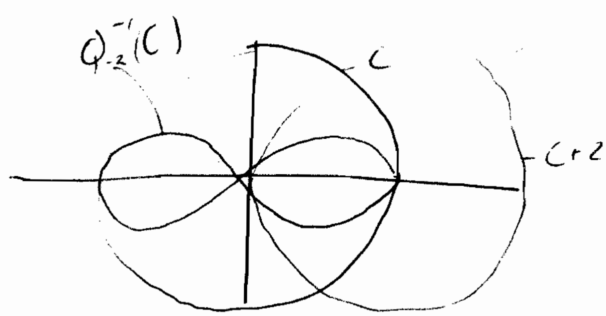
$$\left|(Q_i^2)'(-i)\right| = \left|2(-i)\cdot 2\cdot(-1+i)\right| = 4\sqrt{2} > 1 \Rightarrow \text{rep.}$$

11. Give a definition of the Filled Julia set of the complex map $z \to z^2 + 2$. Also define the Julia set of the same map. (You need not determined what the sets are, just give the definitions.)

$$K_2 = \{ z \in \mathbb{C} : Q_2^n(z) \not\to \infty \}$$

$$J_2 = \partial K_2$$

12. Let C be the circle in the complex plane with center at the origin and radius 2. Let $Q_{-2}(z) = z^2 - 2$. Sketch and label C and $Q_{-2}^{-1}(C)$ on the same set of axes. What does this sketch have to do with the filled Julia set of Q_{-2}?



$K_c \subsetneq \infty.$

For problems 14 – 15, consider the function $F_\lambda(z) = \lambda z - z^2$. (Both λ and z are complex.)

13. Determine all fixed points for F_λ (in terms of λ).

$$F_\lambda(z) = z \iff \lambda z - z^2 = z$$

$$z(\lambda - 1) - z^2 = 0$$

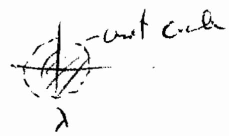
$$z(\lambda - 1 - z) = 0$$

$$\underline{z = 0}, \quad z = \underline{\lambda - 1}$$

14. Give a formula and sketch of the λ values whose corresponding maps F_λ have an attracting fixed point.

$$F'_\lambda(z) = \lambda - 2z$$

$|F'_\lambda(0)| = |\lambda| < 1$ if $\lambda \in$



$|F'_\lambda(\lambda - 1)| = |\lambda - 2(\lambda - 1)| = |2 - \lambda| = |\lambda - 2| < 1$ if

$\lambda \in$