Name ________________________________ Math 5260, Dynamical Systems
Fall 2011 MIDTERM 2
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Total: 100 points. Directions: Do all problems. You may choose any two parts of problems to receive FREE full credit. (Each part is worth between 4 and 6 points.) Clearly indicate both here and by the two problem parts which two you are designating as FREE: ________________________________

1. Consider the differential equation: $\dot{x} = x(2-x)(1+x)$.
   (a) (5pts) Sketch the corresponding phase line.
   (b) (4pts) Sketch solution curves (x vs t) corresponding to initial conditions $x(0) = 1/2$ and $x(0) = -1/2$. The solution need not be exact, but must be consistent with the phase line.

2. Construct a bifurcation diagram in the parameter $\times$ phase space for the family of differential equations:
$$\dot{x} = (a + x^2)(a + 1 - x^2)$$
(6pts) Include all equilibria (solid lines for attracting and dashed for repelling) and some representative phase lines on the bifurcation diagram. (4pts) Locate on the diagram any bifurcations. If are any of them are saddle-node, transcritical, or pitchfork bifurcations, label them S,T, or P, respectively. Label B for any others.

3. Consider the system of differential equations $\dot{\mathbf{x}} = A\mathbf{x}$ where $A = \begin{pmatrix} -1 & 2 \\ -1 & -1 \end{pmatrix}$, and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. All on the same phase plane, sketch
   (a) (5pts) nullclines and arrows indicating the direction of the flow across them,
   (b) (5pts) two phase curves, with one of them corresponding to the solution having initial conditions: $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Make sure the curves cross the nullclines with the appropriate slopes.

   (c) (4pts) On a separate graph, Sketch a possible graph for x_1 versus t for the initial conditions: $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

4. Consider the following system of differential equations:
$$\dot{x} = -x + xy, \quad \dot{y} = 2y - y^2 - xy/2$$
   (a) (5pts) Locate all equilibrium points.
   (b) (6pts) Find matrix of linearization at each equilibrium point. Classify each equilibrium point as sink, saddle, source, or "other".
   (c) (4pts) Are the axes invariant? Justify.

(d) (5pts) Describe the dynamics of population y in the absence of population x.

(e) (5pts Extra Credit) Sketch the phase plane for the first quadrant. Label any stable and unstable manifolds of saddles.

5. (5pts) List 3 advantages and/or disadvantages of using phase portraits instead of looking for analytic solutions to families of differential equations.

6. (5pts) Show that if λ is an eigenvalue of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\lambda^2 - T(A)\lambda + D(A) = 0,$$

where $T(A)$ is the trace of A, and $D(A)$ is the determinant of A.

7. (5pts) Given the two-dimensional system $\dot{\mathbf{x}} = A\mathbf{x}$, and the change of variables $\mathbf{y} = K\mathbf{x}$, determine the differential equation in the variable $\mathbf{y}$. (A and K are 2×2 matrices; $\mathbf{x}$ and $\mathbf{y}$ are 2×1 column vectors. Assume K is invertible.)

8. (5pts) In the complex plane, sketch the first four points: z_0, z_1, z_2, z_3 on the orbit starting at $z_0 = 0 + i$ for the iteration function given by $f(z) = (1/2)e^{\frac{\pi i}{4}} z$. As usual, assume $z_{n+1} = f(z_n)$.

9. (6pts) Describe the full dynamics for the complex map $z \mapsto z^2$. Describe the filled Julia set and the Julia set. Describe the fate for all points in the complex plane.

10. (5pts) Consider the complex map $z \to z^2 + 1$. Find any one point that is

    (a) not in the filled Julia set

    (b) in the filled Julia set

    (c) in the Julia set.

    Justify briefly.

11. (5pts) Let R be the "pie-shaped" sector defined by the portion of the disk of radius 4, centered at the origin, and in the second quadrant of the complex plane. Let $Q(z) = z^2$. On the same axes, sketch R and $Q^{-1}(R)$.

    For problems 12 – 13, consider the function $F_\lambda(z) = \lambda z + z^3$. (Both λ and z are complex.)

12. (5pts) Determine all fixed points for F_λ (in terms of λ).

13. (6pts) Give a formula and sketch of the λ values whose corresponding maps F_λ have an attracting fixed point.