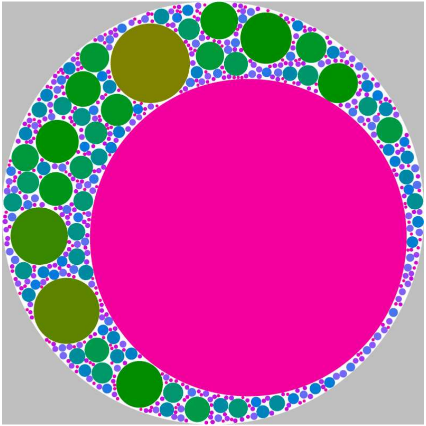
JMM 2018, San Diego, California

A Property of Area and Perimeter

Douglas Dunham
Dept. of Computer Science
Univ. of Minnesota, Duluth
Duluth, MN 55812, USA

John Shier
6935 133rd Court
Apple Valley, MN 55124 USA



Outline

- Background
- The original algorithm
- The new algorithm
- Variations
- Conclusions and future work
- Contact information

Background

Our original goal was to randomly fill a region R of area A with successively smaller copies of a motif, creating a fractal pattern.

We achieved that goal with our original algorithm, which we describe below. For that algorithm. The size of the motifs was specified by an inverse power law.

Recently we used a modified version of the algorithm in which the size of the motifs was determined using an estimate of the amount of "room" left after each placement.

We have evidence that the size of the motifs in the new algorithm also obeys and inverse power law.

The Original Algorithm

We found experimentally that we could fill the region R if for $i = 0, 1, 2, \ldots$, the area A_i of i-th motif obeyed an inverse power law:

$$A_i = \frac{A}{\zeta(c, N)(N + i)^c}$$

where where $c > 1$ and $N > 0$ are parameters, and $\zeta(c, N)$ is the Hurwitz zeta function: $\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$ (and thus $\sum_{k=0}^{\infty} A_i = A$).

We call this the **Area Rule**

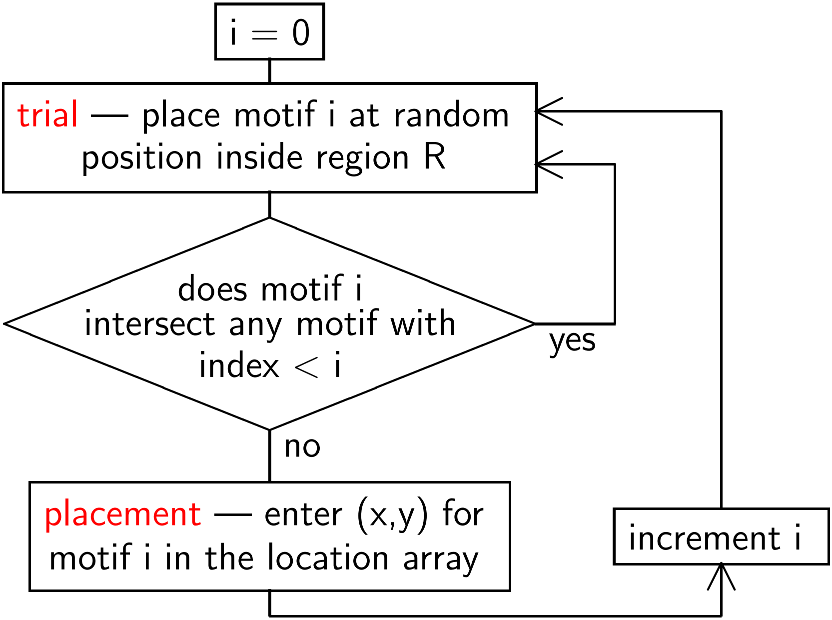
Algorithm Details

The algorithm works by successively placing copies m_i of the motif at locations inside the bounding region R.

This is done by repeatedly picking a random **trial** location (x, y) inside R until the motif m_i placed at that location doesn't intersect any previously placed motifs.

We call such a successful location a **placement**. We store that location in an array so that we can find successful locations for subsequent motifs.

A Flowchart for the Algorithm



A Sample Pattern of Peppers

The New Algorithm

The new algorithm is the same as the original algorithm except for the determination of the size of the next motif. Rather than specifying the size by the area rule, we use information about the part of the region R not yet filled to give the size of the next motif.

Now, for discussion, we will restrict both R the motifs to be disks, Although the algorith works in the general case too.

We use the term **gasket** for the unfilled part of R after placing the first i motifs. So the gasket looks like Swiss cheese when the motifs are disks. Now we let A_i be the area of the gasket and P_i be its perimeter, then the radius r_{i+1} of the next disk is given by:

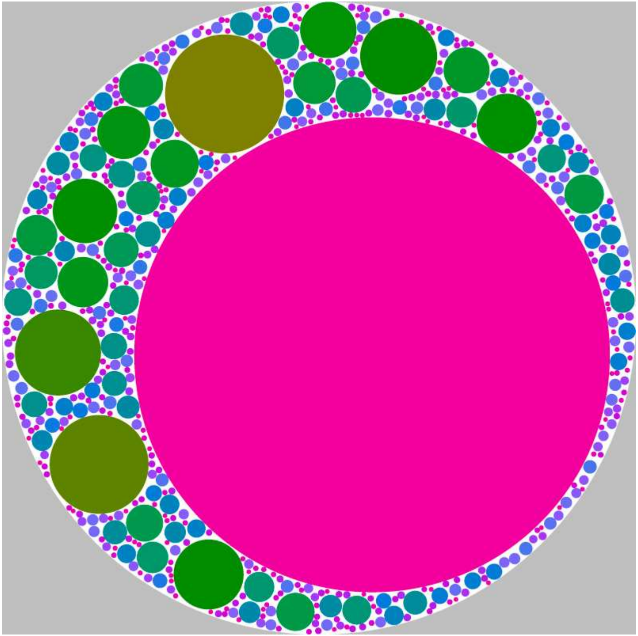
$$r_{i+1} = \gamma \frac{A_i}{P_i}$$

where $\gamma > 0$ is a dimensionless parameter which can be as large as 3 (but if $\gamma > 2$ the algorith is modified as discussed below).

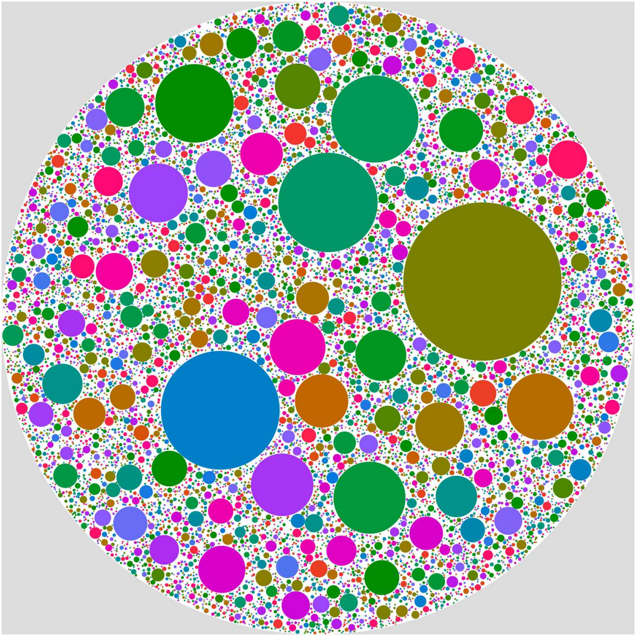
Notes

1. In the formula above, the size of the next motif is certainly proportional to A_i since that is the remaining area. But P_i measures the amount of "fragmentation" of that area, hence the inverse relationship.
2. To get started, the area and perimeter of the bounding circle are πR^2 and $2\pi R$ respecitively, so the initial area/perimeter ratio is $R/2$. Thus if $\gamma > 2$ the first disk doesn't fit within the bounding circle.

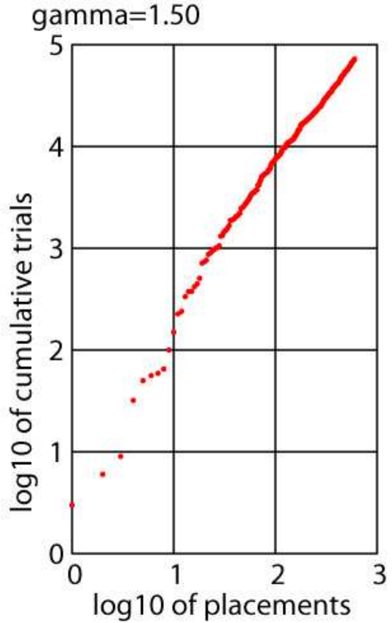
A Sample Pattern ($\gamma = 3/2$)



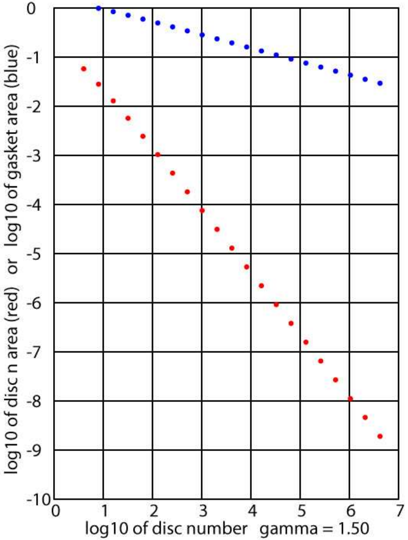
Another Pattern ($\gamma = 1/2$)



Log-log plot of trials versus placements



Log-log plot of disk areas and the gasket area



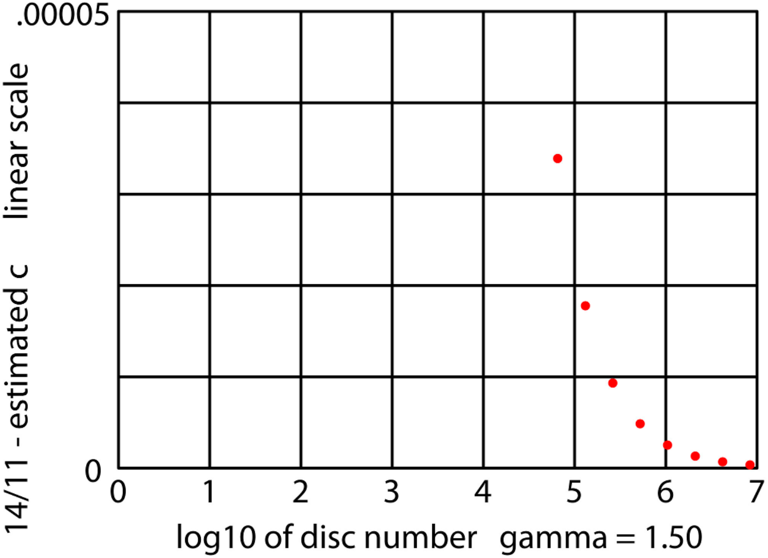
Notes

1. The plot above was again for $\gamma = 3/2$. The plot of the areas appears to be close to linear, indicating a power relationship. The slope of the last pair is -1.272727, which appears to be the beginning of a repeating decimal, and thus hints at a rational value for the exponent.
2. By running the algorithm with different values of γ, it was found that the exponent seemed to be given by

$$c = -\frac{4 + 2\gamma}{4 + \gamma}$$

3. The next plot shows how fast the estimated values of c (from successive disk area values at $i =$ a power of 2) converge to the value $14/11$ given by the formula above when $\gamma = 3/2$.

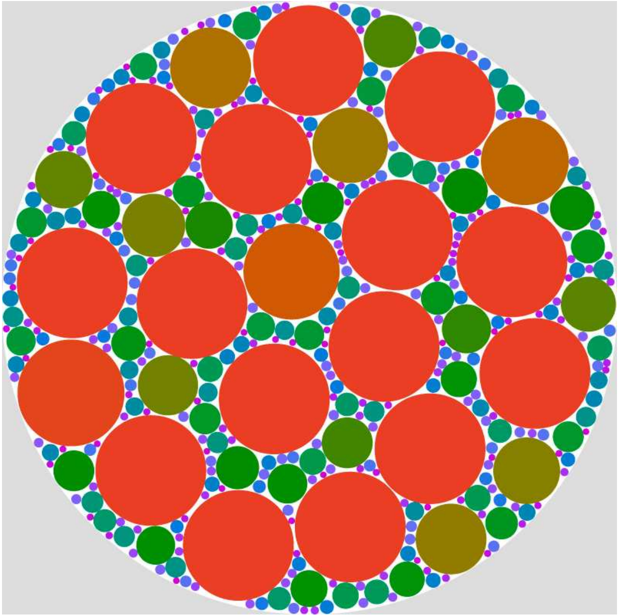
Convergence of c to $14/11$



Variation 1

- We have noted that the new algorithm as given will not work if $\gamma > 2$. But we can start with disks of a fixed size (smaller than the bounding circle of course) and keep placing them until the disks given by the radius formula are smaller, then switch to the new algorithm. The next slide shows a sample pattern when $\gamma = 5/2$.
- It can be shown that the fractal dimension of the disk pattern is given by $D = 2/c$ in both the new algorithm and this variant.

A Pattern using the modified algorithm



Variation 2 (3 dimensions)

- Another variation is to consider spheres within a bounding sphere in 3 dimensions. Here the next radius is given by the formula

$$r_{i+1} = \gamma \frac{V_i}{A_i}$$

  where V_i is the gasket volume and A_i is the gasket surface area.
- By running several samples with different γ values, we discovered that the sphere volumes seemed to follow a power law whose exponent is given by:

$$c = -\frac{36 + 3\gamma}{36 + 2\gamma}$$

Future Work

- Prove that the new algorithm always converges for all γ less than some minimum value.
- Prove that the areas of the disks, or spheres in 3D, decrease according to a power law.
- Prove that the power laws are given by the formulas above.
- One guesses that an appropriate form of the new algorithm should also work in dimensions 1 and greater than 3. This could be investigated.

Thank You!

To the JMM organizers

Contact Information:
Doug Dunham
Email: ddunham@d.umn.edu
Web: http://www.d.umn.edu/~ddunham

John Shier
Email: johnpf99@frontiernet.net
Web: http://www.john-art.com/stat_geom_linkpage.html