PERIODIC FRACTAL PATTERNS

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ABSTRACT: We present an algorithm that can create patterns that are locally fractal in nature, but repeat in two independent directions in the Euclidean plane — in other word "wallpaper patterns". The goal of the algorithm is to randomly place progressively smaller copies of a basic sub-pattern or *motif* within a fundamental region for one of the 17 wallpaper groups. This is done in such a way as to completely fill the region in the limit of infinitely many motifs. This produces a fractal pattern of motifs within that region. Then the fundamental region is replicated by the defining relations of the wallpaper group to produce a repeating pattern. The result is a pattern that is locally fractal, but repeats globally — a mixture of both randomness and regularity. We show several such patterns.

Keywords: Fractals, wallpaper groups, algorithm

1. INTRODUCTION

We have previously described how to fill a planar region with a series of progressively smaller randomly-placed sub-patterns or *motifs* producing pleasing patterns [2, 5]. In this paper we explain how the basic algorithm can be modified to fill fundamental regions of wallpaper groups, and thus create wallpaper patterns. Figure 1 shows a random circle pattern with symmetry group $p1$. Thus to create our patterns, we first fill a fundamental region for one of the 17 2-dimensional crystallographic groups (or wallpaper groups) with randomly placed, progressively smaller copies of a motif, such as the circles in Figure 1. Then we apply transformations of the wallpaper group to the fundamental region in order to tile the plane. This produces a locally random, but globally symmetric pattern.

In the next section we describe how the basic algorithm works. Then we recall a few facts about wallpaper groups, and indicate how the alorithm can be modified to produce filled fundamental regions for those groups. Next we exhibit some patterns. Finally, we draw conclusions and summarize the results.

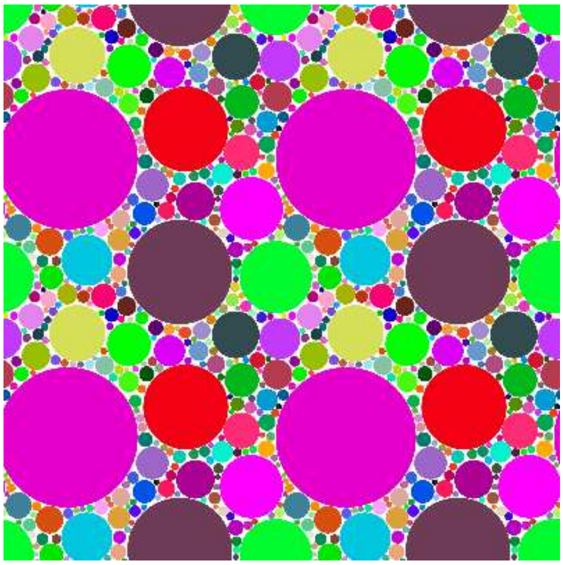

Figure 1: A circle pattern with p1 symmetry.

2. THE ALGORITHM

The algorithm works by placing a sequence of progressively smaller sub-patterns or *motifs* m_i within a region R so that a motif does not overlap any previously placed motif. Random locations are tried until a non-overlapping one is found. This process continues indefinitely if the motifs adhere to an "area rule" which is described following the algorithm:

For each $i = 0, 1, 2, \ldots$
  Repeat:
    Randomly choose a point within R to place the i-th motif m_i.
  Until (m_i doesn't intersect any of $m_0, m_1, \ldots, m_{i-1}$)
  Add m_i to the list of successful *placements*
Until some stopping condition is met, such as a maximum value of i or a minimum value of A_i.

It has been found experimentally by the second author that the algorithm does not halt for a wide range of choices of shapes of R and the motifs provided that the motifs obey an inverse power *area rule*: if A is the area of R, then for $i = 0, 1, 2, \ldots$ the area of m_i, A_i, can be taken to be:

$$A_i = \frac{A}{\zeta(c,N)(N+i)^c} \qquad (1)$$

where $c > 1$ and $N > 1$ are parameters, and $\zeta(c,N)$ is the Hurwitz zeta function: $\zeta(s,q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s}$. Thus $\lim_{n\to\infty} \sum_{i=0}^{n} A_i = A$, that is, the process is space-filling if the algorithm continues indefinitely. In the limit, the fractal dimension D of the placed motifs can be computed to be $D = 2/c$ [5]. Examples of the algorithm written in C code can be found at the second author's web site [6]. It is conjectured by the authors that the algorithm does not halt for non-pathological shapes of R and m_i, and "reasonable" choices of c and N (depending on the shapes of R and the m_is). In fact this has been proved for $1 < c < 1.0965\ldots$ and $N \geq 1$ by Christopher Ennis when R is a circle and the motifs are also circles [3].

3. WALLPAPER PATTERNS

It has been known for more than a century that there 17 different kinds of patterns in the Euclidean plane that repeat in two independent directions. Such patterns are called *wallpaper patterns* and their symmetry groups are called plane crystallographic groups or *wallpaper groups*. In 1952 the International Union of Crystallography (IUC) established a notation for these groups. In 1978 Schattschneider wrote a paper clarifying the notation and giving an algorithm for identifying the symmetry group of a wallpaper pattern [4]. Later, Conway popularized the more general orbifold notation [1]. So, as mentioned above, we create fractal wallpaper patterns by first filling a fundamental region R for a wallpaper group with motifs, then extend the pattern using transformations of the wallpaper group. Figures 1 (above) and 2 show patterns that have $p1$ (or **o** in orbifold notation) symmetry, the simplest kind of wallpaper symmetry, with only translations in two independent directions. In Figure 2 notice that the peppers on the left edge "wrap around" and are continued on the right edge; similarly peppers on the top edge "wrap around" to the bottom.

Figure 2: A pattern of peppers with $p1$ symmetry.

We consider two issues that arise with motifs being placed in a fundamental region for a wallpaper group: (1) what to do if a motif crosses a mirror line that is the boundary of the region, and (2) what to do if the motif overlaps a center of rotation. One solution is to just let that happen, as is shown in Figure 3 for circles overlapping mirror lines. Another solution is to simply reject all

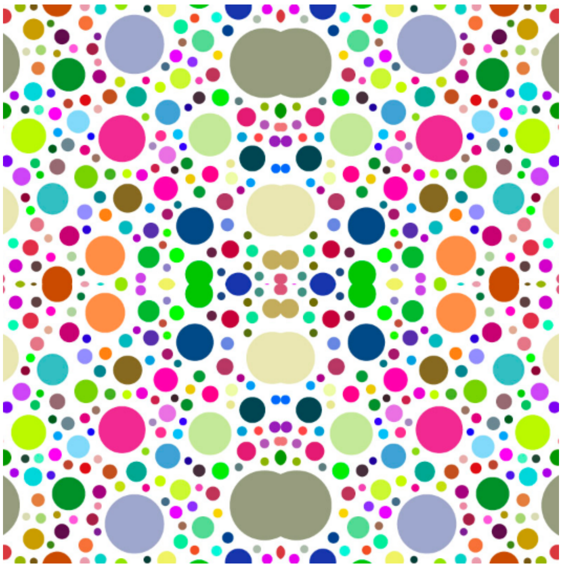

Figure 3: Overlapping circles on mirror boundaries.

motifs that cross a mirror boundary or overlap a rotation point. Figure 4 shows a pattern of hearts with *p*2*mm* symmetry that avoids the reflection lines.

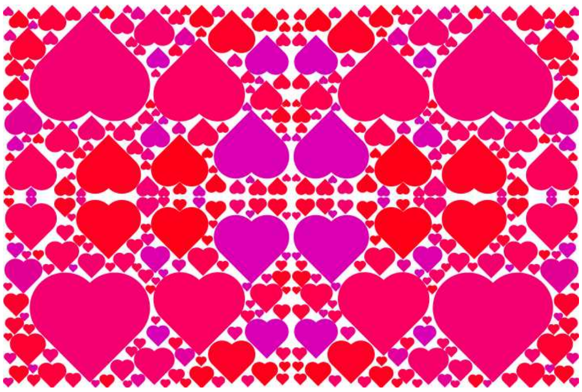

Figure 4: A pattern of hearts that avoids mirror boundaries.

If a fundamental region has a mirror boundary and the overlapping motif itself has mirror symmetry, then a more satisfactory solution is that we move the motif (perpendicularly) onto the boundary so that the mirror of the motif aligns with the boundary mirror. Also, the area rule calculation needs to be adjusted each time this happens since only half of the motif is placed within the fundamental region. Figure 5 shows a fractal pattern of flowers with *p*6*mm* symmetry with flowers centered on the mirror lines.

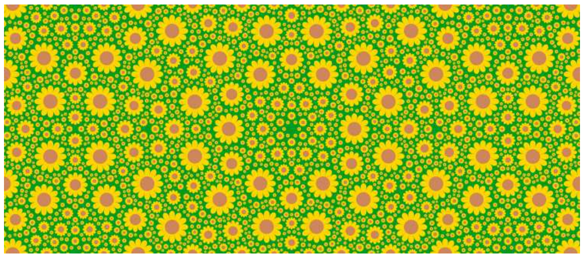

Figure 5: A pattern of flowers with some on mirror boundaries.

Similarly if the fundamental region has a rotation point and the overlapping motif has that kind of rotational symmetry, we move the motif to be centered on the rotation point. Figure 6 shows a *p*4 pattern of circles with magenta circles on one kind of 4-fold rotation point (those 4-fold points are at the corners, centers of the edges, and center of the figure). Again the area

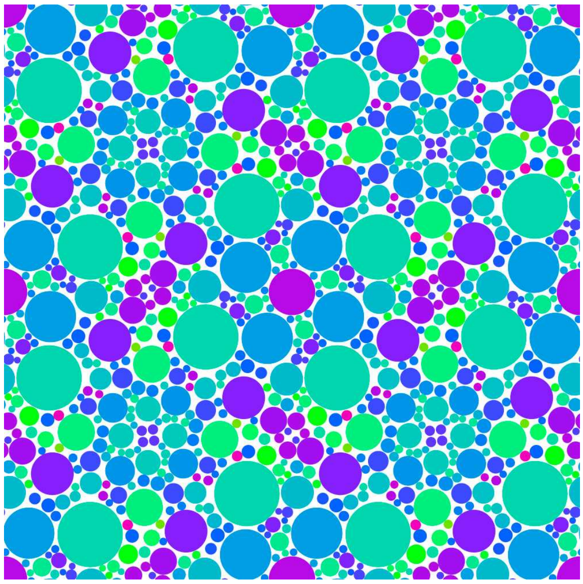

Figure 6: A *p*4 pattern of circles with some centered on 4-fold rotation points.

rule calculation needs to be adjusted if a motif overlaps one of the 4-fold points or the 2-fold point. Of course circles make excellent motifs for testing these solutions since they have mirror symmetry across any axis through the center and n-fold rotational symmetry for any n.

4. SAMPLE PATTERNS

In this section, we present a few more patterns. First, Figure 7 shows a pattern of black and white triangles on a blue background with $p4mm$ symmetry. Figure 8 shows another triangle pattern, but with $p6mm$ symmetry. Finally, Figure 9 shows a pattern of arrows with $p6mm$ symmetry.

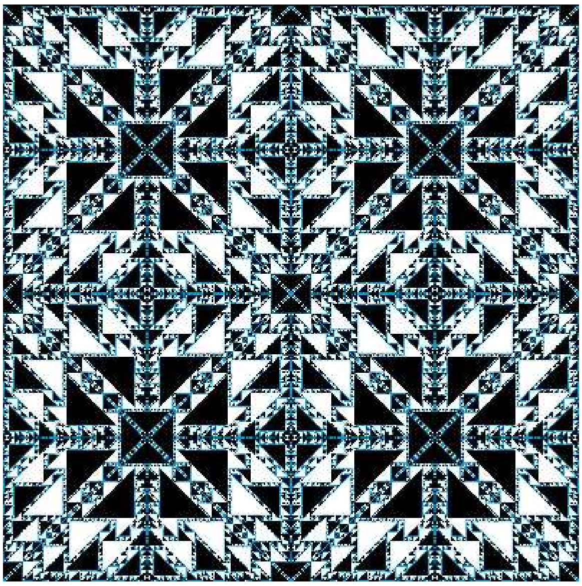

Figure 7: A triangle pattern with $p4mm$ symmetry.

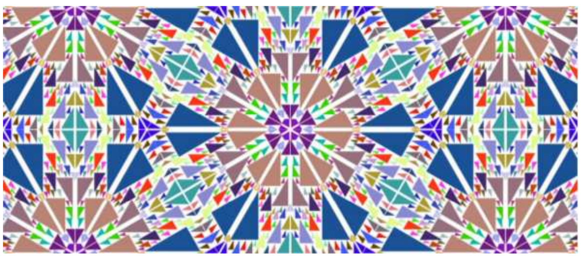

Figure 8: A triangle pattern with $p6mm$ symmetry.

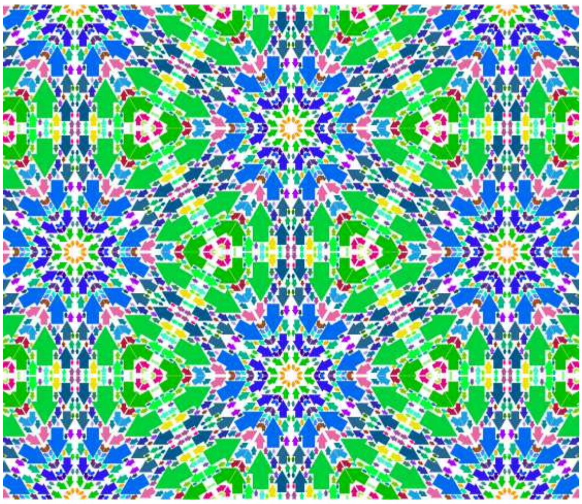

Figure 9: An arrow pattern with $p6mm$ symmetry.

5. CONCLUSIONS

We have presented a geometric algorithm that can be used to fill the fundamental region of one of the 17 wallpaper groups with a sequence of progressively smaller motifs. In turn that filled fundamental region can be replicated about the Euclidean plane to produce a repeating pattern that is locally fractal.

In the future we hope to extend this concept to hyperbolic patterns and patterns with color symmetry.

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