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A Family of Butterfly Patterns

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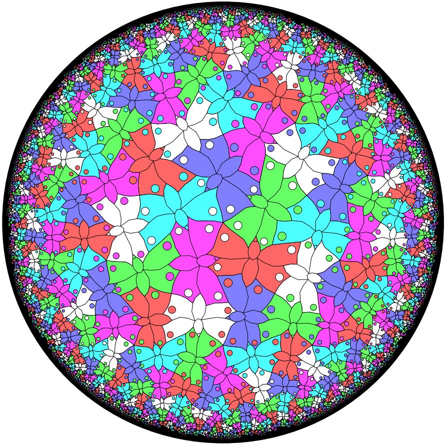


Outline

- A detour: A papercrafted fish pattern on the triply periodic polyhedron $\{6, 6 \,|\, 3\}$ — joint work with Lisa Shier
- Families of patterns - the basic idea
- Some theory
- The family of butterfly patterns
- Other families of patterns
- Future research

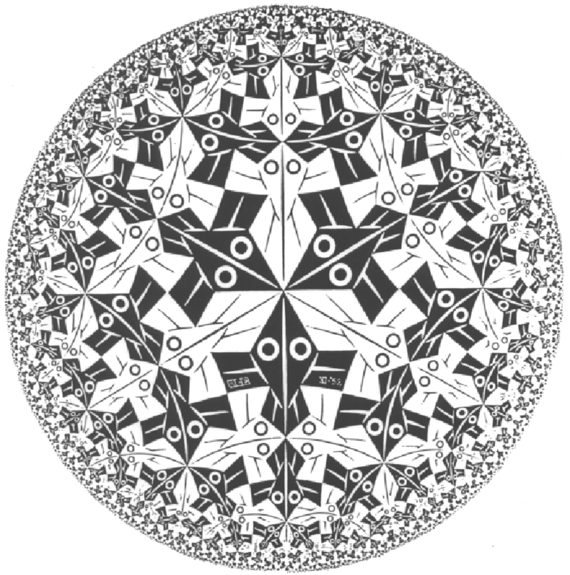
Detour: A Fish Pattern on the $\{6, 6 \,|\, 3\}$ Poyhedron

Joint work with Lisa Shier

Detour Outline

- Background and motivation
  - M.C. Escher's *Circle Limit I* and *Circle Limit III*
  - Regular $\{p, q \mid r\}$ triply periodic polyhedra
  - Previous polyhedra and their aesthetic problems
- The papercrafted part of a $\{4, 6 \mid 4\}$ polyhedron
- A part of the $\{6, 6 \mid 3\}$ polyhedron that solves all the problems

Escher's Woodcut Circle Limit I



Aesthetic Problems with Circle Limit I per Escher

1. The fish were not consistently colored along backbone lines — they alternated from black to white and back every two fish lengths.
2. The fish also changed direction every two fish lengths — thus there was no “traffic flow” (Escher’s words) in a single direction along the backbone lines.
3. The fish are very angular and not “fish-like”

Escher's Woodcut Circle Limit III

— solved the problems

Regular Triply Repeating Polyhedra

In 1926 H.S.M. Coxeter defined *regular skew polyhedra* (apeirohedra) to be infinite polyhedra repeating in three independent directions in Euclidean 3-space, with the symmetry group of isometries being transitive on flags.

Coxeter denoted them by the extended Schläfli symbol $\{p, q \mid r\}$ which denotes the polyhedron composed of p-gons meeting q at each vertex, with regular r-sided polygonal holes.

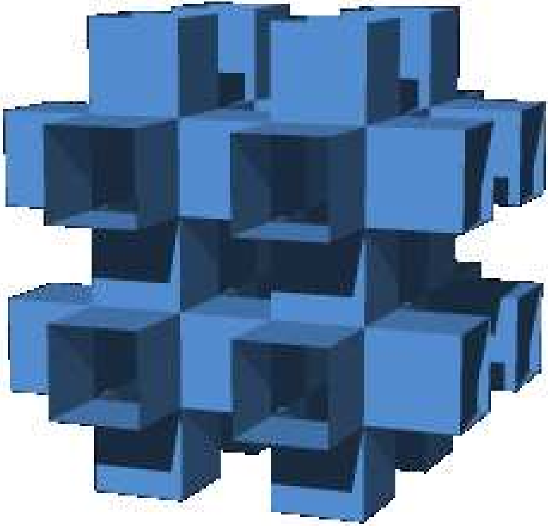
Coxeter and John Flinders Petrie proved that there are exactly three of them: $\{4, 6 \mid 4\}$, $\{6, 4 \mid 4\}$, and $\{6, 6 \mid 3\}$.

Since the sum of the vertex angles is greater than 2π, they are considered to be the hyperbolic analogs of the Platonic solids and the regular Euclidean tessellations $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$

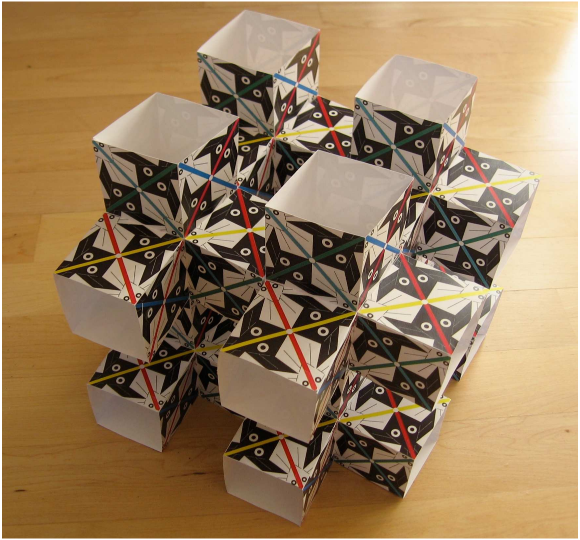
In 2012 Dunham was the first person to decorate those solids with Escher-inspired patterns.

The simplest regular skew polyhedron: $\{4, 6 \,|\, 4\}$

Also called the *Mucube* (for Multi-cube). It consists of invisible "hub" cubes connected by "strut" cubes, hollow cubical cylinders with their open ends connecting neighboring hubs.



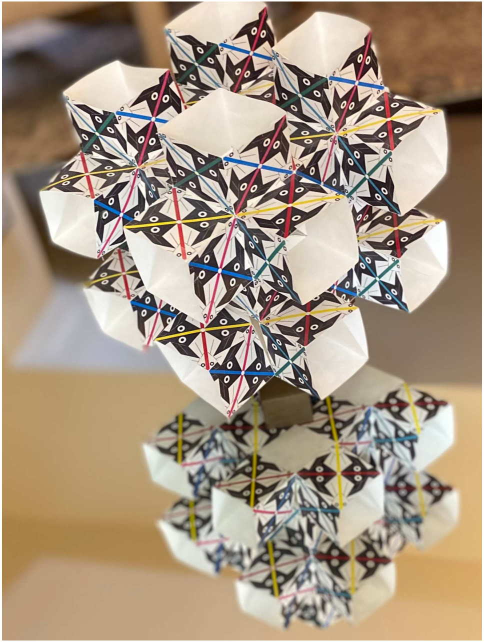
An old patterned $\{4, 6\,|\,4\}$ with fish



Problems with the old fish polyhedron

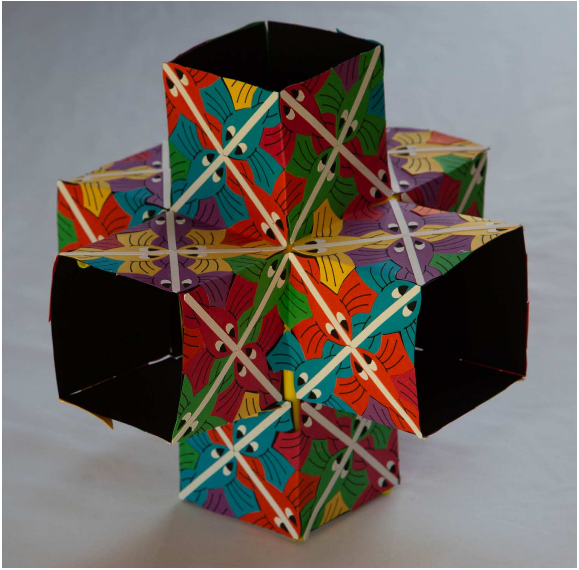
1. The same three problems Escher saw in *Circle Limit I*.
2. A fourth problem: the backbone lines of a particular color are not parallel — which can be seen in a mirror.

The old fish polyhedron on a mirror



A new papercrafted fish pattern on the $\{4,6\,|\,4\}$ polyhedron

Fixes the first and third problems.



The papercrafted $\{4, 6 \,|\, 4\}$ polyhedron on a mirror

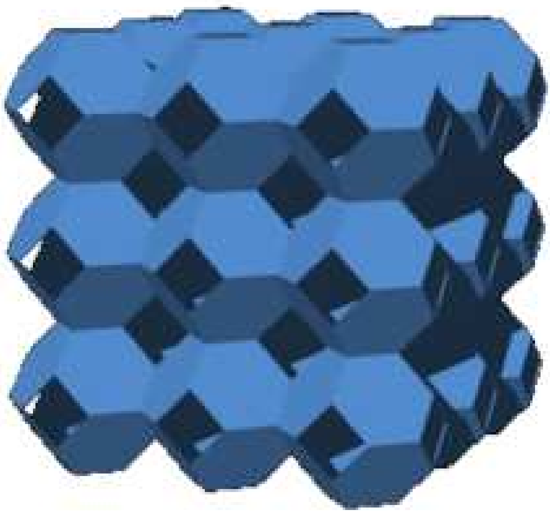
Fixes the fourth problem too, but not the second one.

Colors of fish on the $\{4, 6 \,|\, 4\}$ polyhedron

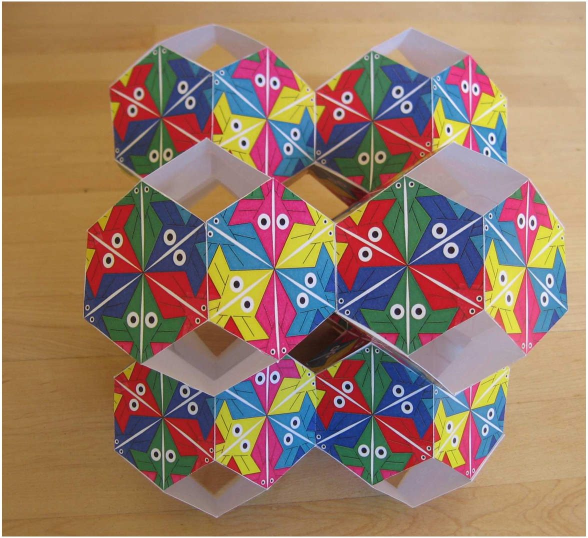
1. There are six families of fish backbone lines that are parallel to the face diagonals of a cube.
2. All the fish in one family are the same color.

The dual of the Mucube is the $\{6,4\,|\,4\}$ polyhedron

Also called the *Muoctahedron* (for Multi-octahedron). It consists of truncated octahedra in a cubic lattice arrangement, connected on their invisible square faces (which are also the square holes between the truncated octahedra).

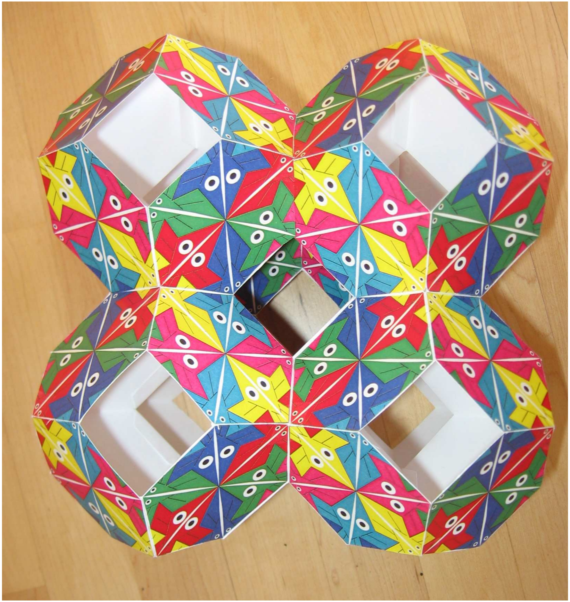


An angular fish pattern on the $\{6, 4 \mid 4\}$ polyhedron



A top view of the fish pattern on the $\{6,4\,|\,4\}$ polyhedron

It solves Escher's first problem, but still has problems two and three.



The $\{6, 6 \mid 3\}$ polyhedron is self-dual

Also called the *Mutetrahedron* (for Multi-tetrahedron). It consists of truncated tetrahedra in a diamond lattice arrangement, connected by their missing triangular faces to faces of invisible regular tetrahedra between them.

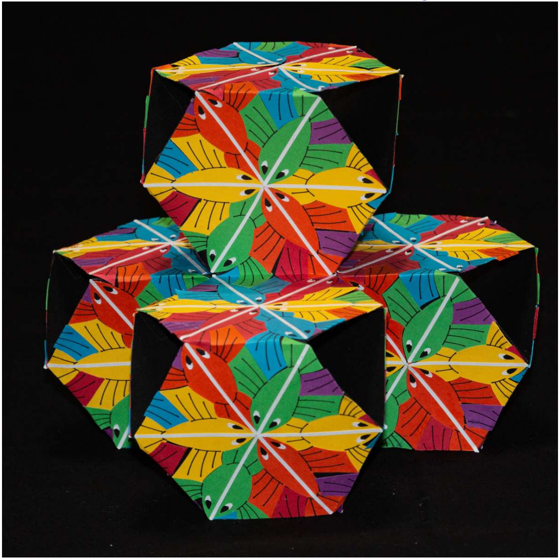
The new $\{6, 6 \mid 3\}$ patterned polyhedron

Also fixes the second, "traffic flow", problem.

Colors of fish on the $\{6, 6 \,|\, 3\}$ polyhedron

1. Again, there are six families of fish backbone lines that go through the centers of the hexagon faces of the $\{6, 6 \,|\, 3\}$ polyhedron.
2. And again, the fish in one family are the same color.
3. Each of the families is parallel to one of the sides of a tetrahedron — which can be one of the truncated tetrahedra, since all the (patterned) truncated tetrahedra in the $\{6, 6 \,|\, 3\}$ polyhedron are translates of one another.
4. In each family half the lines of fish go one direction, and the other half go the opposite direction — so that fish of one color on one truncated tetrahedron go in opposite directions on adjacent faces.

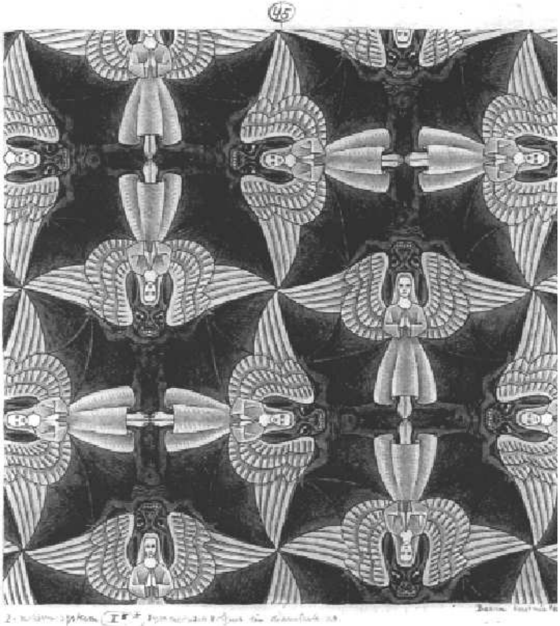
A Papercrafted Version of the New Polyhedron: End Detour



Families of Patterns

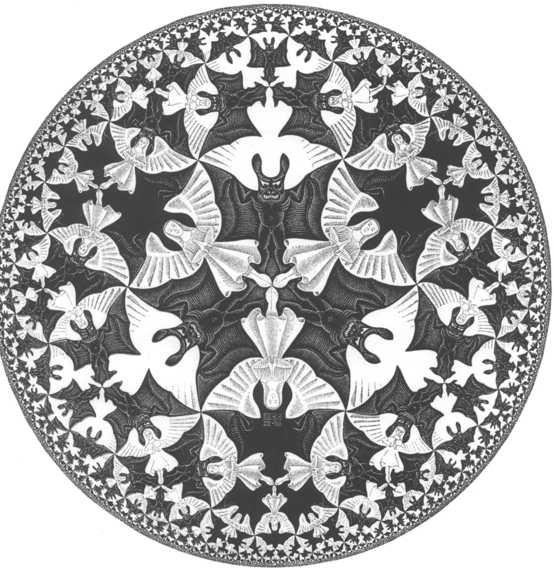
- Many artists have created related works.
- Example: the paintings of Picasso's "Blue Period"
- We will examine works that are related in a more precise mathematical way.
- We will consider patterns that can be classified by two integer parameters.
- M.C. Escher was most likely the first artist to create patterns related in this way.
- These patterns can exist in any of the three "Classical Geometries": the Euclidean plane, the sphere, and the hyperbolic plane.

Escher's (Euclidean) Regular Division Drawing 45



Escher's "Angles and Devils" carved sphere

Escher's (hyperbolic) Circle Limit IV



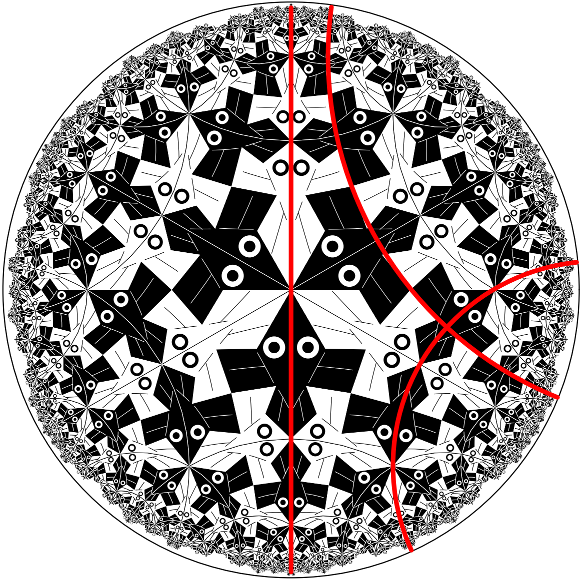
The Classical Geometries

- Euclidean geometry — zero curvature, i.e. flat
- Spherical geometry — constant positive curvature
- Hyperbolic geometry — constant negative curvature
- The hyperbolic plane cannot be smoothly embedded in Euclidean 3-space, unlike the sphere (proved in 1901 by David Hilbert).
- Therefore we must rely on **models** hyperbolic geometry — Euclidean constructs that can be interpreted as hyperbolic objects.

A Model of Hyperbolic Geometry

- M.C. Escher (and other artists) used the **Poincaré circle model** of hyperbolic geometry.
- Hyperbolic points are represented by Euclidean points within a (Euclidean) **bounding circle**.
- Hyperbolic lines are represented by (Euclidean) circular arcs that are orthogonal to the bounding circle (including diameters as special cases).
- Preferred by M.C. Escher and other artists since (1) it is contained in a finite area of the Euclidean plane, and (2) it is *conformal*, the hyperbolic measure of an angle is the same as its Euclidean measure, so that motifs retain approximately the same shape as they get smaller toward the bounding circle.

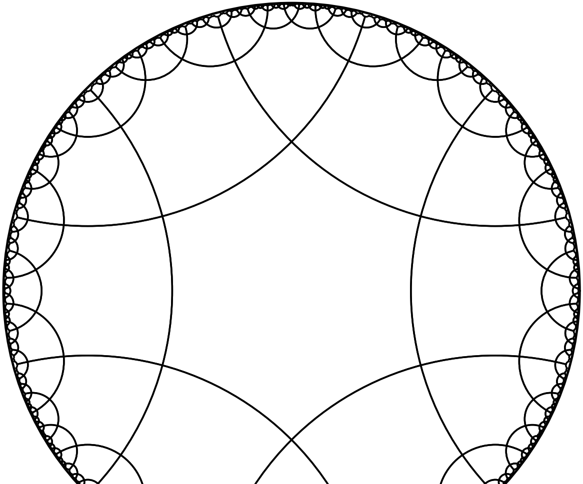
Escher's Circle Limit I Showing Hyperbolic Lines.



Repeating Patterns and Regular Tessellations

- A *repeating pattern* in any of the 3 "classical geometries" (Euclidean, spherical, and hyperbolic geometry) is composed of congruent copies of a basic subpattern or *motif*.
- For example if we ignore color, one butterfly is a motif for the butterfly pattern on the title page.
- The *regular tessellation*, $\{p, q\}$, is an important kind of repeating pattern composed of regular p-sided polygons meeting q at a vertex.
- If $(p-2)(q-2) < 4$, $\{p, q\}$ is a spherical tessellation (assuming $p > 2$ and $q > 2$ to avoid special cases).
- If $(p-2)(q-2) = 4$, $\{p, q\}$ is a Euclidean tessellation.
- If $(p-2)(q-2) > 4$, $\{p, q\}$ is a hyperbolic tessellation. The next slide shows the $\{6, 4\}$ tessellation.
- Escher based his 4 "Circle Limit" patterns, and many of his spherical and Euclidean patterns on regular tessellations.

The $\{6, 4\}$ tessellation.



A Table of the Regular Tessellations

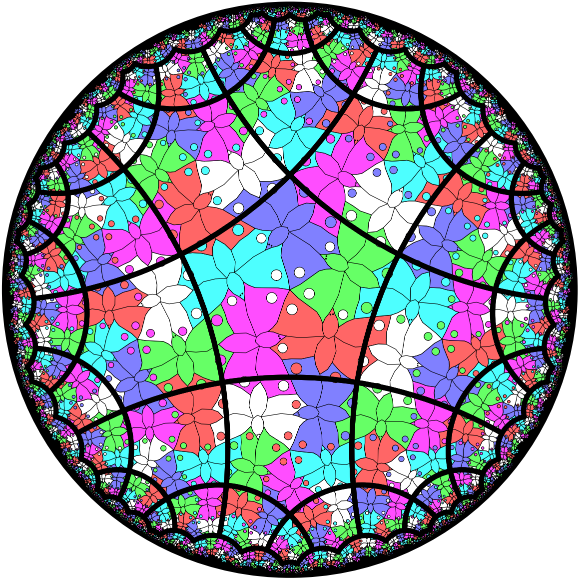
q \ p	3	4	5	6	7	8	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
8	$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$
7	$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$
6	$\square$	$*$	$*$	$*$	$*$	$*$	$\cdots$
5	$\bigcirc$	$*$	$*$	$*$	$*$	$*$	$\cdots$
4	$\bigcirc$	$\square$	$*$	$*$	$*$	$*$	$\cdots$
3	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\square$	$*$	$*$	$\cdots$

$\square$ - Euclidean tessellations

$\bigcirc$ - spherical tessellations

$*$ - hyperbolic tessellations

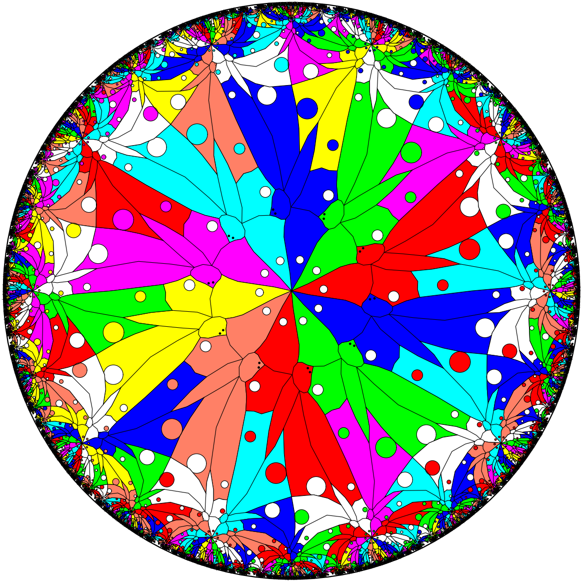
The $\{5, 4\}$ tessellation overlaying a butterfly pattern:
5 butterflies meet at left front wingtips; 4 meet at right rear wingtips



Parameterized Families of Patterns

- If a pattern is based on an underlying $\{p, q\}$ tessellation, we can conceive of other patterns with the same motif (actually somewhat distorted) based on a different tessellation $\{p', q'\}$.
- This observation leads us to consider a whole *family* of such patterns indexed by p and q.
- We use (p, q) to denote the pattern of the family that is based on the tessellation $\{p, q\}$.
- For example, the butterfly pattern above would be denoted $(5, 4)$.
- Unfortunately, large values of p or q or both usually do not produce aesthetically appealing patterns, since such values lead to distortion of the motif and/or push most of the pattern outward near the bounding circle.

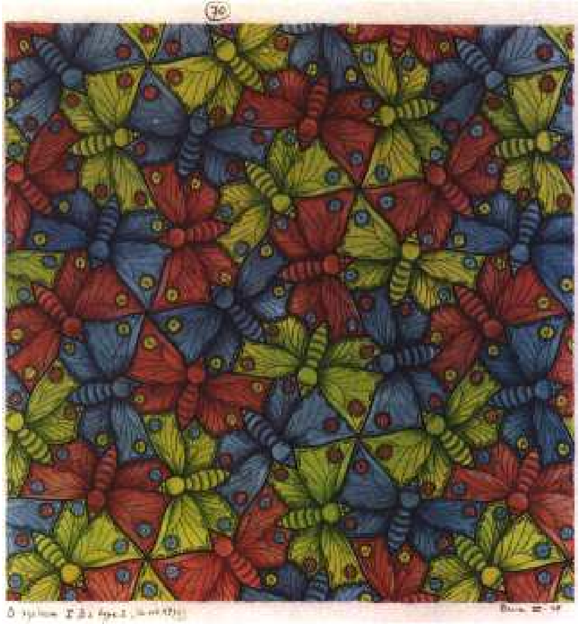
A $(10, 4)$ butterfly pattern showing distortion



The Family of Butterfly Patterns

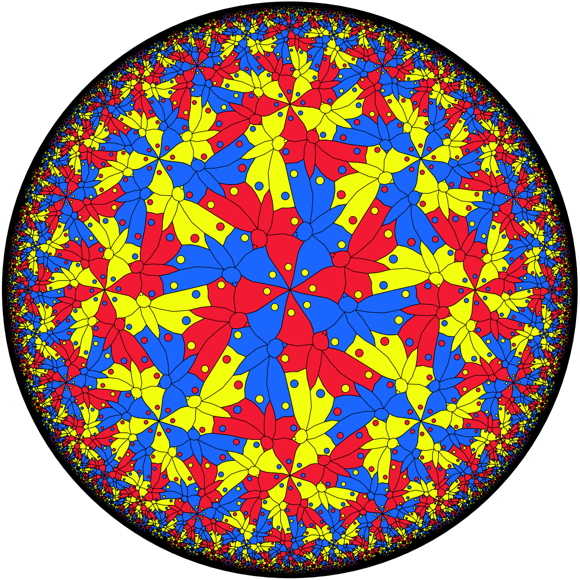
- Theoretically, we can create a butterfly pattern, denoted (p, q), that is based on $\{p, q\}$ like the one above for any values of p and q provided $p \geq 3$ and $q \geq 3$.
- For these patterns, p butterflies meet at their left front wing tips and q butterflies meet at their right rear wings.
- Escher created only one member of this family of patterns, his Regular Division Drawing Number 70, based on the Euclidean hexagon tessellation $\{6, 3\}$ (which we would denote $(6, 3)$). At least 3 colors are needed to satisfy the map-coloring principle at the meeting points of right rear wings.
- Following Escher, we add the restriction to our patterns that all circles on the butterfly wings around a p-fold meeting point of left wingtips be a different color from the butterflies meeting there.
- The hyperbolic butterfly pattern $(5, 4)$ requires at least five colors for color symmetry since five is prime, and six colors if the circles on the wings are to be a different color.

Escher's 3-colored butterfly pattern
Regular Division Drawing Number 70

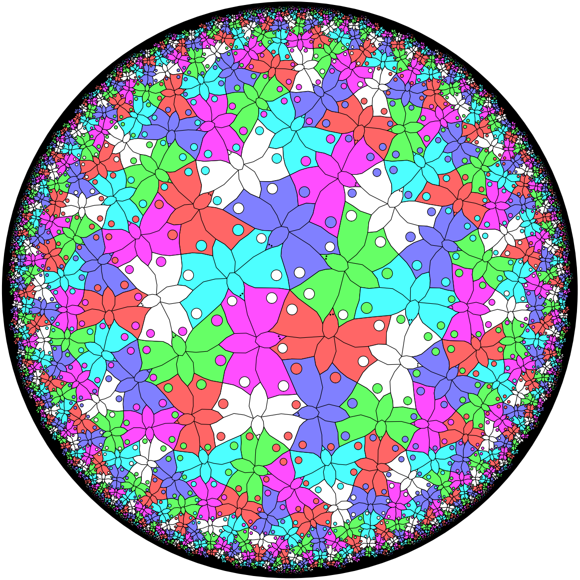


Schattschneider and Walker's $(3, 5)$ butterfly pattern on an icosahedron

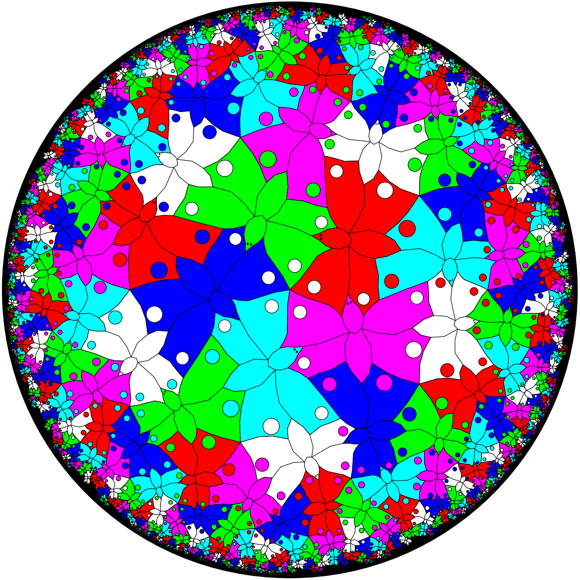
A 3-colored $(8, 3)$ butterfly pattern



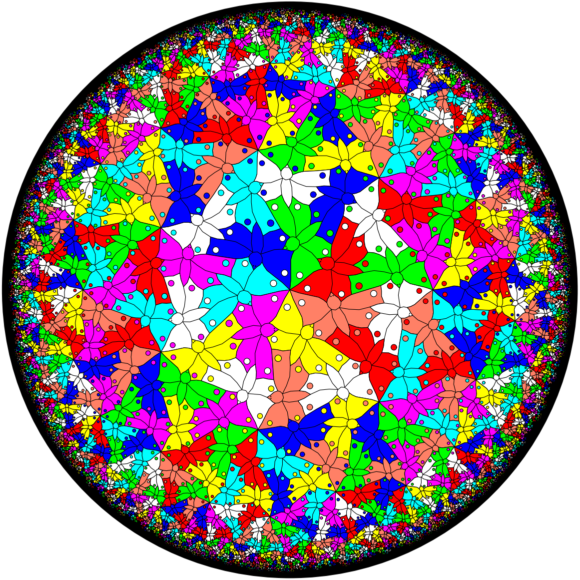
The 6-colored $(5, 4)$ pattern



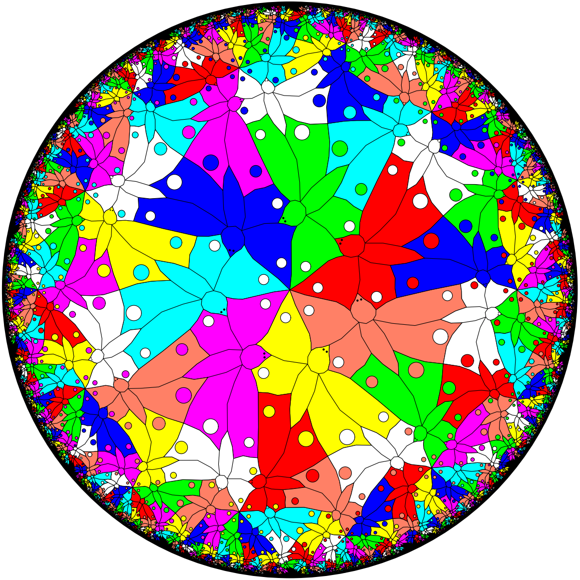
A 6-colored $(5, 5)$ butterfly pattern



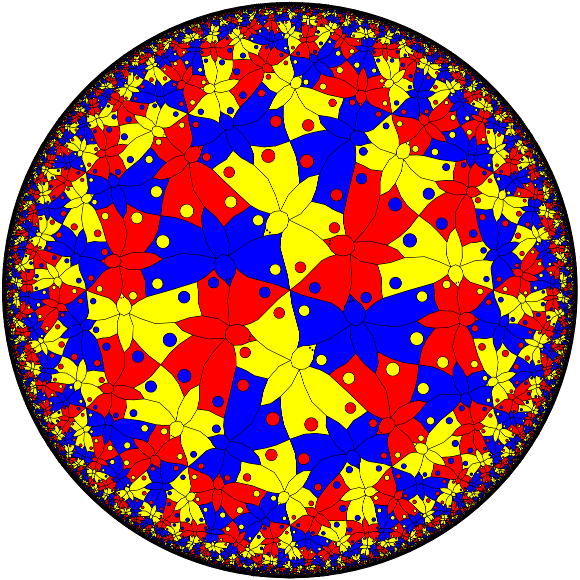
An 8-colored $(7, 3)$ butterfly pattern



An 8-colored $(7, 4)$ butterfly pattern



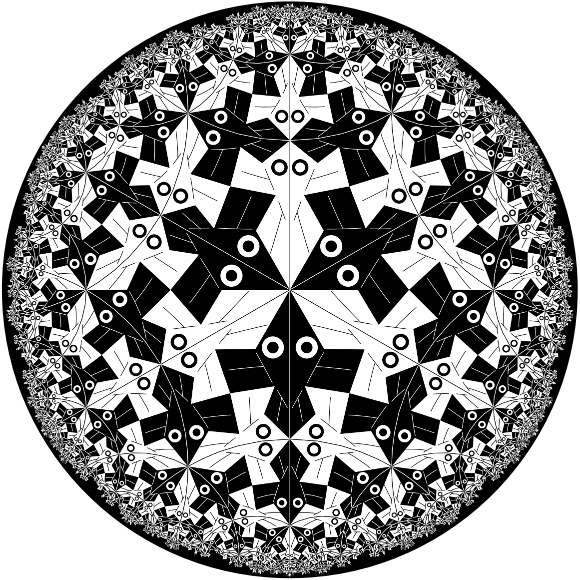
A 3-colored $(6, 4)$ butterfly pattern
that violates the color of circles convention



Other Families of Patterns

- We have seen Escher's three "Angles and Devils" patterns, all based on $\{p, q\}$ tessellations with p even:
  - Regular Division Drawing 45, which is based on the "square" $\{4, 4\}$ tessellation.
  - The carved sphere, based on the $\{4, 3\}$ tessellation.
  - The hyperbolic *Circle Limit IV*, based on the $\{6, 4\}$ tessellation.
- The fact that p must be even is a *divisibility condition*.
- Escher's *Circle Limit I* (shown below) is the only pattern he made in that family. These patterns are based on on $\{p, q\}$ tessellations with both p and q even, since the backbones of both the black and white fish are lines of reflection.

Escher's hyperbolic *Circle Limit I* pattern
Based on the $\{6, 4\}$ tessellation.



Future Work

- Investigate other families of Escher-like patterns, and draw such patterns, including 3-parameter families.
- Automatically generate the colors so that the pattern is symmetrically colored. Currently this must be done manually for each pattern in a family.
- Generate patterns of a family on a polyhedron of genus ≥ 2, since such polyhedra have the hyperbolic plane as their universal covering space.

Thank You

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