Group distance magic and antimagic graphs *

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Abstract

Given a graph G with n vertices and an Abelian group A of order n, an A-distance antimagic labelling of G is a bijection from $V(G)$ to A such that the vertices of G have pairwise distinct weights, where the weight of a vertex is the sum (under the operation of A) of the labels assigned to its neighbours. An A-distance magic labelling of G is a bijection from $V(G)$ to A such that the weights of all vertices of G are equal to the same element of A. In this paper we study these new labellings under a general setting with a focus on product graphs. We prove among other things several general results on group antimagic or magic labellings for Cartesian, direct and strong products of graphs. As applications we obtain several families of graphs admitting group distance antimagic or magic labellings with respect to elementary Abelian groups, cyclic groups or direct products of such groups.

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1 Introduction

We study two graph labellings that belong to a large family of magic and antimagic-type labellings. Vaguely speaking, a magic-type labelling of a graph $G = (V(G), E(G))$ is an injective mapping from $V(G) \cup E(G)$ to a set of integers such that the sum of the labels of the elements adjacent and/or incident to an element of $V(G) \cup E(G)$ is equal to the same constant. An introductory text on magic-type labellings was published by W. Wallis [19], and a book dealing with edge-antimagic labellings of graphs was written by M. Baca and M. Miller [2]. A thorough dynamic survey of graph labellings including other kinds of labellings is being maintained by J. Gallian [9].

In particular, a *distance magic labelling* of a graph G with $|V(G)| = n$ is a bijection from $V(G)$ to $\{1, 2, \ldots, n\}$ such that the sum of the labels of all neighbours of every vertex $x \in V(G)$, called the *weight* of x, is equal to the same constant, called the *magic constant*. This labelling has been also called a 1-*vertex magic vertex labelling* or a Σ-*labelling* by various authors. A graph that admits a distance magic labelling is often called a *distance magic graph*. The concept of distance magic labelling has been motivated by the construction of magic squares. Although the first results were probably obtained by V. Vilfred in a Ph.D. thesis [18] in 1994, it gained more attention only recently. For a survey, we refer the reader to [1].

A related concept is the notion of *distance antimagic labelling*. This is again a bijection from $V(G)$ to $\{1, 2, \ldots, n\}$ but this time different vertices are required to have distinct weights. A more restrictive version of this labelling is the (a, d)-*distance antimagic labelling*. It is a distance antimagic labelling with the additional property that the weights of vertices form an arithmetic progression with difference d and first term a.

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In [8] the second author introduced the concept of group distance magic labelling in the case when the group involved is cyclic. In the present paper we deal with the general case when the group involved is Abelian. Given a graph G with n vertices and an Abelian group A of order n, an *A-distance magic labelling* of G is defined to be a bijection f from $V(G)$ to A such that the weight of every vertex $x \in V(G)$ is equal to the same element of A, called the *magic constant* of G with respect to f. Here the *weight* of x under f is defined as

$$w_f(x) = \sum_{y \in N_G(x)} f(y)$$

with the understanding that the addition is performed in the group A, and $N_G(x)$ is the (open) neighbourhood of x in G. We usually write $w(x)$ in place of $w_f(x)$ and $N(x)$ instead of $N_G(x)$ if there is no risk of confusion. Obviously, every graph with n vertices and a distance magic labelling [19] also admits a $\mathbb{Z}_n$-distance magic labelling. However, the converse is not necessarily true.

Besides group distance magic labelling, we will also study group distance antimagic labelling in this paper. With G and A above, an *A-distance antimagic labelling* of G is a bijection f from $V(G)$ to A such that the weights of all vertices of G are pairwise distinct. Since G and A have the same cardinality, this is to say that every element of A occurs as the weight of precisely one vertex of G.

The main results in this paper are as follows. In Section 2 we give a necessary condition for a graph with an even number of vertices to be distance antimagic with respect to an Abelian group with a unique involution (Theorem 2.2). We then give sufficient conditions for a Cayley graph on an Abelian group to be distance antimagic or magic with respect to the same group (Theorem 2.5), and discuss the consequences of these results to Cayley graphs on elementary Abelian groups.

In Section 3 we study group distance antimagic and magic labellings of Cartesian products of graphs. Among other things we prove two general results, Theorems 3.4 and 3.12, which provide machineries for constructing new group distance antimagic or magic graphs. These can be applied to many special cases, and we illustrate this by a few corollaries with a focus on hypercubes, Hamming graphs and Cartesian products of cycles.

In Section 4 we prove a general result (Theorem 4.1) that can be used to construct group distance antimagic graphs by means of direct products of graphs. We then give a few families of such graphs in four corollaries.

In Section 5 we concentrate on distance antimagic graphs with respect to cyclic groups $\mathbb{Z}_n$. We obtain several general results (Theorems 5.2, 5.5, 5.7 and 5.11) for Cartesian, direct and strong products, and we then use them to give concrete families of $\mathbb{Z}_n$-distance antimagic groups in a few corollaries. In Section 6 we give two sufficient conditions for a circulant graph with n vertices to be antimagic with respect to $\mathbb{Z}_n$. We conclude the paper with remarks and open problems.

We notice that graph labelling by Abelian groups has been studied in the literature under different settings. For example, in [7] vertex-magic and edge-magic total labellings of graphs by Abelian groups were studied. In [4], edge-magic total-labellings of countably infinite graphs by $\mathbb{Z}$ were investigated. In [13, 14, 16], vertex-magic edge-labellings of graphs by Abelian groups were studied.

We refer the reader to [17] for group-theoretic terminology. We use $A_1 \times \cdots \times A_d$ to denote the direct product of groups $A_1, \ldots, A_d$. In particular, $\mathbb{Z}_q^d = \mathbb{Z}_q \times \cdots \times \mathbb{Z}_q$ (d factors), and when $q = p$ is a prime $\mathbb{Z}_p^d$ is an elementary Abelian p-group. As usual we write the operation of an Abelian group A additively, denote its identity element by 0, and represent the inverse element of $x \in A$ by $-x$. All groups in the paper are finite and Abelian, and all graphs considered are finite, simple and undirected. As usual the cardinality of a set X is denoted by $|X|$.

2 A necessary condition, and sufficient conditions for Cayley graphs

An element of a group with order 2 is called an *involution*. A well-known result [17] due to Feit and Thompson asserts that a finite group has involutions if and only if its order is even. Let $\sigma(A) = \sum_{x \in A} x$ be the sum of the elements of A. Then $\sigma(A)$ is equal to the sum of the involutions of A. The following lemma, which can be easily verified, will be used in the proof of Theorem 2.2.

Lemma 2.1. *([7, Lemma 8]) Let A be a finite Abelian group.*

(a) *If A has exactly one involution, say, a, then $\sigma(A) = a$.*

(b) *If A has no involutions, or more than one involution, then $\sigma(A) = 0$.*

Theorem 2.2. *Let G be an r-regular graph on n vertices, where n is even. Then for any Abelian group A of order n with exactly one involution, G cannot be A-distance antimagic unless r is odd.*

Proof. Let A be an Abelian group of order n with exactly one involution, say, a. Suppose r is even and G is A-distance antimagic with f the desired labelling. Denote $w(G) = \sum_{x\in V(G)} w(x)$. Since f is an A-antimagic labelling, we have $w(G) = \sigma(A)$. On the other hand, since G is r-regular, $w(G) = \sum_{x\in V(G)} \left(\sum_{y\in N(x)} f(y)\right) = r\sigma(A)$. Hence $r\sigma(A) = \sigma(A)$. Since r is even and by Lemma 2.1 $\sigma(A) = a$ is the involution, we have $r\sigma(A) = 0$, which implies $\sigma(A) = 0$, a contradiction. □

Since the cyclic group $\mathbb{Z}_n$ has exactly one involution when n is even, Theorem 2.2 implies the following result.

Corollary 2.3. *Let G be an r-regular graph on n vertices such that both n and r are even. Then G is not $\mathbb{Z}_n$-distance antimagic.*

The fundamental theorem of finite Abelian groups [17] asserts that any finite Abelian group A can be decomposed into the direct product of cyclic subgroups of prime power orders, and this decomposition is unique up to the order of the cyclic subgroups. It can be verified that, if in this decomposition there are exactly t cyclic subgroups with order a power of 2, then A has exactly $2^t - 1$ involutions. In particular, if $|A| = n \equiv 2 \pmod 4$, then A has exactly one involution. Thus, by Theorem 2.2, we obtain:

Corollary 2.4. *Let G be an r-regular graph on n vertices such that $n \equiv 2 \pmod 4$ and r is even. There does not exists an Abelian group A of order n such that G is A-distance antimagic.*

The *exponent* [17] of a finite Abelian group A, denoted $\exp(A)$, is the least positive integer m such that $mx = 0$ for every $x \in A$. In particular, $\exp(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$ is equal to the least common multiple of $n_1, \ldots, n_k$.

Given an Abelian group A and a subset $S \subseteq A \setminus \{0\}$ such that $-S = S$, where $-S = \{-s : s \in S\}$, the *Cayley graph* $\mathrm{Cay}(A, S)$ of A with respect the *connection set* S is defined to have vertex set A such that $x, y \in A$ are adjacent if and only if $x - y \in S$. It is evident that $\mathrm{Cay}(A, S)$ is an $|S|$-regular graph.

Theorem 2.5. *Let A be a finite Abelian group of order $n = |A|$, and $G = \mathrm{Cay}(A, S)$ a Cayley graph on A of degree $r = |S|$.*

(a) *If n and r are coprime, then G is A-distance antimagic, and any automorphism of A is an A-distance antimagic labelling of G.*

(b) *If $\exp(A)$ is a divisor of r, then G is A-distance magic, and any automorphism f of A is an A-distance magic labelling of G with magic constant $\sum_{s\in S} f(s)$.*

Proof. By the definition of G, the neighbourhood in G of each vertex $x \in A$ is $\{x + s : s \in S\}$. Let f be an automorphism of A. Since A is Abelian, the weight of $x \in A$ under f is given by

$$\begin{aligned} w(x) &= \sum_{s\in S} f(x+s) \\ &= \sum_{s\in S} (f(x) + f(s)) \\ &= rf(x) + \sum_{s\in S} f(s). \end{aligned}$$

(a) Suppose n and r are coprime. If $w(x) = w(y)$ for distinct $x, y \in A$, then $rf(x) = rf(y)$ and hence $r(f(x) - f(y)) = 0$. Since $f(x) \neq f(y)$, this implies that the order $o(f(x) - f(y))$ of $f(x) - f(y)$ in A, which is greater than 1 as $f(x) - f(y) \neq 0$, is a divisor of r. On the other hand, $o(f(x) - f(y))$ is a divisor of n. Thus $o(f(x) - f(y))$ is a common divisor of n and r that is greater than 1, contradicting our assumption $\gcd(n, r) = 1$. Therefore, $w(x) \neq w(y)$ for distinct $x, y \in A$ and so f is an A-distance antimagic labelling of G.

(b) Suppose $\exp(A)$ divides r. Then for distinct $x, y \in A$, $o(f(x) - f(y))$ is a divisor of r and as such $r(f(x) - f(y)) = 0$. Thus the computation above yields $w(x) = w(y)$. Therefore, f is an A-distance magic labelling of G with magic constant $\sum_{s \in S} f(s)$. □

In part (b) of Theorem 2.5, the magic constant $\sum_{s \in S} f(s)$ relies on the automorphism f of A. This shows that a graph may have several magic labellings with respect to the same group but with distinct magic constants.

Since $\exp(\mathbb{Z}_p^d) = p$, Theorem 2.5 implies the following corollary.

Corollary 2.6. *Let $p \geq 2$ be a prime, and let $r \geq 2$ and $d \geq 1$ be integers.*

(a) *If p is not a divisor of r, then any Cayley graph on $\mathbb{Z}_p^d$ with degree r is $\mathbb{Z}_p^d$-distance antimagic, and any automorphism of $\mathbb{Z}_p^d$ is a $\mathbb{Z}_p^d$-distance antimagic labelling.*

(b) *If p is a divisor of r, then any Cayley graph $\mathrm{Cay}(\mathbb{Z}_p^d, S)$ on $\mathbb{Z}_p^d$ with degree r is $\mathbb{Z}_p^d$-distance magic, and any automorphism f of $\mathbb{Z}_p^d$ is a $\mathbb{Z}_p^d$-distance magic labelling with magic constant $\sum_{s \in S} f(s)$.*

In particular, for the elementary Abelian 2-groups $\mathbb{Z}_2^d$, Corollary 2.6 yields the following result.

Corollary 2.7. *Let $d \geq 2$ be an integer.*

(a) *Any Cayley graph on $\mathbb{Z}_2^d$ with an odd degree is $\mathbb{Z}_2^d$-distance antimagic, with any automorphism of $\mathbb{Z}_2^d$ as a $\mathbb{Z}_2^d$-distance antimagic labelling.*

(b) *Any Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^d, S)$ on $\mathbb{Z}_2^d$ with an even degree $|S|$ is $\mathbb{Z}_2^d$-distance magic, with any automorphism f of $\mathbb{Z}_2^d$ as a $\mathbb{Z}_2^d$-distance magic labelling with magic constant $\sum_{s \in S} f(s)$.*

The d-dimensional *hypercube* Q_d is defined as the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^d, S)$ with

$$S = \{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}.$$

In other words, Q_d is the graph with binary strings of length d as its vertices such that two vertices are adjacent if and only if the corresponding strings differ in precisely one position. Since Q_d is a Cayley graph on $\mathbb{Z}_2^d$ with degree d, Corollary 2.7 implies:

Corollary 2.8. *Let $d \geq 2$ be an integer.*

(a) *If d is odd, then Q_d is $\mathbb{Z}_2^d$-distance antimagic.*

(b) *If d is even, then Q_d is $\mathbb{Z}_2^d$-distance magic with magic constant $(1, \dots, 1)$ ([6]).*

As we will see in the next section, both parts of this corollary are special cases of some more general results, namely Corollary 3.8 and Theorem 3.3, respectively. In Theorem 3.9 and Remark 3.10 we will give more sufficient conditions for Q_d to be $\mathbb{Z}_2^d$-distance antimagic. The magic constant $(1, \dots, 1)$ in (b) above is obtained from the trivial automorphism of $\mathbb{Z}_2^d$. Different choices of the automorphism f in (b) of Corollary 2.7 may result in different magic constants for Q_d.

3 Labelling Cartesian products

Given d graphs $G_1, \dots, G_d$, the *Cartesian product* [12] of them, denoted $G_1 \Box \cdots \Box G_d$, is the graph with vertex set $V(G_1) \times \cdots \times V(G_d)$ such that $(x_1, \dots, x_d)$ and $(y_1, \dots, y_d)$ are adjacent if and only if $x_i \neq y_i$ for exactly one i, and for this i, x_i and y_i are adjacent in G_i. The Cartesian product $H_{q_1,\dots,q_d} = K_{q_1} \Box \cdots \Box K_{q_d}$ of complete graphs is called a *Hamming graph*, where $q_1, \dots, q_d \geq 2$ are integers. In the case where $q_1 = \cdots = q_d = q$, we usually write $H(d, q)$ in place of $H_{q,\dots,q}$. Equivalently, $H_{q_1,\dots,q_d}$ is the Cayley graph on the group $\mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_d}$ with respect to the connection set $S = \{(s_1, \ldots, s_d) : s_i \in \mathbb{Z}_{q_i}, 1 \leq i \leq d$, there exists exactly one i such that $s_i \neq 0\}$. In particular, $H(d, q)$ can be viewed as the Cayley graph on $\mathbb{Z}_q^d$ in which $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ are adjacent if and only if they differ at exactly one coordinate. Note that $H(d, q)$ has q^d vertices and is $d(q-1)$-regular, and $H(d, 2)$ is exactly the hypercube Q_d.

Theorem 3.1. *Let $d \geq 1$ and $q \geq 2$ be integers.*

(a) *If d and q are coprime, then $H(d, q)$ is $\mathbb{Z}_{q^d}$-distance antimagic. In particular, K_q is $\mathbb{Z}_q$-distance antimagic for any $q \geq 2$.*

(b) *If both d and q are even, then $H(d, q)$ is not $\mathbb{Z}_{q^d}$-distance antimagic.*

Proof. (a) Define

$$f(x) = \sum_{i=1}^{d} x_i q^{i-1},\ x = (x_1, \ldots, x_d),\ x_i \in \{0, 1, \ldots, q-1\}.$$

Then $0 \leq f(x) \leq q^d - 1$ and f is a bijection from $\mathbb{Z}_q^d$ to $\mathbb{Z}_{q^d}$. The $d(q-1)$ neighbours of x are $y_{j,t} = (x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_d)$, $1 \leq j \leq d$, $t \in \mathbb{Z}_q \setminus \{x_j\}$. Note that the label of $y_{j,t}$ is $f(y_{j,t}) = f(x) - x_j q^{j-1} + t q^{j-1}$. Thus the weight of x under f is given by

$$\begin{aligned} w(x) &= d(q-1)f(x) + \sum_{j=1}^{d} \left(\sum_{t \neq x_j} (t - x_j) q^{j-1} \right) \\ &= d(q-1)f(x) + \sum_{j=1}^{d} \left(\sum_{t=0}^{q-1} (t - x_j) q^{j-1} \right) \\ &= d(q-1)f(x) + \sum_{j=1}^{d} \left(\frac{q(q-1)}{2} \cdot q^{j-1} - x_j q^j \right) \\ &= d(q-1)f(x) + \frac{q(q-1)}{2} \cdot \frac{q^d - 1}{q - 1} - q f(x) \\ &= (d(q-1) - q)\, f(x) + \frac{q(q^d - 1)}{2}. \end{aligned}$$

Since $\gcd(d, q) = 1$ by our assumption, we have $\gcd(d(q-1) - q, q^d) = 1$. Since f is a bijection from $\mathbb{Z}_q^d$ to $\mathbb{Z}_{q^d}$, it follows that for different $x, x' \in \mathbb{Z}_q^d$ we have $w(x) \neq w(x')$. In other words, f is a $\mathbb{Z}_{q^d}$-distance antimagic labelling of $H(d, q)$.

In particular, since $K_q \cong H(1, d)$ and $\gcd(1, q) = 1$, it follows from what we proved above that K_q is $\mathbb{Z}_q$-distance antimagic for any $q \geq 2$.

(b) Since both d and q are even, the degree $d(q-1)$ and the order q^d of $H(d, q)$ are even. Thus, by Corollary 2.3, $H(d, q)$ is not $\mathbb{Z}_{q^d}$-distance antimagic. ☐

Since $Q_d = H(d, 2)$, Theorem 3.1 implies the following result.

Corollary 3.2. *The d-dimensional hypercube Q_d is $\mathbb{Z}_{2^d}$-distance antimagic if and only if d is odd.*

Theorem 3.3. *Suppose $d, q \geq 2$ are integers such that q is a divisor of d. Then $H(d, q)$ is $\mathbb{Z}_q^d$-distance magic, with magic constant $\left(\frac{q}{2}, \ldots, \frac{q}{2}\right)$ when q is even and $(0, \ldots, 0)$ when q is odd.*

Proof. Define

$$f(x) = (x_1, \ldots, x_d),\ x = (x_1, \ldots, x_d) \in \mathbb{Z}_q^d.$$

Trivially, f is a bijection from the vertex set $\mathbb{Z}_q^d$ of $H(d,q)$ to the group $\mathbb{Z}_q^d$. The weight of x under f is given by

$$\begin{aligned}
w(x) &= \sum_{i=1}^{d} \sum_{t \in \mathbb{Z}_q \setminus \{x_i\}} (x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_d) \\
&= \sum_{i=1}^{d} \left((q-1)x_1, \ldots, \frac{q(q-1)}{2} - x_i, \ldots, (q-1)x_d \right) \\
&= \sum_{i=1}^{d} \left\{ (q-1)(x_1, \ldots, x_d) + \left(0, \ldots, \frac{q(q-1)}{2} - qx_i, \ldots, 0\right) \right\} \\
&= \sum_{i=1}^{d} \left\{ -(x_1, \ldots, x_d) + \left(0, \ldots, \underbrace{\frac{q(q-1)}{2}}_{i-th}, \ldots, 0\right) \right\} \\
&= -d(x_1, \ldots, x_d) + \frac{q(q-1)}{2}(1, \ldots, 1).
\end{aligned}$$

Notice that all operations are taken modulo q and that in the i-th summand of the summation the entry $\frac{q(q-1)}{2}$ in $\left(0, \ldots, 0, \frac{q(q-1)}{2}, 0, \ldots, 0\right)$ appears in the i-th position.

Since q is a divisor of d by our assumption, we have $d(x_1, \ldots, x_d) = (0, \ldots, 0)$ in $\mathbb{Z}_q^d$. Therefore, $w(x) = \frac{q(q-1)}{2}(1, \ldots, 1)$ for every vertex $x \in \mathbb{Z}_q^d$ of $H(d,q)$, and hence $H(d,q)$ is $\mathbb{Z}_q^d$-distance magic. If q is even, then the magic constant is $\frac{q(q-1)}{2}(1, \ldots, 1) = \frac{q}{2}(q-1)(1, \ldots, 1) = -\frac{q}{2}(1, \ldots, 1) = \frac{q}{2}(1, \ldots, 1)$. If q is odd, then the magic constant is $\frac{q-1}{2}q(1, \ldots, 1) = (0, \ldots, 0)$. $\square$

Since $H(d,q)$ has degree $d(q-1)$, in the special case when $q = p$ is a prime factor of d, the fact that $H(d,p)$ is $\mathbb{Z}_p^d$-distance magic is also implied by part (b) of Corollary 2.6.

In the special case when $q = 2$, Theorem 3.3 gives (b) of Corollary 2.8.

Theorem 3.4. *Let G_i be an r_i-regular graph with $n_i \geq 2$ vertices, $1 \leq i \leq k$. Let A_i be an Abelian group of order n_i such that $\exp(A_i)$ is a divisor of $r - r_i$, $1 \leq i \leq k$, where $r = \sum_{i=1}^{k} r_i$.*

(a) *If G_i is A_i-distance antimagic for $1 \leq i \leq k$, then $G_1 \Box \cdots \Box G_k$ is $A_1 \times \cdots \times A_k$-distance antimagic.*

(b) *If G_i is A_i-distance magic for $1 \leq i \leq k$, then $G_1 \Box \cdots \Box G_k$ is $A_1 \times \cdots \times A_k$-distance magic.*

Proof. (a) Since G_i is A_i-distance antimagic, it admits an A_i-distance antimagic labelling, say, $g_i : V(G_i) \to A_i$, $1 \leq i \leq k$. Define $f : V(G_1) \times \cdots \times V(G_k) \to A_1 \times \cdots \times A_k$ by

$$f(x_1, \ldots, x_k) = (g_1(x_1), \ldots, g_k(x_k)),\ x_i \in V(G_i), 1 \leq i \leq k. \tag{1}$$

Then f is a bijection from $V(G_1) \times \cdots \times V(G_k)$ to $A_1 \times \cdots \times A_k$. Denote by $w_{G_i}(x_i)$ the weight of x_i under g_i. Then the weight of $x = (x_1, \ldots, x_k)$ under f is given by

$$\begin{aligned}
w(x) &= \sum_{i=1}^{k} \sum_{x_i' :\, x_i x_i' \in E(G_i)} f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_k) \\
&= \sum_{i=1}^{k} \sum_{x_i' :\, x_i x_i' \in E(G_i)} (g_1(x_1), \ldots, g_{i-1}(x_{i-1}), g_i(x_i'), g_{i+1}(x_{i+1}), \ldots, g_k(x_k)) \\
&= \sum_{i=1}^{k} (r_i g_1(x_1), \ldots, r_i g_{i-1}(x_{i-1}), w_{G_i}(x_i), r_i g_{i+1}(x_{i+1}), \ldots, r_i g_k(x_k)) \\
&= r(g_1(x_1), \ldots, g_k(x_k)) + \sum_{i=1}^{k} (0, \ldots, 0, w_{G_i}(x_i) - r_i g_i(x_i), 0, \ldots, 0) \\
&= (w_{G_1}(x_1) + (r - r_1)g_1(x_1), \ldots, w_{G_k}(x_k) + (r - r_k)g_k(x_k)).
\end{aligned} \tag{2}$$

Since $\exp(A_i)$ is a divisor of $r - r_i$ by our assumption, we have $(r - r_i)g_i(x_i) = 0$ in A_i for each i. Hence $w(x) = (w_{G_1}(x_1), \ldots, w_{G_k}(x_k))$. (In all computations, the operation on the ith coordinate is performed in A_i.) Since each G_i is assumed to be A_i-distance antimagic, it follows that f is an $A_1 \times \cdots \times A_k$-distance antimagic labelling of $G_1 \Box \cdots \Box G_k$.

(b) Define f as in (a) with the understanding that each g_i is an A_i-distance magic labelling of G_i. The computation above yields, for any x,

$$w(x) = (w_{G_1}(x_1), \ldots, w_{G_k}(x_k)).$$

Since g_i is an A_i-distance magic labelling of G_i, say, with magic constant $\mu_i \in A_i$, we have $w_{G_i}(x_i) = \mu_i$ for all $x_i \in V(G_i)$. Therefore, $w(x) = (\mu_1, \ldots, \mu_k)$ for all x, and hence $G_1 \Box \cdots \Box G_k$ is $A_1 \times \cdots \times A_k$-distance magic. □

Since $\exp(\mathbb{Z}_n^d) = n$, Theorem 3.4 implies:

Corollary 3.5. *Let $d_i, r_i \geq 1$ and $n_i \geq 2$ be integers, $1 \leq i \leq k$. Let G_i be an r_i-regular graph with $n_i^{d_i}$ vertices, $1 \leq i \leq k$. Suppose n_i is a divisor of $r - r_i$, $1 \leq i \leq k$, where $r = \sum_{i=1}^{k} r_i$.*

(a) *If G_i is $\mathbb{Z}_{n_i}^{d_i}$-distance antimagic for $1 \leq i \leq k$, then $G_1 \Box \cdots \Box G_k$ is $\mathbb{Z}_{n_1}^{d_1} \times \cdots \times \mathbb{Z}_{n_k}^{d_k}$-distance antimagic.*

(b) *If G_i is $\mathbb{Z}_{n_i}^{d_i}$-distance magic for $1 \leq i \leq k$, then $G_1 \Box \cdots \Box G_k$ is $\mathbb{Z}_{n_1}^{d_1} \times \cdots \times \mathbb{Z}_{n_k}^{d_k}$-distance magic.*

Setting $k = 2$ and $n_1 = n_2 = 2$ in Corollary 3.5, we obtain:

Corollary 3.6. *Let G, H be regular graphs with $2^d, 2^e$ vertices, respectively. Suppose both G and H have even degrees.*

(a) *If G is $\mathbb{Z}_2^d$-distance antimagic and H is $\mathbb{Z}_2^e$-distance antimagic, then $G \Box H$ is $\mathbb{Z}_2^{d+e}$-distance antimagic.*

(b) *If G is $\mathbb{Z}_2^d$-distance magic and H is $\mathbb{Z}_2^e$-distance magic, then $G \Box H$ is $\mathbb{Z}_2^{d+e}$-distance magic.*

Theorem 3.4 and its corollaries above enable us to construct group distance antimagic/magic graphs from known ones. As an example, from Corollary 2.6 and Corollary 3.5 we obtain the following result (a general result involving more than two factors can be formulated similarly).

Corollary 3.7. *Let $p_i \geq 2$ be a prime and $d_i, r_i \geq 1$ be integers with $2 \leq r_i \leq p_i^{d_i}$ for $i = 1, 2$. Let G_i be any Cayley graph on $\mathbb{Z}_{p_i}^{d_i}$ with degree r_i for $i = 1, 2$.*

(a) *If p_1 divides r_2 but not r_1, and p_2 divides r_1 but not r_2, then $G_1 \Box G_2$ is $\mathbb{Z}_{p_1}^{d_1} \times \mathbb{Z}_{p_2}^{d_2}$-distance antimagic.*

(b) *If both p_1 and p_2 divide each of r_1 and r_2, then $G_1 \Box G_2$ is $\mathbb{Z}_{p_1}^{d_1} \times \mathbb{Z}_{p_2}^{d_2}$-distance magic.*

Theorem 3.1 and part (a) of Theorem 3.4 together imply the following result.

Corollary 3.8. *Let $d_i \geq 1$ and $q_i \geq 2$ be integers which are coprime, $1 \leq i \leq k$. If $q_i^{d_i}$ is a divisor of $\sum_{j \neq i} d_j(q_j - 1)$ for $1 \leq i \leq k$, then $H(d_1, q_1) \Box \cdots \Box H(d_k, q_k)$ is $\mathbb{Z}_{q_1^{d_1}} \times \cdots \times \mathbb{Z}_{q_k^{d_k}}$-distance antimagic.*

We will use a special case of this corollary, with all $(d_i, q_i) = (1, 2)$, in the proof of the following result.

Theorem 3.9. *Let $d \geq 3$ be an integer. If d is odd or $d \equiv 0 \pmod 4$, then Q_d is $\mathbb{Z}_2^d$-distance antimagic.*

Proof. Choosing all $d_i = 1$ and $q_i = 2$, we have $Q_k = H(d_1, q_1) \Box \cdots \Box H(d_k, q_k)$, and $q_i^{d_i}$ divides $\sum_{j \neq i} d_j(q_j - 1)$ if and only if k is odd. Thus, by Corollary 3.8, if k is odd, then Q_k is $\mathbb{Z}_2^k$-distance antimagic.

Next we prove that Q_4 is $\mathbb{Z}_2^4$-distance antimagic. In fact, Q_4 can be viewed as $C_4 \Box C_4$, and one can verify that it admits a $\mathbb{Z}_2^4$-distance antimagic labelling as given in the following table:

(0,0,0,0)	(0,1,0,0)	(1,1,1,1)	(1,1,0,0)
(0,0,0,1)	(1,0,1,1)	(1,1,1,0)	(0,1,0,1)
(0,0,1,1)	(1,0,1,0)	(1,0,0,1)	(0,1,1,0)
(0,0,1,0)	(0,1,1,1)	(1,0,0,0)	(1,1,0,1)

(In the table every entry represents a vertex label, and the vertex neighbours are the entries immediately above, below, on the left and on the right, where the top row is considered to be next to the bottom and the first column next to the last.)

Since Q_4 is $\mathbb{Z}_2^4$-distance antimagic, by part (a) of Corollary 3.6, $Q_4 \square Q_4$ is $\mathbb{Z}_2^8$-distance antimagic since $\mathbb{Z}_2^4 \times \mathbb{Z}_2^4 \cong \mathbb{Z}_2^8$. Similarly, by induction and part (a) of Corollary 3.6, one can see that $Q_{4t} \cong Q_4 \square \cdots \square Q_4$ (t factors) is $\mathbb{Z}_2^{4t}$-distance antimagic. In other words, if $k \geq 4$ is a multiple of 4, then Q_k is $\mathbb{Z}_2^k$-distance antimagic. □

Remark 3.10. *If Q_k is $\mathbb{Z}_2^k$-distance antimagic for some integer $k \geq 6$ with $k \equiv 2 \pmod 4$, then Q_d is $\mathbb{Z}_2^d$-distance antimagic for every integer $d \geq k$ with $d \equiv 2 \pmod 4$.*

In particular, if we can prove that Q_6 is $\mathbb{Z}_2^6$-distance antimagic, then by Theorem 3.9, Q_d is $\mathbb{Z}_2^d$-distance antimagic for all integers $d \geq 3$.

In fact, we have $d = 4t + k$ for some t and so $Q_d \cong Q_{4t} \square Q_k$. We may assume $t > 0$. Since Q_{4t} is $\mathbb{Z}_2^{4t}$-distance antimagic by Theorem 3.9, if Q_k is $\mathbb{Z}_2^k$-distance antimagic, then by part (a) of Corollary 3.6, Q_d is $\mathbb{Z}_2^d$-distance antimagic.

Unfortunately, at the time of writing we do not known whether Q_6 is $\mathbb{Z}_2^6$-distance antimagic.

Applying part (a) of Corollary 3.5 to cycles, we obtain the following result, where C_n denotes the cycle of length n.

Theorem 3.11. (a) *If $n \geq 3$ is an odd integer, then C_n is $\mathbb{Z}_n$-distance antimagic.*

(b) *Let $n_1, \ldots, n_k \geq 3$ be odd integers, where $k \geq 2$. If each n_i is a divisor of $k-1$, then $C_{n_1} \square \cdots \square C_{n_k}$ is $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$-distance antimagic.*

In particular, for any prime p and any integer $d \geq 1$, $C_p \square \cdots \square C_p$ ($pd+1$ factors) is $\mathbb{Z}_p^{pd+1}$-distance antimagic.

Proof. (a) Label the vertices along C_n by $0, 1, \ldots, n-1$ consecutively. Since n is odd, one can verify that this is a $\mathbb{Z}_n$-distance antimagic labelling of C_n.

(b) By (a), each C_{n_i} is $\mathbb{Z}_{n_i}$-distance antimagic as n_i is odd. Applying Corollary 3.5 and noting that all $d_i = 1$ and $r_i = 2$, we have $r - r_i = 2(k-1)$ and so the result follows. □

In part (b) of Theorem 3.11, k ought to be relatively large comparable to $n_1, \ldots, n_k$. It would be interesting to find other conditions under which the same result holds.

We now give a generalisation of part (a) of Theorem 3.4. To this end we introduce the following concept which will also be used in the next section. Given an r-regular graph G with n vertices and an Abelian group A of order n, a bijection $f: V(G) \to A$ is called an *A-balanced labelling* if $w(x) = rf(x)$ for every $x \in V(G)$. We refer the reader to Lemmas 5.1 and 5.8 for examples of graphs admitting $\mathbb{Z}_n$-balanced labellings.

Theorem 3.12. *Let G_i be an r_i-regular graph with $n_i \geq 2$ vertices, $1 \leq i \leq k$. Let A_i be an Abelian group of order n_i, $1 \leq i \leq k$, and let $r = \sum_{i=1}^k r_i$. Suppose $\{1, \ldots, k\}$ is partitioned into, say, $I = \{1, \ldots, k_1\}$, $J = \{k_1 + 1, \ldots, k_2\}$ and $L = \{k_2 + 1, \ldots, k\}$ for some $0 \leq k_1 \leq k_2 \leq k$, possibly with one or two of I, J, L to be empty, such that the following conditions are satisfied:*

(a) *for each $i \in I$, G_i is A_i-distance magic and $r - r_i$ is coprime to n_i;*

(b) *for each $j \in J$, G_j is A_j-distance antimagic and $\exp(A_j)$ is a divisor of $r - r_j$;*

(c) *for each $l \in L$, G_l admits an A_l-balanced labelling and r is coprime to n_l.*

Then $G_1 \square \cdots \square G_k$ is $A_1 \times \cdots \times A_k$-distance antimagic.

Proof. For $i \in I$, let g_i be an A_i-distance magic labelling of G_i, with corresponding magic constant μ_i. For $j \in J$, let g_j be an A_j-distance antimagic labelling of G_j. For $l \in L$, let g_l be an A_l-balanced labelling of G_l. Then, for every vertex $x = (x_1, \ldots, x_k)$ of $G_1 \square \cdots \square G_k$, we have $w_{G_i}(x_i) = \mu_i$ if $i \in I$, $(r - r_j)g_j(x_j) = 0$ if $j \in J$ as $\exp(A_j)$ is a divisor of $r - r_j$, and $w_{G_l}(x_l) = r_l g_l(x_l)$ if $l \in L$, where $w_{G_t}(x_t)$ is the weight of x_t with respect to g_t. Define f as in (1). Then the weight of x under f is given by (2). Thus the tth component of $w(x)$ is equal to $\mu_t + (r - r_t)g_t(x_t)$ if $t \in I$, $w_{G_t}(x_t)$ if $t \in J$, and $rg_t(x_t)$ if $t \in L$. For any vertex $x' = (x'_1, \ldots, x'_k) \neq x$ of $G_1 \square \cdots \square G_k$, there exists at least one t such that $x_t \neq x'_t$. If $t \in I$, then $(r - r_t)g_t(x_t) \neq (r - r_t)g_t(x'_t)$ since $r - r_t$ is coprime to n_t and the order of $g_t(x_t) - g_t(x'_t)$ $(\neq 0)$ in A_t is a divisor of n_t. If $t \in J$, then $w_{G_t}(x_t) \neq w_{G_t}(x'_t)$ as g_t is A_t-distance antimagic. If $t \in L$, then $rg_t(x_t) \neq rg_t(x'_t)$ as r is coprime to n_t. In any case we have $w(x) \neq w(x')$. Therefore, f is an $A_1 \times \cdots \times A_k$-distance antimagic labelling of $G_1 \square \cdots \square G_k$. □

Note that part (a) of Theorem 3.4 can be obtained from Theorem 3.12 by setting $I = L = \emptyset$.

Theorem 3.12 enables us to construct new families of distance antimagic graphs based on known distance antimagic graphs and distance magic graphs. We illustrate this by the following corollary, which is obtained by setting $n_t = 2^{d_t}$ and $A_t = \mathbb{Z}_2^{d_t}$ for $1 \leq t \leq s + t$, and $I = \{1, \ldots, s\}$, $J = \{s + 1, \ldots, s + t\}$ and $L = \emptyset$, in Theorem 3.12.

Corollary 3.13. *Suppose G_i is an r_i-regular $\mathbb{Z}_2^{d_i}$-distance magic graph, $1 \leq i \leq s$, and G_j an r_j-regular $\mathbb{Z}_2^{d_j}$-distance antimagic graph, $s + 1 \leq j \leq s + t$. Let $r = \sum_{i=1}^{s+t} r_i$ and $d = \sum_{i=1}^{s+t} d_i$. If r_i and r have different parity for $i = 1, \ldots, s$ and the same parity for $i = s + 1, \ldots, s + t$, then $G_1 \square \cdots \square G_{s+t}$ is $\mathbb{Z}_2^d$-distance antimagic.*

In particular, if G_1 is a regular $\mathbb{Z}_2^{d_1}$-distance magic graph with even degree, and G_2 a regular $\mathbb{Z}_2^{d_2}$-distance antimagic graph with odd degree, then $G_1 \square G_2$ is $\mathbb{Z}_2^{d_1+d_2}$-distance antimagic.

The reader is invited to compare the last statement in Corollary 3.13 with Corollary 3.6. To satisfy the conditions in Corollary 3.13, s must be even when r is even, and t must be odd when r is odd.

4 Labelling direct products

The *direct product* $G_1 \times \cdots \times G_d$ of d graphs $G_1, \ldots, G_d$ is defined [12] to have vertex set $V(G_1) \times \cdots \times V(G_d)$, such that two vertices $(x_1, \ldots, x_d)$ and $(y_1, \cdots, y_d)$ are adjacent if and only if x_i is adjacent to y_i in G_i for $i = 1, \ldots, d$.

Theorem 4.1. *Let G_i be an r_i-regular graph with n_i vertices and A_i an Abelian group of order n_i, $i = 1, \ldots, d$. Denote $r = r_1 \cdots r_d$. If G_i is A_i-distance antimagic and n_i and r/r_i are coprime for $i = 1, \ldots, d$, then $G_1 \times \cdots \times G_d$ is $A_1 \times \cdots \times A_d$-distance antimagic.*

Proof. Since G_i is A_i-distance antimagic, it admits at least one A_i-distance antimagic labelling, say, $g_i \colon V(G_i) \to A_i$, $i = 1, \ldots, d$. Denote $G = G_1 \times \cdots \times G_d$ and $A = A_1 \times \cdots \times A_d$. Define $f : V(G) \to A$ by

$$f(x) = (g_1(x_1), \ldots, g_d(x_d)), \text{ for } x = (x_1, \ldots, x_d).$$

Obviously, f is a bijection. The weight of x under f is given by

$$\begin{aligned} w(x) &= \sum_{(y_1,\ldots,y_d) \in N_G(x)} (g_1(y_1), \ldots, g_d(y_d)) \\ &= \left(\sum_{y_i \in N_{G_i}(x_i),\, i \neq 1} g_1(y_1), \ldots, \sum_{y_i \in N_{G_i}(x_i),\, i \neq d} g_d(y_d) \right) \\ &= ((r/r_1) w_{G_1}(x_1), \ldots, (r/r_d) w_{G_d}(x_d)). \end{aligned}$$

Since $\gcd(r/r_i, n_i) = 1$ for each i, the mapping defined by $(a_1, \ldots, a_d) \mapsto ((r/r_1)a_1, \ldots, (r/r_d)a_d)$, $(a_1, \ldots, a_d) \in A$, is a bijection from A to itself. Combining this with the assumption that each g_i is an A_i-distance antimagic labelling, we obtain that f is an A-distance antimagic labelling of G. □

Denote $G_\times^d = G \times \cdots \times G$ (d factors) and $A^d = A \times \cdots \times A$ (d factors). By Theorem 4.1 we obtain the following result.

Corollary 4.2. *Suppose G is an r-regular A-distance antimagic graph with n vertices, where A is an Abelian group of order n. If n and r are coprime, then for any integer $d \geq 1$, $G_\times^d$ is A^d-distance antimagic.*

Since C_n is $\mathbb{Z}_n$-distance antimagic when n is odd (Theorem 3.11), Theorem 4.1 implies the following result.

Corollary 4.3. *For any odd integers $n_1, \ldots, n_d \geq 3$, $C_{n_1} \times \cdots \times C_{n_d}$ is $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$-distance antimagic.*
In particular, for any integer $d \geq 1$ and odd integer $n \geq 3$, $(C_n)_\times^d$ is $\mathbb{Z}_n^d$-distance antimagic.

Since K_n is $\mathbb{Z}_n$-distance antimagic (Theorem 3.1), by Theorem 4.1 we obtain:

Corollary 4.4. *Let $n_1, \ldots, n_d \geq 3$ be integers such that n_i and $n_j - 1$ are coprime for distinct i, j. Then $K_{n_1} \times \cdots \times K_{n_d}$ is $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$-distance antimagic.*
In particular, for any integers $d \geq 1$ and $n \geq 3$, $(K_n)_\times^d$ is $\mathbb{Z}_n^d$-distance antimagic.

Denote by $D_n = C_n \Box P_2$ the *prism* of $2n \geq 6$ vertices.

Lemma 4.5. *Let $n \geq 4$ be an integer not divisible by 3. Then D_n is $\mathbb{Z}_{2n}$-distance antimagic.*

Proof. Denote the vertices of D_n by $x_{i,j}$ such that $x_{i,j}x_{i+1,j}$ and $x_{i,1}x_{i,2}$ are edges of D_n for $i = 0, 1, \ldots, n-1$ and $j = 1, 2$, where the first subscript is taken modulo n. Define

$$f(x_{i,j}) = \begin{cases} 2i, & \text{if } j = 1 \\ 2i+1, & \text{if } j = 2 \end{cases}$$

Then

$$w(x_{i,1}) = f(x_{i-1}, 1) + f(x_{i+1}, 1) + f(x_i, 2) \equiv 6i + 1 \pmod{2n}$$
$$w(x_{i,2}) = f(x_{i-1}, 2) + f(x_{i+1}, 2) + f(x_i, 1) \equiv 6i + 2 \pmod{2n}.$$

Since $n \not\equiv 0 \pmod 3$, $g(x) = 3x+1$, $x \in \mathbb{Z}_{2n}$ defines a bijection from $\mathbb{Z}_{2n}$ to itself. Therefore, the weights are all different elements of $\mathbb{Z}_{2n}$. ☐

Combining Theorem 4.1 and Lemma 4.5, we obtain the following result.

Corollary 4.6. *Let $n_1, \ldots, n_d \geq 4$ be integers not divisible by 3. Then $D_{n_1} \times \cdots \times D_{n_d}$ is $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$-distance antimagic.*
In particular, for any $d \geq 1$, and any $n \geq 4$ not divisible by 3, $(D_n)_\times^d$ is $\mathbb{Z}_n^d$-distance antimagic.

5 $\mathbb{Z}_n$-distance antimagic product graphs

For brevity, a $\mathbb{Z}_n$-balanced labelling of a regular graph is called a *balanced labelling*. In other words, a balanced labelling of an r-regular graph G with n vertices is a bijection $f : V(G) \to \mathbb{Z}_n$ such that $w(x) \equiv rf(x) \pmod n$ for every $x \in V(G)$.

Lemma 5.1. (a) *The cycle C_n for any $n \geq 3$ admits a balanced labelling.*

(b) *The complete graph K_n on $n \geq 2$ vertices admits a balanced labelling if and only if n is odd.*

Proof. (a) The labelling that sequentially assigns $0, 1, \ldots, n-1$ to the vertices of C_n along the cycle is balanced.

(b) Denote the vertices of K_n by x_i, $1 \leq i \leq n$. Define $f : V(G) \to \mathbb{Z}_n$ by $f(x_i) = i - 1$. Then $(n-1)f(x_i) \equiv 1 - i \pmod n$ for each i. If n is odd, then $\sum_{i=1}^n f(x_i) \equiv n(n-1)/2 \equiv 0 \pmod n$ and the weight of x_i is $w(x_i) = \sum_{j=1}^n f(x_j) - f(x_i) \equiv 1 - i \equiv (n-1)f(x_i)$. Thus f is a balanced labelling of K_n when n is odd.

Conversely, suppose K_n admits a balanced labelling $f : V(G) \to \mathbb{Z}_n$. Then $w(x_i) = (n-1)f(x_i) \equiv -f(x_i) \pmod n$ for each i, and on the other hand $w(x_i) = \sum_{j=1}^{n} f(x_j) - f(x_i)$. Since f is a bijection, we have

$$\sum_{j=1}^{n} f(x_j) \equiv \sum_{j=1}^{n} j \equiv \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

It follows that n must be odd. □

Theorem 5.2. *Suppose G_i is an r_i-regular graph with n_i vertices which admits a balanced labelling, $1 \le i \le k$. Suppose further that $r_i \le n_j$ for any $1 \le i, j \le k$ and that $r = \sum_{i=1}^{k} r_i$ is coprime to $n_1 \cdots n_k$. Then $G_1 \Box \cdots \Box G_k$ is $\mathbb{Z}_{n_1 \cdots n_k}$-distance antimagic.*

Proof. Let $g_i : V(G_i) \to \{0, 1, \dots, n_i - 1\}$ be a balanced labelling of G_i, $1 \le i \le k$. Define $f : V(G_1) \times \cdots \times V(G_k) \to \mathbb{Z}_{n_1 \cdots n_k}$ by

$$\begin{aligned} f(x) \quad = \quad & (g_1(x_1) \bmod n_1) + (g_2(x_2) \bmod n_2)\ n_1 + (g_3(x_3) \bmod n_3)\ n_1 n_2 \\ & + \cdots + (g_k(x_k) \bmod n_k)\ n_1 \cdots n_{k-1} \end{aligned} \tag{3}$$

for any $x = (x_1, \dots, x_k) \in V(G_1) \times \cdots \times V(G_k)$. This is indeed a mapping from $V(G_1) \times \cdots \times V(G_k)$ to $\mathbb{Z}_{n_1 \cdots n_k}$ because f takes minimum value 0 and maximum value $(n_1 - 1) + (n_2 - 1)n_1 + \cdots + (n_k - 1)n_1 \cdots n_{k-1} = n_1 \cdots n_k - 1$. Moreover, f is injective, and hence must be bijective since $|V(G_1) \times \cdots \times V(G_k)| = n_1 \cdots n_k$. In fact, if $f(x) = f(x')$ but $x \ne x'$, then $g_i(x_i) \ne g_i(x_i')$ for at least one i as all mappings g_j are bijective. Let t be the largest subscript such that $g_t(x_t) \ne g_t(x_t')$. Then $g_1(x_1) + g_2(x_2)n_1 + \cdots + g_t(x_t)n_1 \cdots n_{t-1} = g_1(x_1') + g_2(x_2')n_1 + \cdots + g_t(x_t')n_1 \cdots n_{t-1}$. Without loss of generality we may assume $g_t(x_t) > g_t(x_t')$. Then $(g_t(x_t) - g_t(x_t'))n_1 \cdots n_{t-1} = (g_1(x_1') - g_1(x_1)) + (g_2(x_2') - g_2(x_2))n_1 + \cdots + (g_{t-1}(x_{t-1}') - g_{t-1}(x_{t-1}))n_1 \cdots n_{t-2}$. However, the right-hand side of this equality is no more than $(n_1 - 1) + (n_2 - 1)n_1 + \cdots + (n_{t-1} - 1)n_1 \cdots n_{t-2} = n_1 \cdots n_{t-1} - 1$, but the left-hand side of it is no less than $n_1 \cdots n_{t-1}$. This contradiction shows that f is a bijection.

Denote $w_{G_i}(x_i) = \sum_{x_i' :\, x_i x_i' \in E(G_i)} g_i(x_i')$ for $x_i \in V(G_i)$, $1 \le i \le k$. With congruence modulo $n_1 \cdots n_k$, we have

$$\begin{aligned} w(x) \quad & \equiv \quad \sum_{i=1}^{k} \sum_{x_i' :\, x_i x_i' \in E(G_i)} f(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_k) \\ & \equiv \quad \sum_{i=1}^{k} \sum_{x_i' :\, x_i x_i' \in E(G_i)} \{(g_1(x_1) \bmod n_1) + \cdots + (g_i(x_i') \bmod n_i)\ n_1 \cdots n_{i-1} \\ & \qquad\qquad + \cdots + (g_k(x_k) \bmod n_k)\ n_1 \cdots n_{k-1}\} \\ & \equiv \quad \sum_{i=1}^{k} \{(r_i g_1(x_1) \bmod n_1)\ + \cdots + (w_{G_i}(x_i) \bmod n_i)\ n_1 \cdots n_{i-1} \\ & \qquad\qquad + \cdots + (r_i g_k(x_k) \bmod n_k)\ n_1 \cdots n_{k-1}\} \\ & \equiv \quad r f(x) + \sum_{i=1}^{k} \{(w_{G_i}(x_i) - r_i g_i(x_i)) \bmod n_i\} n_1 \cdots n_{i-1} \\ & \equiv \quad r f(x). \end{aligned}$$

In the last two steps we used the assumption that $r_i \le n_j$ for each pair i, j and that g_i is a balanced labelling of G_i. (Since $r_i \le n_j$, we have $(r_i g_j(x_j)) \bmod n_j = r_i \cdot (g_j(x_j) \bmod n_j)$.) If $w(x) \equiv w(x') \pmod{n_1 \cdots n_k}$, then $r(f(x) - f(x')) \equiv 0 \pmod{n_1 \cdots n_k}$, which implies $f(x) \equiv f(x')$ since r is coprime to $n_1 \cdots n_k$. Since f is bijective as shown above, it follows that f is a $\mathbb{Z}_{n_1 \cdots n_k}$-distance antimagic labelling of $G_1 \Box \cdots \Box G_k$. □

Denote $G_{\Box}^{k} = G \Box \cdots \Box G$ (k factors). We have the following corollary of Theorem 5.2 and Corollary 2.3.

Corollary 5.3. *Let $n_1, \dots, n_k \geq 3$ be (not necessarily distinct) integers such that k is coprime to $n_1 \cdots n_k$. Then $C_{n_1} \square \cdots \square C_{n_k}$ is $\mathbb{Z}_{n_1 \cdots n_k}$-distance antimagic if and only if all $n_1, \dots, n_k$ are odd.*

In particular, for any integer $k \geq 1$ and odd integer $n \geq 3$ that are coprime, $(C_n)^k_\square$ is $\mathbb{Z}_{n^k}$-distance antimagic.

Proof. Since $C_{n_1} \square \cdots \square C_{n_k}$ has even degree, namely $2k$, by Corollary 2.3, $C_{n_1} \square \cdots \square C_{n_k}$ cannot be $\mathbb{Z}_{n_1 \cdots n_k}$-distance antimagic unless $n_1, \dots, n_k$ are all odd.

Suppose $n_1, \dots, n_k \geq 3$ are all odd. Applying Theorem 5.2 to $C_{n_1}, \dots, C_{n_k}$, we have $r_i = 2 < n_j$, $r = 2k$, and each C_{n_i} admits a balanced labelling by Lemma 5.1. Since $n_1, \dots, n_k$ are odd and k is coprime to $n_1 \cdots n_k$ by our assumption, r is coprime to $n_1 \cdots n_k$. Thus, by Theorem 5.2, $C_{n_1} \square \cdots \square C_{n_k}$ is $\mathbb{Z}_{n_1 \cdots n_k}$-distance antimagic. $\square$

In particular, when k is a prime, Corollary 5.3 yields the following result.

Corollary 5.4. *Let p be a prime, and let $n_1, \dots, n_p \geq 3$ be (not necessarily distinct) integers none of which has p as a factor. Then $C_{n_1} \square \cdots \square C_{n_p}$ is $\mathbb{Z}_{n_1 \cdots n_p}$-distance antimagic if and only if all $n_1, \dots, n_p$ are odd.*

In particular, for any integers $n_1, n_2 \geq 3$, $C_{n_1} \square C_{n_2}$ is $\mathbb{Z}_{n_1 n_2}$-distance antimagic if and only if both n_1 and n_2 are odd.

Moreover, if a prime p is not a divisor of an odd integer $n \geq 3$, then $(C_n)^p_\square$ is $\mathbb{Z}_{n^p}$-distance antimagic.

Applying Theorem 5.2 to d copies of K_q and using Lemma 5.1, we obtain that $H(d, q)$ ($\cong (K_q)^d_\square$) is $\mathbb{Z}_{q^d}$-distance antimagic for any $d \geq 1$ and odd $q \geq 3$ that are coprime. It is interesting to note that this is also given in part (a) of Theorem 3.1 where q is not required to be odd.

Theorem 5.5. *Let G_i be an r_i-regular graph with n_i vertices, $i = 1, \dots, k$. Suppose n_i and r/r_i are coprime for $i = 1, \dots, k$, where $r = r_1 \cdots r_k$.*

(a) *If each G_i is $\mathbb{Z}_{n_i}$-distance antimagic, then $G_1 \times \cdots \times G_k$ is $\mathbb{Z}_{n_1 \dots n_k}$-distance antimagic.*

(b) *If each G_i is $\mathbb{Z}_{n_i}$-distance magic, then $G_1 \times \cdots \times G_k$ is $\mathbb{Z}_{n_1 \dots n_k}$-distance magic.*

Proof. Denote $G = G_1 \times \cdots \times G_k$ and $n = n_1 \dots n_k$.

(a) Define f as in (3) with the understanding that each g_i is a $\mathbb{Z}_{n_i}$-distance antimagic labelling of G_i. As shown in the proof of Theorem 5.2, f is a bijection from $V(G)$ to $\mathbb{Z}_n$. The weight of x under f is given by

$$\begin{aligned} w(x) &\equiv \sum_{(y_1,\dots,y_k) \in N_G(x)} f(y_1, \dots, y_k) \\ &\equiv \sum_{(y_1,\dots,y_k) \in N_G(x)} \left\{ \sum_{i=1}^{k} (g_i(y_i) \bmod n_i)\ n_1 \cdots n_{i-1} \right\} \\ &\equiv \sum_{i=1}^{k} \left\{ \sum_{(y_1,\dots,y_k) \in N_G(x)} (g_i(y_i) \bmod n_i)\ n_1 \cdots n_{i-1} \right\} \\ &\equiv \sum_{i=1}^{k} \left\{ \left(\frac{r}{r_i} w_{G_i}(x_i) \bmod n_i \right)\ n_1 \cdots n_{i-1} \right\}, \end{aligned}$$

where the congruence is modulo n. Similar to the proof after (3), one can show that for $x = (x_1, \dots, x_k)$, $x' = (x'_1, \dots, x'_k) \in V(G)$, $w(x) \equiv w(x') \pmod{n_1 \cdots n_k}$ if and only if $\frac{r}{r_i} w_{G_i}(x_i) \bmod n_i \equiv \frac{r}{r_i} w_{G_i}(x'_i) \bmod n_i$ for each i. Since $\gcd(r/r_i, n_i) = 1$, the latter holds if and only if $w_{G_i}(x_i) \bmod n_i \equiv w_{G_i}(x'_i) \bmod n_i$ for each i. However, since g_i is a $\mathbb{Z}_{n_i}$-distance antimagic labelling, we have $w_{G_i}(x_i) \bmod n_i \not\equiv w_{G_i}(x'_i) \bmod n_i$ for $x_i \neq x'_i$. Therefore, $w(x) \not\equiv w(x') \pmod{n_1 \cdots n_k}$ for distinct vertices x, x' of G, and hence f is a $\mathbb{Z}_n$-distance antimagic labelling of G.

(b) Define f as above with the understanding that each g_i is a $\mathbb{Z}_{n_i}$-distance magic labelling of G_i. Denote by μ_i the magic constant of G_i with respect to g_i. The computation above yields

$$w(x) \equiv \sum_{i=1}^{k} \left\{ \left(\frac{r\mu_i}{r_i} \bmod n_i \right) \; n_1 \cdots n_{i-1} \right\}.$$

Since this is independent of x, f is a $\mathbb{Z}_n$-distance magic labelling of G. □

Combining part (a) of Theorem 5.5, Corollary 2.3 and part (a) of Theorem 3.11, we obtain the following result.

Corollary 5.6. *Let $n_1, \ldots, n_k \geq 3$ be (not necessarily distinct) integers. Then $C_{n_1} \times \cdots \times C_{n_k}$ is $\mathbb{Z}_{n_1 \cdots n_k}$-distance antimagic if and only if all $n_1, \ldots, n_k$ are odd.*

In particular, for any integer $d \geq 1$ and odd integer $n \geq 3$, $(C_n)^d_\times$ is $\mathbb{Z}_{n^d}$-distance antimagic.

The *strong product* $G_1 \boxtimes \cdots \boxtimes G_d$ of d graphs $G_1, \ldots, G_d$ is defined [12] to have vertex set $V(G_1) \times \cdots \times V(G_d)$, such that two vertices $(x_1, \ldots, x_d)$ and $(y_1, \cdots, y_d)$ are adjacent if and only if, for $i = 1, \ldots, d$, either x_i is adjacent to y_i in G_i or $x_i = y_i$.

Theorem 5.7. *Let G_i be an r_i-regular graph with n_i vertices, for $i = 1, 2$.*

(a) *Suppose both G_1 and G_2 have balanced labellings and $r_1 r_2 + r_1 + r_2$ is coprime to $n_1 n_2$. Then $G_1 \boxtimes G_2$ is $\mathbb{Z}_{n_1 n_2}$-distance antimagic.*

(b) *Suppose G_1 is $\mathbb{Z}_{n_1}$-distance magic, G_2 is $\mathbb{Z}_{n_2}$-distance magic, r_1 is coprime to n_2, and r_2 is coprime to n_1. Then $G_1 \boxtimes G_2$ is $\mathbb{Z}_{n_1 n_2}$-distance antimagic.*

Proof. Denote $G = G_1 \boxtimes G_2$ and $n = n_1 n_2$.

(a) Let $g_i : V(G_i) \to \mathbb{Z}_{n_i}$ be a balanced labelling of G_i, $i = 1, 2$. Define, for $x = (x_1, x_2) \in V(G_1) \times V(G_2)$,

$$f(x) = (g_1(x_1) \bmod n_1) + (g_2(x_2) \bmod n_2) \; n_1.$$

Then f is a bijection from $V(G)$ to $\mathbb{Z}_n$, as we saw in the paragraph after (3). With congruence modulo n, we have

$$\begin{aligned}
w(x) \;\; &\equiv \sum_{(y_1,y_2) \in N_G(x)} f(y_1, y_2) \\
&\equiv \sum_{y_1 \in N_{G_1}(x_1)} f(y_1, x_2) + \sum_{y_2 \in N_{G_2}(x_2)} f(x_1, y_2) + \sum_{y_1 \in N_{G_1}(x_1),\, y_2 \in N_{G_2}(x_2)} f(y_1, y_2) \\
&\equiv \sum_{y_1 \in N_{G_1}(x_1)} \{(g_1(y_1) \bmod n_1) + (g_2(x_2) \bmod n_2) n_1\} \\
&\quad + \sum_{y_2 \in N_{G_2}(x_2)} \{(g_1(x_1) \bmod n_1) + (g_2(y_2) \bmod n_2) n_1\} \\
&\quad + \sum_{y_1 \in N_{G_1}(x_1),\, y_2 \in N_{G_2}(x_2)} \{(g_1(y_1) \bmod n_1) + (g_2(y_2) \bmod n_2) n_1\} \\
&\equiv (w_{G_1}(x_1) \bmod n_1) + (r_1 g_2(x_2) \bmod n_2) n_1 + (r_2 g_1(x_1) \bmod n_1) + (w_{G_2}(x_2) \bmod n_2) n_1 \\
&\quad + (r_2 w_{G_1}(x_1) \bmod n_1) + (r_1 w_{G_2}(x_2) \bmod n_2) n_1 \\
&\equiv \{[(r_2 + 1) w_{G_1}(x_1) + r_2 g_1(x_1)] \bmod n_1\} + \{[(r_1 + 1) w_{G_2}(x_2) + r_1 g_2(x_2)] \bmod n_2\} n_1 \\
&\equiv \{[(r_1 r_2 + r_1 + r_2) g_1(x_1)] \bmod n_1\} + \{[(r_1 r_2 + r_1 + r_2) g_2(x_2)] \bmod n_2\} n_1.
\end{aligned}$$

In the last step we used the assumption that g_1, g_2 are balanced labellings of G_1, G_2 respectively. Since $r_1 r_2 + r_1 + r_2$ is coprime to $n_1 n_2$, the computation above implies $w(x) \neq w(x')$ for distinct $x, x' \in V(G)$. Thus f is a $\mathbb{Z}_n$-distance antimagic labelling of G.

(b) Define $f : V(G) \to \mathbb{Z}_n$ as in (a) with the understanding that g_i is a $\mathbb{Z}_{n_i}$-distance magic labelling of G_i, $i = 1, 2$. Let μ_1, μ_2 be the corresponding magic constants of g_1, g_2 respectively. The computation in (a) implies that, for $x = (x_1, x_2) \in V(G)$, $w(x) = \{[(r_2+1)\mu_1 + r_2 g_1(x_1)] \bmod n_1\} + \{[(r_1+1)\mu_2 + r_1 g_2(x_2)] \bmod n_2\} n_1$. Since both g_1 and g_2 are bijective, and since r_1 is coprime to n_2 and r_2 is coprime to n_1, one can show that $w(x) \neq w(x')$ for distinct $x, x' \in V(G)$. Therefore, f is a $\mathbb{Z}_n$-distance antimagic labelling of G. □

Theorem 5.7 involves only two factor graphs. Nevertheless, with the help of the following lemma (together with the associativity [12] of the strong product), we can obtain results about strong products with more than two factors by recursively applying part (a) of Theorem 5.7.

Lemma 5.8. *Let G_1 and G_2 be regular graphs. If both G_1 and G_2 have balanced labellings, then $G_1 \boxtimes G_2$ has a balanced labelling.*

Proof. Denote $n_i = |V(G_i)|$, and let $g_i : V(G_i) \to \mathbb{Z}_{n_i}$ be a balanced labelling of G_i, $i = 1, 2$. Define $g : V(G_1) \times V(G_2) \to \mathbb{Z}_{n_1 n_2}$ by $g(x_1, x_2) = g_1(x_1) + g_2(x_2) n_1$. Then g is a bijection from $V(G_1) \times V(G_2)$ to $\mathbb{Z}_{n_1 n_2}$. One can verify that g is a balanced labelling of $G_1 \boxtimes G_2$. □

Since any cycle C_n admits a $\mathbb{Z}_n$-balanced labelling, in view of Corollary 2.3, we obtain the following result from part (a) of Theorem 5.7.

Corollary 5.9. *Let $m, n \geq 3$ be (not necessarily distinct) integers. Then $C_m \boxtimes C_n$ is $\mathbb{Z}_{mn}$-distance antimagic if and only if both m and n are odd.*

Similarly, since K_n admits a $\mathbb{Z}_n$-balanced labelling if n is odd (Lemma 5.1), by part (a) of Theorem 5.7 we obtain the following corollary.

Corollary 5.10. *For any odd integers $m, n \geq 3$ (not necessarily distinct), $K_m \boxtimes K_n$ is $\mathbb{Z}_{mn}$-distance antimagic.*

We conclude this section with the following theorem, the second part of which is obtained by using a known result on magic rectangles. A *magic (m,n)-rectangle* is an $m \times n$ array in which each of the integers $1, 2, \ldots, mn$ occurs exactly once such that the sum over each row is a constant and the sum over each column is also a constant.

Theorem 5.11. *Let G be a regular graph on m vertices other than the empty graph $\overline{K}_m$.*

(a) *If n is even, then $K_n \boxtimes G$ is $\mathbb{Z}_{nm}$-distance antimagic.*

(b) *If both n and m are odd, then $K_n \boxtimes G$ is $\mathbb{Z}_{nm}$-distance antimagic.*

Proof. Let $r \geq 1$ be the degree of G. Then $K_n \boxtimes G$ is $(n(r+1)-1)$-regular with mn vertices. Denote $V(K_n) = \{x_1, x_2, \ldots, x_n\}$ and $V(G) = \{v_1, v_2, \ldots, v_m\}$.

(a) Let n be even. As in [3], define $f : V(K_n \boxtimes G) \to \{1, 2, \ldots, mn\}$ by

$$f(x_j, v_i) = \begin{cases} \frac{i-1}{2} n + j - 1, & \text{if } j \text{ is odd,} \\ mn - 1 - f(x_{j-1}, v_i), & \text{if } j \text{ is even} \end{cases}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. It can be verified that f is a bijection and $w(x) = (r+1)\frac{n}{2} - f(x)$ for each $x \in V(K_n \boxtimes G)$. Hence f is a $\mathbb{Z}_{nm}$-distance antimagic labelling of $K_n \boxtimes G$.

(b) Suppose both n and m are odd. Then there exists [10, 11] a magic (n,m)-rectangle, say, with (j,i)-entry $a_{j,i}$, $1 \leq j \leq n$, $1 \leq i \leq m$ and $\sum_{j=1}^{n} a_{j,i} = C$ for some constant C and every i. Define $f \colon V(K_n \boxtimes G) \to \{1, 2, \ldots, nm\}$ by $f(x_j, v_i) = a_{j,i}$ for each pair (j,i). Obviously, f is a bijection and moreover $\sum_{j=1}^{n} f(x_j, v_i) = C$ for every i. Therefore, for any $x \in V(K_n \boxtimes G)$, we have $w(x) = (r+1)C - f(x)$. It follows that f is a $\mathbb{Z}_{nm}$-distance antimagic labelling of $K_n \boxtimes G$. □

6 $\mathbb{Z}_n$-distance antimagic circulants

A *circulant graph* is a Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, S)$ on a cyclic group $\mathbb{Z}_n$, where $S \subseteq \mathbb{Z}_n \setminus \{0\}$ such that $S = -S = \{-s : s \in S\}$ with operations modulo n. Note that the degree $|S|$ of $\mathrm{Cay}(\mathbb{Z}_n, S)$ is odd if n is even and $n/2 \in S$, and even otherwise.

The purpose of this section is to show the following result on $\mathbb{Z}_n$-distance antimagic circulants.

Theorem 6.1. *The circulant graph* $\mathrm{Cay}(\mathbb{Z}_n, S)$ *is* $\mathbb{Z}_n$*-distance antimagic if one of the following holds:*

(a) n *is even,* $n/2 \in S$*,* S *contains* $2s$ *even integers and* $2t$ *odd integers other than* $n/2$*, and* $2(s-t)+1$ *and* $2(t-s)+1$ *are both coprime to* n*;*

(b) $|S|$ *and* n *are coprime.*

On the other hand, if $|S|$ *and* n *are both even, then* $\mathrm{Cay}(\mathbb{Z}_n, S)$ *is not* $\mathbb{Z}_n$*-distance antimagic.*

Proof. To prove (a), assume $n = 2m$ is even and $m \in S$. Denote $\mathbb{Z}_n = \{0, 1, \ldots, 2m-1\}$ and let $S = \{\pm i_1, \pm i_2, \ldots, \pm i_s, \pm j_1, \pm j_2, \ldots, \pm j_t, m\}$, with operations modulo $2m$, where $0 < i_1 < i_2 < \cdots < i_s$ $(< m)$ are even and $0 < j_1 < j_2 < \cdots < j_t$ $(< m)$ are odd. Define the labelling as $f(k) = k$ for k odd and $f(k) = -k$ for k even.

If k is even, then the weight of vertex k under f is given by

$$\begin{aligned} w(k) =& \sum_{\ell=1}^{s} (f(k+i_\ell) + f(k-i_\ell)) + \sum_{\ell=1}^{t} (f(k+j_\ell) + f(k-j_\ell)) + f(k+m) \\ =& \sum_{\ell=1}^{s} (-(k+i_\ell) - (k-i_\ell)) + \sum_{\ell=1}^{t} ((k+j_\ell) + (k-j_\ell)) + f(k+m) \\ =& 2(t-s)k + f(k+m). \end{aligned}$$

Thus $w(k) = (2(t-s)-1)k - m$ if m is even, and $w(k) = (2(t-s)+1)k + m$ if m is odd. Since both $2(s-t)+1$ and $2(t-s)+1$ are coprime to $2m$, if m is even, then w induces a bijection from even elements of $\mathbb{Z}_{2m}$ to even elements of $\mathbb{Z}_{2m}$; and if m is odd, then w induces a bijection from even elements of $\mathbb{Z}_{2m}$ to odd elements of $\mathbb{Z}_{2m}$.

Similarly, when k is odd, we have

$$\begin{aligned} w(k) =& \sum_{\ell=1}^{s} ((k+i_\ell) + (k-i_\ell)) + \sum_{\ell=1}^{t} (-(k+j_\ell) - (k-j_\ell)) + f(k+m) \\ =& 2(s-t)k + f(k+m). \end{aligned}$$

Thus $w(k) = (2(s-t)+1)k + m$ if m is even, and $w(k) = (2(s-t)-1)k - m$ if m is odd. Similar to the above, if m is even, then w induces a bijection from odd elements of $\mathbb{Z}_{2m}$ to odd elements of $\mathbb{Z}_{2m}$; and if m is odd, then w induces a bijection from odd elements of $\mathbb{Z}_{2m}$ to even elements of $\mathbb{Z}_{2m}$. Combining this with what we proved in the previous paragraph, we conclude that w is a bijection from $\mathbb{Z}_{2m}$ to $\mathbb{Z}_{2m}$, and this proves part (a).

Part (b) follows from Theorem 2.5. By Corollary 2.3, if both n and $|S|$ are even, then $\mathrm{Cay}(\mathbb{Z}_n, S)$ is not $\mathbb{Z}_n$-distance antimagic. □

7 Remarks and questions

This paper represents the first attempt towards a systematic study of group distance antimagic and group distance magic labellings of graphs under a general setting. These notions are natural extensions of related concepts in the domain of graph labelling that have been extensively investigated in the past decades. A major theme for these new labellings is to answer the following question for various families of graph-group pairs (G, A) with $|V(G)| = |A|$: Under what conditions does G admit an A-distance antimagic/magic labelling?

In this paper we proved among other things several general results on group antimagic/magic labellings for product graphs. As applications we obtained concrete families of graphs admitting distance antimagic/magic labellings with respect to elementary Abelian groups, cyclic groups or direct products of such groups.

Many problems and questions arise naturally from our studies. As examples we list a few of them below that are relevant to this paper, with no attempt to be exhaustive.

1. When does a Cayley graph on $\mathbb{Z}_2^d$ with even degree (odd degree, respectively) admit a $\mathbb{Z}_2^d$-distance antimagic (magic, respectively) labelling? (See Corollary 2.7.)

2. Give a necessary and sufficient condition for the Hamming graph $H(d,q)$ $(d \geq 1, q \geq 3)$ to be $\mathbb{Z}_{q^d}$-distance antimagic. (Partial results were obtained in Theorem 3.1, and when $q = 2$ the answer was given in Corollary 3.2.)

3. Prove or disprove Q_6 is $\mathbb{Z}_2^6$-distance antimagic. (See Remark 3.10.)

4. Give a necessary and sufficient condition for $C_{n_1} \Box \cdots \Box C_{n_d}$ to be $\mathbb{Z}_{n_1 \cdots n_d}$-distance antimagic when d is not coprime to $n_1 \cdots n_d$. (A necessary condition is that all n_i's must be odd. See Corollary 5.3 for the case when d and $n_1 \cdots n_d$ are coprime.)

5. Give a necessary and sufficient condition for a circulant graph on n vertices to be $\mathbb{Z}_n$-distance antimagic. (See Section 6.)

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