Complex Exponentials and their Significance in LTI Systems

Euler's Identity

$$e^{j2\pi f_F t} = \cos(2\pi f_F t) + j\sin(2\pi f_F t)$$

Also

$$e^{-j2\pi f_F t} = \cos(2\pi f_F t) - j\sin(2\pi f_F t)$$

$$\Rightarrow \cos(2\pi f_F t) = \frac{e^{j2\pi f_F t} + e^{-j2\pi f_F t}}{2}$$

and

$$\Rightarrow \sin(2\pi f_F t) = \frac{e^{j2\pi f_F t} - e^{-j2\pi f_F t}}{2j}$$

Review of Euler's Identity – Complex valued sinusoidal signals

Euler's Identity

$$Ce^{j(2\pi f_F t+\theta)} = C\cos(2\pi f_F t+\theta) + jC\sin(2\pi f_F t+\theta)$$

Also

$$Ce^{-j(2\pi f_F t+\theta)} = C\cos(2\pi f_F t+\theta) - jC\sin(2\pi f_F t+\theta)$$

$$\Rightarrow C\cos(2\pi f_F t+\theta) = \frac{Ce^{j(2\pi f_F t+\theta)} + Ce^{-j(2\pi f_F t+\theta)}}{2}$$

and

$$\Rightarrow C\cos(2\pi f_F t+\theta) = \frac{Ce^{j\theta}}{2}e^{j2\pi f_F t} + \frac{Ce^{-j\theta}}{2}e^{-j2\pi f_F t}$$

Three ways to represent a sinusoidal frequency f_F

Method 1:

$$\frac{Ce^{j\theta}}{2}e^{j2\pi f_F t}+\frac{Ce^{-j\theta}}{2}e^{-j2\pi f_F t}$$

Method 2:

$$C\cos(2\pi f_F t+\theta)$$

Method 3:

$$A\cos(2\pi f_F t)+B\sin(2\pi f_F t)$$

Where

$$C=\sqrt{A^2+B^2} \qquad \theta=\tan^{-1}-\frac{B}{A}$$

Derivatives of Complex Exponentials

$$y(t) = Be^{j2\pi f_F t}$$

$$y^1(t) = B(j2\pi f_F)e^{j2\pi f_F t}$$

$$y^2(t) = B(j2\pi f_F)^2 e^{j2\pi f_F t}$$

$$\vdots$$

$$y^n(t) = B(j2\pi f_F)^n e^{j2\pi f_F t}$$

What does it mean regarding LTI systems?

Here is the Answer

$$x(t) = ? \longrightarrow \boxed{H} \longrightarrow y(t) = Be^{j2\pi f_F t}$$

$$a_n y^n(t) + \ldots + a_1 y^1(t) + a_0 y(t) = x(t)$$

$$a_n B(j2\pi f_F)^n e^{j2\pi f_F t} + \ldots + a_1 B(j2\pi f_F) e^{j2\pi f_F t} + a_0 B e^{j2\pi f_F t} = x(t)$$

$$e^{j2\pi f_F t}[a_n B(j2\pi f_F)^n + \ldots + a_1 B(j2\pi f_F) + a_0 B] = x(t)$$

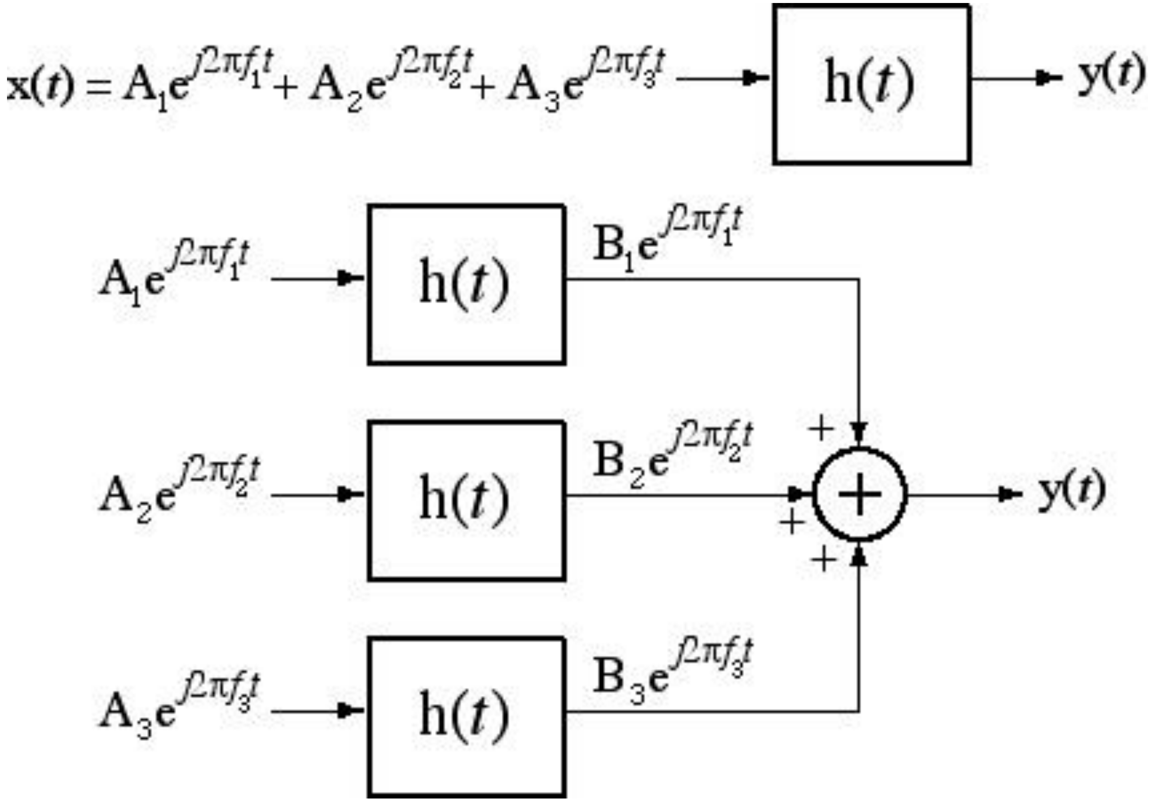
$$\Rightarrow Ae^{j2\pi f_F t} = x(t)$$

Complex exponential output means complex exponential input

=> ***Complex exponential input will produce complex exponential output***

Linearity and Superposition

If an excitation can be expressed as a sum of complex sinusoids the response can be expressed as the sum of responses to complex sinusoids.



Significance of Complex Exponential

- ***Complex Exponential Functions are called Eigenfunctions of LTI Systems.***

  (If applied at the input, output will remain same with a different constant)

- ***Any periodic function can be represented by weighted summation of complex exponentials.***

  Concept of Fourier Series

Fourier Series

Any periodic function can be represented by weighted summation of complex exponentials

$$x(t) = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kf_F)t}$$

X[k] needs to be found for any periodic signal

Concept of representing a periodic signal with a summation of sinusoids

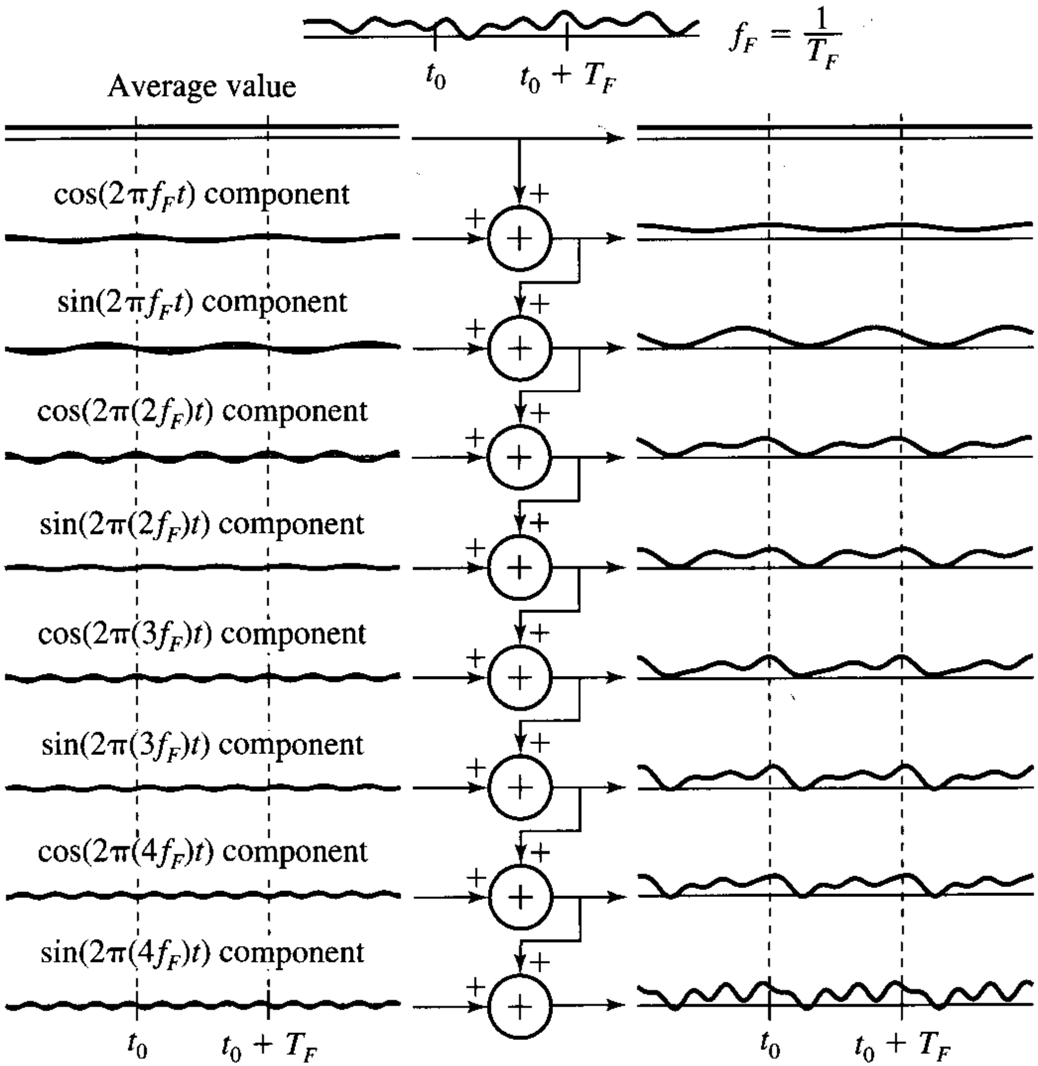
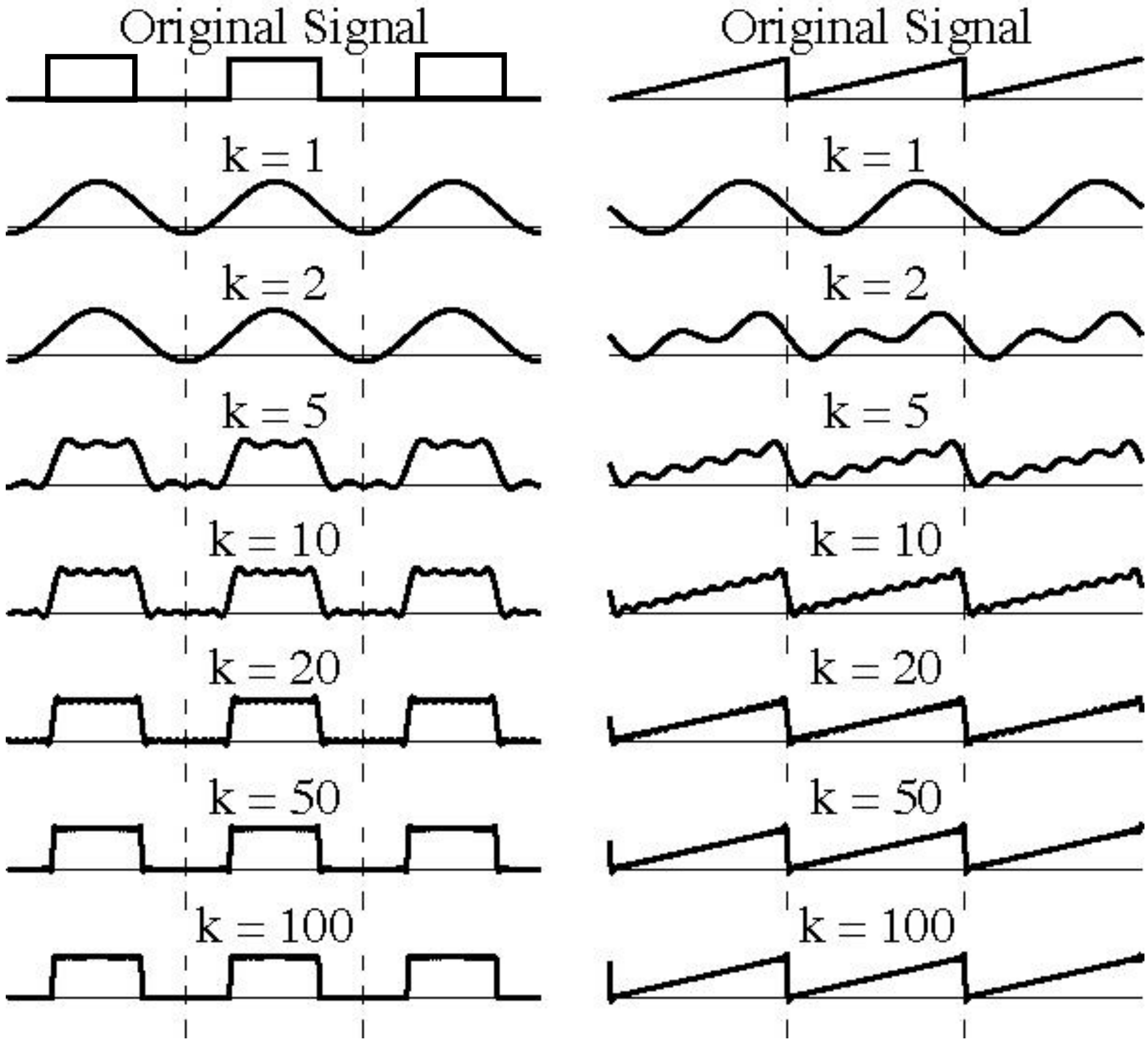


Figure 4.3
Illustration of the concept of representing an arbitrary signal as a linear combination of sinusoids.

Concept of representing a periodic signal with a summation of sinusoids … Cont.



$$x(t) = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kf_F)t}$$

Multiplying both sides with $e^{-j2\pi(qf_F)t}$ ***, where q is an integer***

$$x(t)e^{-j2\pi(qf_F)t} = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kf_F)t} e^{-j2\pi(qf_F)t}$$

$$x(t)e^{-j2\pi(qf_F)t} = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi[(k-q)f_F]t}$$

Integrating both sides over one period i.e., from t_0 ***to*** t_0+T_F

$$\int_{t_0}^{t_0+T_F} x(t)e^{-j2\pi(qf_F)t}dt = \int_{t_0}^{t_0+T_F} \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi((k-q)f_F)t}dt$$

$$\int_{t_0}^{t_0+T_F} x(t)e^{-j2\pi(qf_F)t}dt = \int_{t_0}^{t_0+T_F} \sum_{k=-\infty}^{k=\infty} X[k]e^{j2\pi((k-q)f_F)t}dt$$

$$\int_{t_0}^{t_0+T_F} x(t)e^{-j2\pi(qf_F)t}dt = \sum_{k=-\infty}^{k=\infty} X[k] \int_{t_0}^{t_0+T_F} e^{j2\pi((k-q)f_F)t}dt$$

$$\int_{t_0}^{t_0+T_F} x(t)e^{-j2\pi(qf_F)t}dt =$$

$$\sum_{k=-\infty}^{k=\infty} X[k] \int_{t_0}^{t_0+T_F} [\cos(2\pi(k-q)f_F t) + j\sin(2\pi(k-q)f_F t)]dt$$

$$\int_{t_0}^{t_0+T_F} x(t)e^{-j2\pi(qf_F)t}dt =$$

$$\sum_{k=-\infty}^{k=\infty} X[k] \boxed{\int_{t_0}^{t_0+T_F} [\cos(2\pi(k-q)f_F t) + j\sin(2\pi(k-q)f_F t)]dt}$$

If k ≠ q $$\int_{t_0}^{t_0+T_F} [\cos(2\pi(k-q)f_F t) + j\sin(2\pi(k-q)f_F t)]dt = 0$$

If k = q $$\int_{t_0}^{t_0+T_F} [\cos(0) + j\sin(0)]dt = \int_{t_0}^{t_0+T_F} dt = T_F$$

$$\Rightarrow \int_{t_0}^{t_0+T_F} x(t)e^{-j2\pi(qf_F)t}dt = X[q]T_F$$

$$\Rightarrow X[q] = \frac{1}{T_F}\int_{t_0}^{t_0+T_F} x(t)e^{-j2\pi(qf_F)t}dt$$

That is what we wanted to find

$$\boxed{\Rightarrow X[k] = \frac{1}{T_F}\int_{t_0}^{t_0+T_F} x(t)e^{-j2\pi(kf_F)t}dt}$$

Please note:

$$X[0] = \frac{1}{T_F}\int_{t_0}^{t_0+T_F} x(t)dt = \quad average \quad value$$

One more fact about X[k]

$$X[k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t) e^{-j2\pi(kf_F)t} dt$$

$$X[-k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t) e^{-j2\pi(-kf_F)t} dt$$

$$= \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t) e^{+j2\pi(kf_F)t} dt$$

$$= X[k]^*$$

A Step Back! Let's be focused on actual point

$$x(t) = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kf_F)t}$$

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} \left(X[k] e^{j2\pi(kf_F)t} + X[-k] e^{-j2\pi(kf_F)t} \right)$$

However, if we find this, that will be more interesting!

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} \left(X_c[k]\cos(2\pi k f_F t) + X_s[k]\sin(2\pi k f_F t) \right)$$

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} (X[k] e^{j2\pi(kf_F)t} + X[-k] e^{-j2\pi(kf_F)t})$$

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} (X[k][\cos(2\pi k f_F t) + j\sin(2\pi k f_F t)]$$

$$+ X[-k][\cos(2\pi k f_F t) - j\sin(2\pi k f_F t)])$$

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} (\cos(2\pi k f_F t)(X[k] + X[-k])$$

$$+ j\sin(2\pi k f_F t)(X[k] - X[-k]))$$

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} (\cos(2\pi k f_F t)(X[k] + X[-k])$$

$$+ j\sin(2\pi k f_F t)(X[k] - X[-k]))$$

Please note

$$X[k] + X[-k] = X[k] + X[k]^* = 2\,\mathrm{Re}(X[k])$$

$$X[k] - X[-k] = X[k] - X[k]^* = 2j\,\mathrm{Im}(X[k])$$

$$\Rightarrow x(t) = X[0] + \sum_{k=1}^{k=\infty} (\cos(2\pi k f_F t)(2\,\mathrm{Re}(X[k])$$

$$+ j\sin(2\pi k f_F t)(2j\,\mathrm{Im}(X[k]))$$

$$\Rightarrow x(t) = X[0] + \sum_{k=1}^{k=\infty} (\cos(2\pi k f_F t)(2\operatorname{Re}(X[k]))$$

$$+ j\sin(2\pi k f_F t)(2j\operatorname{Im}(X[k]))$$

$$\Rightarrow x(t) = X[0] + \sum_{k=1}^{k=\infty} (\cos(2\pi k f_F t)(2\operatorname{Re}(X[k]))$$

$$+ \sin(2\pi k f_F t)(-2\operatorname{Im}(X[k]))$$

Remember what we wanted to do?

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} (X_c[k]\cos(2\pi k f_F t) + X_s[k]\sin(2\pi k f_F t))$$

$$\Rightarrow x(t) = X[0] + \sum_{k=1}^{k=\infty} (\cos(2\pi k f_F t)(2\,\mathrm{Re}(X[k]))$$

$$+\sin(2\pi k f_F t)(-2\,\mathrm{Im}(X[k]))$$

Remember what we wanted to do?

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} (X_c[k]\cos(2\pi k f_F t) + X_s[k]\sin(2\pi k f_F t))$$

Comparison of the above two equations give us

$$X_c[k] = 2\,\mathrm{Re}(X[k])$$

$$X_s[k] = -2\,\mathrm{Im}(X[k])$$

Let's find Re and Im parts of X[k]

$$X[k]=\frac{1}{T_F}\int_{t_0}^{t_0+T_F} x(t)e^{-j2\pi(kf_F)t}dt$$

$$X[k]=\frac{1}{T_F}\int_{t_0}^{t_0+T_F}[x(t)\cos(2\pi kf_F t)-j\sin((2\pi kf_F t)]dt$$

$$X[k]=\frac{1}{T_F}\int_{t_0}^{t_0+T_F} x(t)\cos(2\pi kf_F t)dt-\frac{j}{T_F}\int_{t_0}^{t_0+T_F} x(t)\sin(2\pi kf_F t)dt$$

$$X[k]= \qquad \mathrm{Re}(X[k]) \qquad + \qquad j\,\mathrm{Im}(X[k])$$

W also know

$$X_c[k]=2\,\mathrm{Re}(X[k]) \qquad X_s[k]=-2\,\mathrm{Im}(X[k])$$

Do we have the answer?

Yes, We do!

$$X_c[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t)\cos(2\pi k f_F t)dt$$

$$X_s[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t)\sin(2\pi k f_F t)dt$$

Trigonometric Fourier Series

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} (X_c[k]\cos(2\pi k f_F t) + X_s[k]\sin(2\pi k f_F t))$$

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} M[k]\cos(2\pi k f_F t + \theta[k])$$

$$M[k] = \sqrt{X_c[k]^2 + X_s[k]^2}$$

$$= 2\sqrt{\mathrm{Re}(X[k])^2 + \mathrm{Im}(X[k])^2}$$

Magnitude

$$\theta[k] = \tan^{-1} - \frac{X_s[k]}{X_c[k]} = \tan^{-1} - \frac{\mathrm{Im}(X[k])}{\mathrm{Re}(X[k])}$$

Phase

Summary – chart 1

Exponential Fourier Series

$$x(t) = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kf_F)t}$$

where

$$X[k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t) e^{-j2\pi(kf_F)t} dt$$

Please note

$$X[0] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t) dt \quad = \quad average \quad value$$

Summary – chart 2

Trigonometric Fourier Series

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} (X_c[k]\cos(2\pi k f_F t) + X_s[k]\sin(2\pi k f_F t))$$

where

$$X_c[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t)\cos(2\pi k f_F t)dt$$

$$X_s[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t)\sin(2\pi k f_F t)dt$$

$$X_c[k] = 2\,\mathrm{Re}(X[k]) \qquad X_s[k] = -2\,\mathrm{Im}(X[k])$$

Summary – chart 3

Compact form of Trigonometric Fourier Series

$$x(t) = X[0] + \sum_{k=1}^{k=\infty} M[k]\cos(2\pi k f_F t + \theta[k])$$

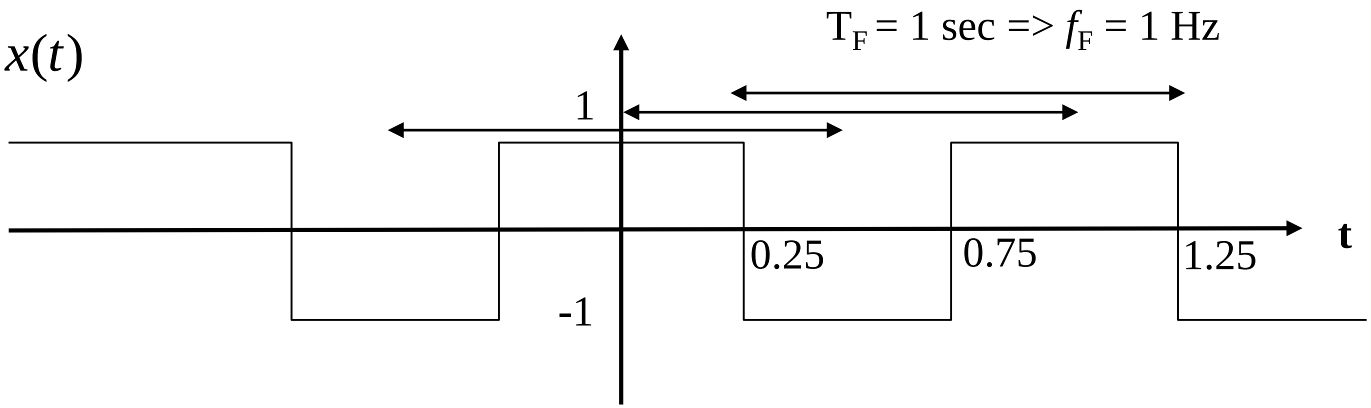
Where

$$M[k] = \sqrt{X_c[k]^2 + X_s[k]^2}$$ **Magnitude**

$$= 2\sqrt{\mathrm{Re}(X[k])^2 + \mathrm{Im}(X[k])^2} = 2|X[k]|$$

$$\theta[k] = \tan^{-1} - \frac{X_s[k]}{X_c[k]} = \tan^{-1} - \frac{\mathrm{Im}(X[k])}{\mathrm{Re}(X[k])}$$ **Phase**

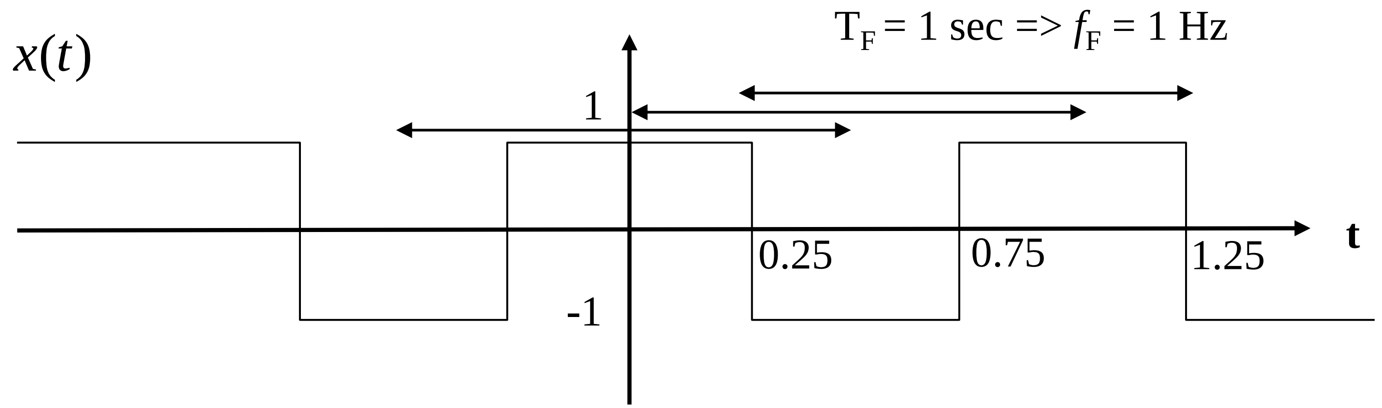
Example 1



$$X[0] = \frac{1}{T_F}\int_{t_0}^{t_0+T_F} x(t)dt = 0$$

$$X_s[k] = \frac{2}{T_F}\int_{t_0}^{t_0+T_F} x(t)\sin(2\pi k f_F t)dt = 0$$

Example 1 … Cont.

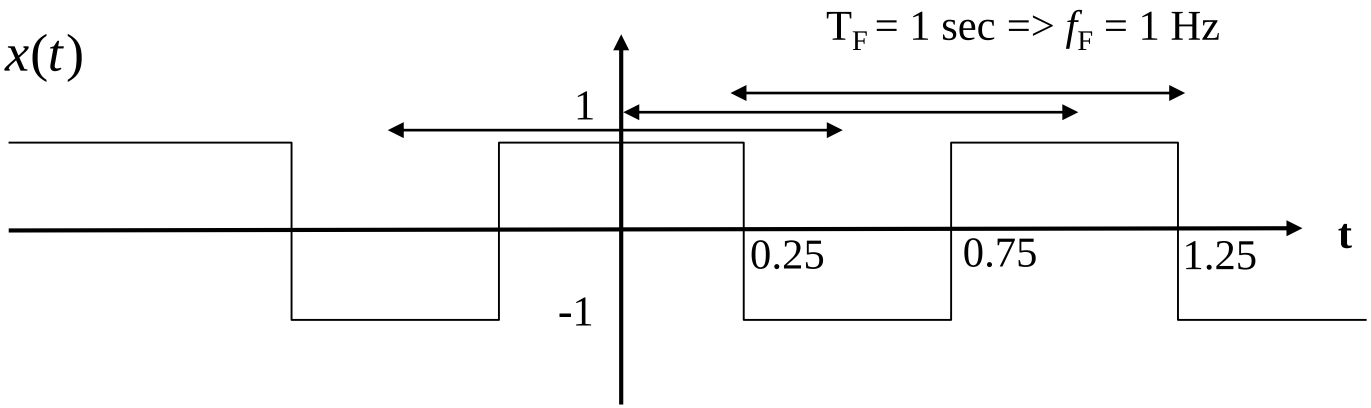


$$X_c[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t)\cos(2\pi k f_F t)dt$$

$$X_c[1] = 2\int_{-0.5}^{0.5} x(t)\cos(2\pi t)dt = 2x2\int_{0}^{0.5} x(t)\cos(2\pi t)dt$$

$$= 4\int_{0}^{0.25}\cos(2\pi t)dt + 4\int_{0.25}^{0.5}(-1)\cos(2\pi t)dt$$

Example 1 … Cont.

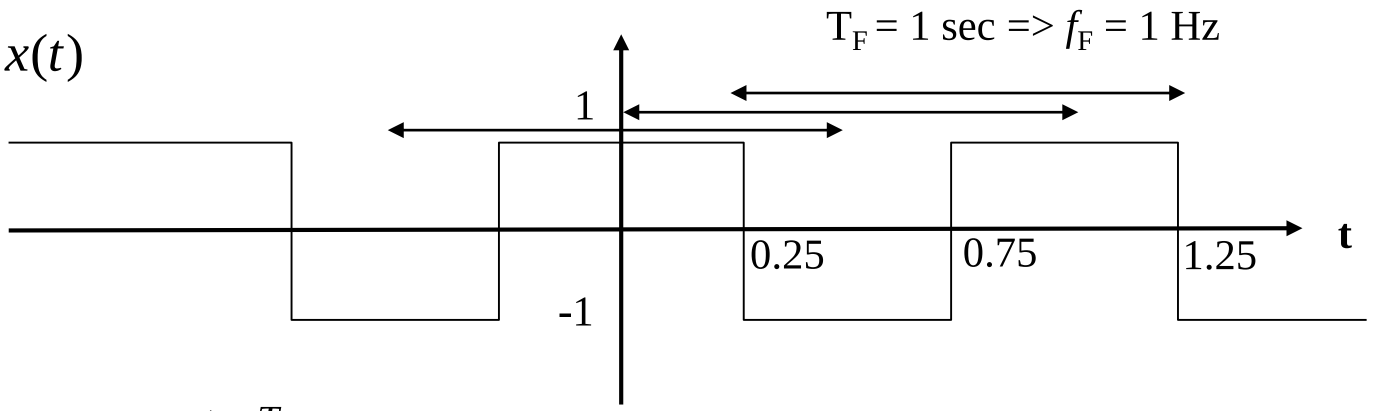


$$= 4\int_{0}^{0.25}\cos(2\pi t)dt + 4\int_{0.25}^{0.5}(-1)\cos(2\pi t)dt$$

$$= \left|\frac{4\sin(2\pi t)}{2\pi}\right|_{0}^{0.25} - \left|\frac{4\sin(2\pi t)}{2\pi}\right|_{0.25}^{0.5}$$

$$= \frac{4}{2\pi} - \left(-\frac{4}{2\pi}\right) = \frac{4}{2\pi} + \frac{4}{2\pi} = \frac{4}{\pi}$$

Example 1 … Cont.

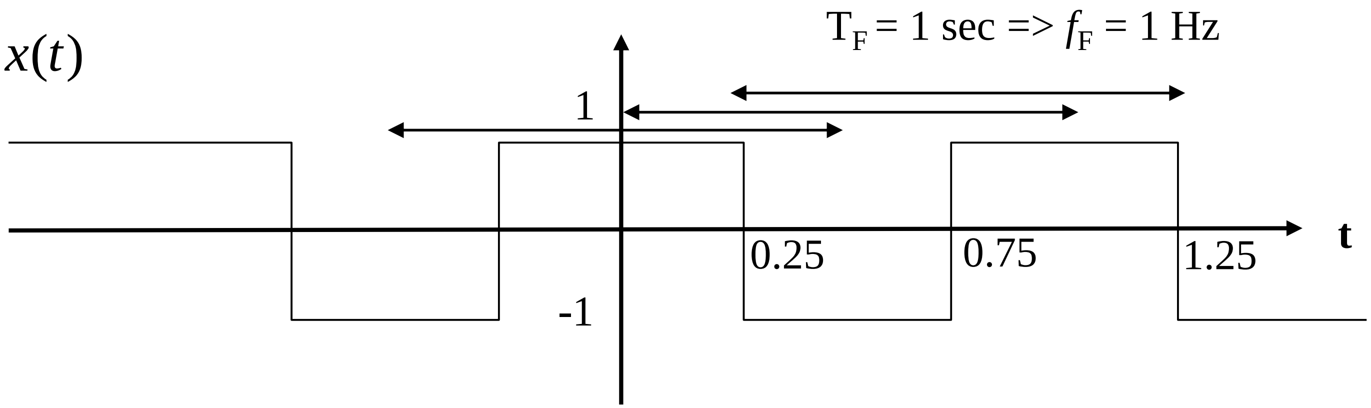


$$X_c[k] = \frac{2}{T_F}\int_{t_0}^{t_0+T_F} x(t)\cos(2\pi k f_F t)dt$$

$$X_c[k] = 2\int_{-0.5}^{0.5} x(t)\cos(2\pi k t)dt = 2x2\int_{0}^{0.5} x(t)\cos(2\pi k t)dt$$

$$= \left|\frac{4\sin(2\pi k t)}{2\pi k}\right|_{0}^{0.25} - \left|\frac{4\sin(2\pi k t)}{2\pi k}\right|_{0.25}^{0.5}$$

Example 1 … Cont.

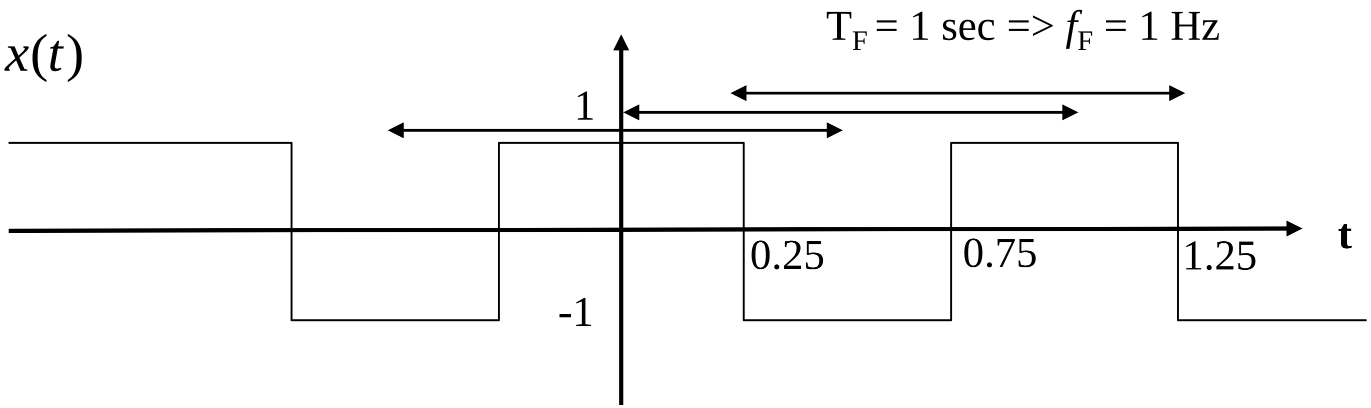


$$= \left| \frac{4\sin(2\pi kt)}{2\pi k} \right|_0^{0.25} - \left| \frac{4\sin(2\pi kt)}{2\pi k} \right|_{0.25}^{0.5}$$

$$= \frac{2}{\pi k}\sin(\frac{\pi k}{2}) - (-\frac{2}{\pi k}\sin(\frac{\pi k}{2})) = \frac{4}{\pi k}\sin(\frac{\pi k}{2})$$

$$\Rightarrow X_c[k] = \frac{4}{\pi k}\sin(\frac{\pi k}{2})$$

Example 1 … Cont.



$$X_c[k] = \frac{4}{\pi k}\sin(\frac{\pi k}{2})$$

$$X_c[1] = \frac{4}{\pi}$$

$$X_c[4] = 0$$

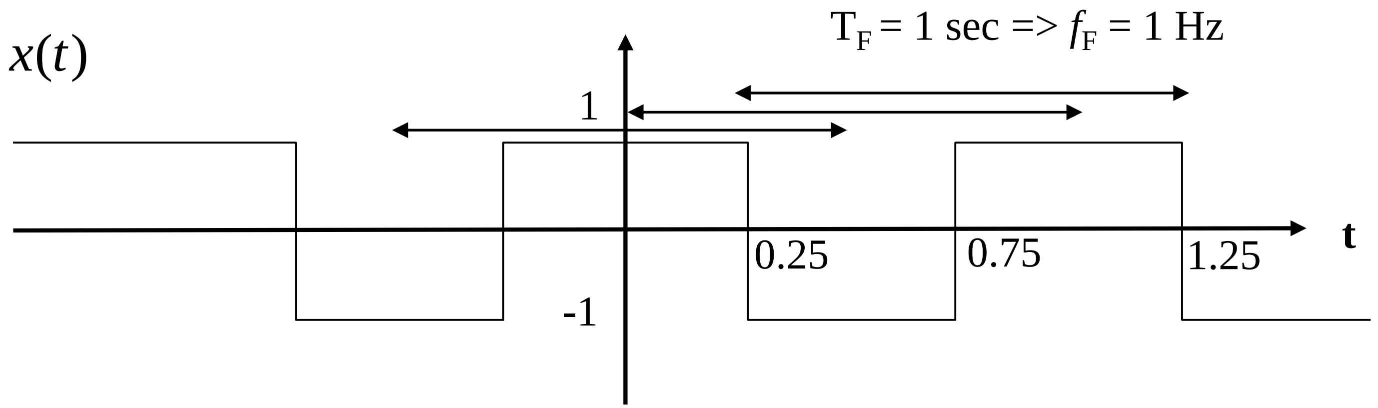
$$X_c[2] = 0$$

$$X_c[5] = \frac{4}{5\pi}$$

$$X_c[3] = -\frac{4}{3\pi}$$

$$X_c[6] = 0$$

Example 1 … Cont.

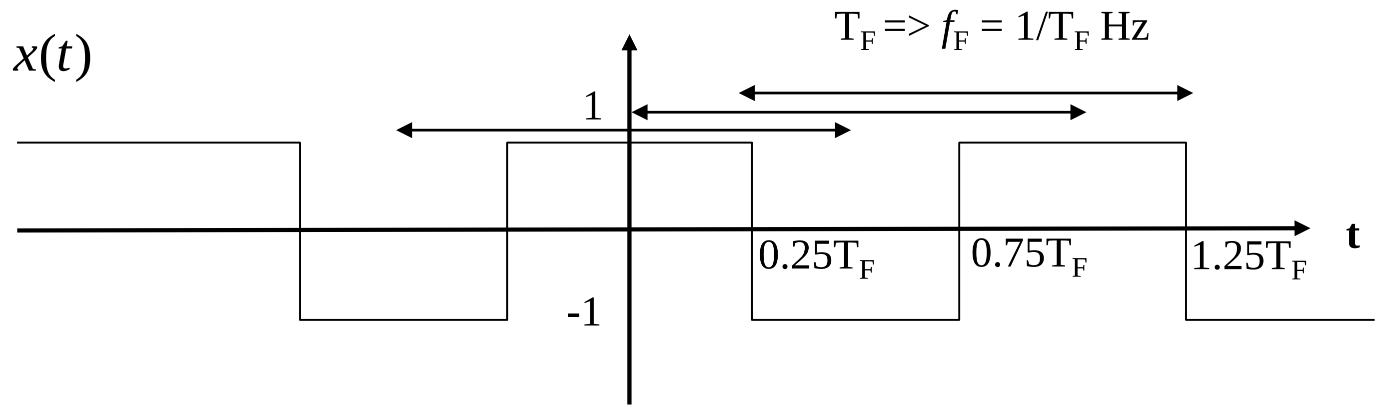


$$x(t) = X[0] + \sum_{k=1}^{k=\infty} \left(X_c[k]\cos(2\pi k f_F t) + X_s[k]\sin(2\pi k f_F t) \right)$$

$$\Rightarrow x(t) = \frac{4}{\pi}\cos(2\pi t) - \frac{4}{3\pi}\cos(2\pi(3)t) + \frac{4}{5\pi}\cos(2\pi(5)t)$$

$$- \frac{4}{7\pi}\cos(2\pi(7)t) + \ldots$$

Example 1 … Cont.

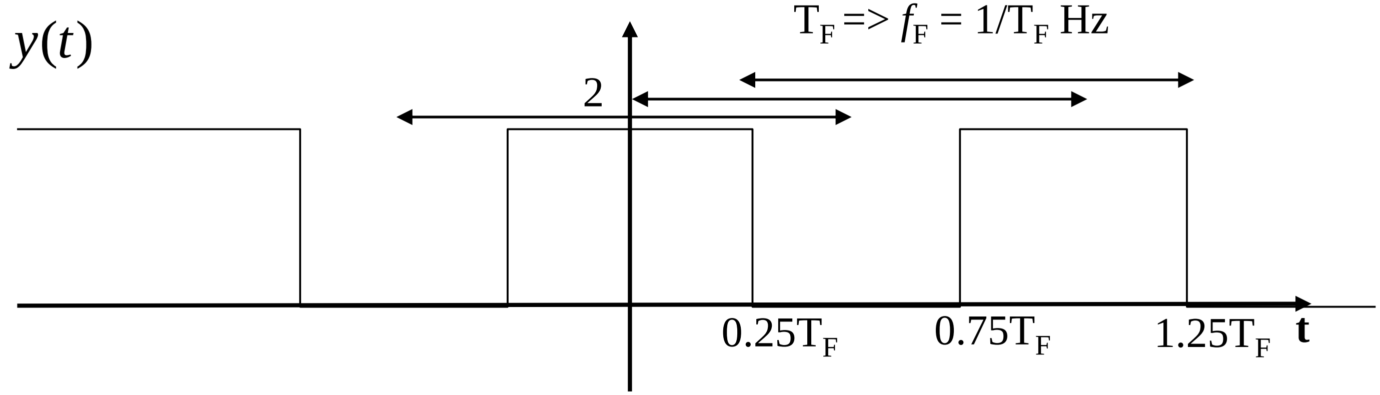


$$x(t) = X[0] + \sum_{k=1}^{k=\infty} (X_c[k]\cos(2\pi k f_F t) + X_s[k]\sin(2\pi k f_F t))$$

$$\Rightarrow x(t) = \frac{4}{\pi}\cos(2\pi f_F t) - \frac{4}{3\pi}\cos(2\pi(3f_F)t) + \frac{4}{5\pi}\cos(2\pi(5f_F)t)$$

$$-\frac{4}{7\pi}\cos(2\pi(7f_F)t) + \dots$$

Example 2



$$y(t) = 1 + x(t)$$

$$Y[0] = 1 + X[0] = 1$$

$$Y_s[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} y(t)\sin(2\pi k f_F t)dt = 0$$

$$Y_c[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} y(t)\cos(2\pi k f_F t)dt = X_c[k]$$

$$Y_s[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} y(t)\sin(2\pi k f_F t)dt$$

$$= \frac{2}{T_F} \int_{t_0}^{t_0+T_F} (1 + x(t))\sin(2\pi k f_F t)dt$$

$$= \frac{2}{T_F} \int_{t_0}^{t_0+T_F} \sin(2\pi k f_F t)dt + \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t)\sin(2\pi k f_F t)dt$$

$$= 0 + X_s[k]$$

$$= 0 + 0 = 0$$

$$Y_c[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} y(t)\cos(2\pi k f_F t)dt$$

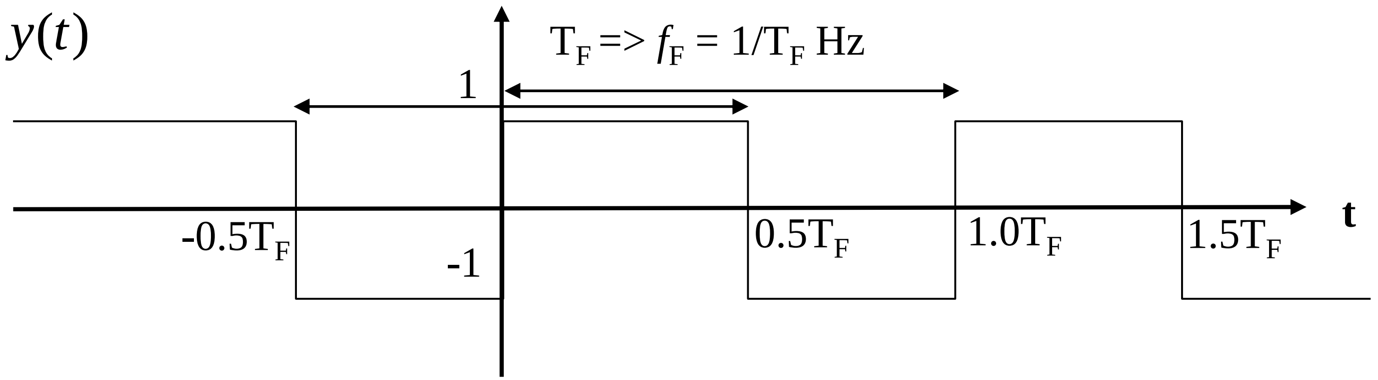
$$= \frac{2}{T_F} \int_{t_0}^{t_0+T_F} (1 + x(t))\cos(2\pi k f_F t)dt$$

$$= \frac{2}{T_F} \int_{t_0}^{t_0+T_F} \cos(2\pi k f_F t)dt + \frac{2}{T_F} \int_{t_0}^{t_0+T_F} x(t)\cos(2\pi k f_F t)dt$$

$$= 0 + X_c[k]$$

$$= X_c[k]$$

Example 3



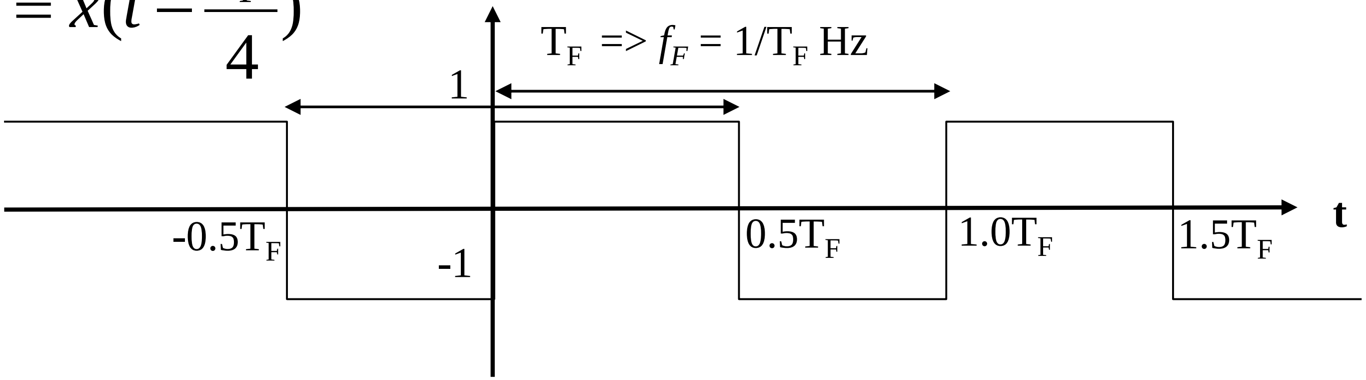
$$y(t) = Y[0] + \sum_{k=1}^{k=\infty} (Y_c[k]\cos(2\pi k f_F t) + Y_s[k]\sin(2\pi k f_F t))$$

$$Y[0] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} y(t)dt = 0$$

$$Y_c[k] = \frac{2}{T_F} \int_{t_0}^{t_0+T_F} y(t)\cos(2\pi k f_F t)dt = 0$$

Example 3 … Cont.

$$y(t) = x(t - \frac{T_F}{4})$$



$$x(t) = \frac{4}{\pi}\cos(2\pi f_F t) - \frac{4}{3\pi}\cos(2\pi(3f_F)t) + \frac{4}{5\pi}\cos(2\pi(5f_F)t)$$

$$-\frac{4}{7\pi}\cos(2\pi(7f_F)t) + \ldots$$

$$\Rightarrow y(t) = \frac{4}{\pi}\cos(2\pi f_F(t - \frac{T_F}{4})) - \frac{4}{3\pi}\cos(2\pi(3f_F)(t - \frac{T_F}{4}))$$

$$+\frac{4}{5\pi}\cos(2\pi(5f_F)(t - \frac{T_F}{4})) - \frac{4}{7\pi}\cos(2\pi(7f_F)(t - \frac{T_F}{4})) + \ldots$$

$$y(t)=\frac{4}{\pi}\cos(2\pi f_F(t-\frac{T_F}{4}))-\frac{4}{3\pi}\cos(2\pi(3f_F)(t-\frac{T_F}{4}))$$
$$+\frac{4}{5\pi}\cos(2\pi(5f_F)(t-\frac{T_F}{4}))-\frac{4}{7\pi}\cos(2\pi(7f_F)(t-\frac{T_F}{4}))+\ldots$$

$$y(t)=\frac{4}{\pi}\cos(2\pi f_F t-\frac{\pi f_F T_F}{2})-\frac{4}{3\pi}\cos(2\pi(3f_F)t-\frac{\pi 3f_F T_F}{2})$$
$$+\frac{4}{5\pi}\cos(2\pi(5f_F)t-\frac{\pi 5f_F T_F}{2})-\frac{4}{7\pi}\cos(2\pi(7f_F)t-\frac{\pi 7f_F T_F}{2})+\ldots$$

$$y(t)=\frac{4}{\pi}\cos(2\pi f_F t-\frac{\pi}{2})-\frac{4}{3\pi}\cos(2\pi(3f_F)t-\frac{\pi 3}{2})$$
$$+\frac{4}{5\pi}\cos(2\pi(5f_F)t-\frac{\pi 5}{2})-\frac{4}{7\pi}\cos(2\pi(7f_F)t-\frac{\pi 7}{2})+\ldots$$

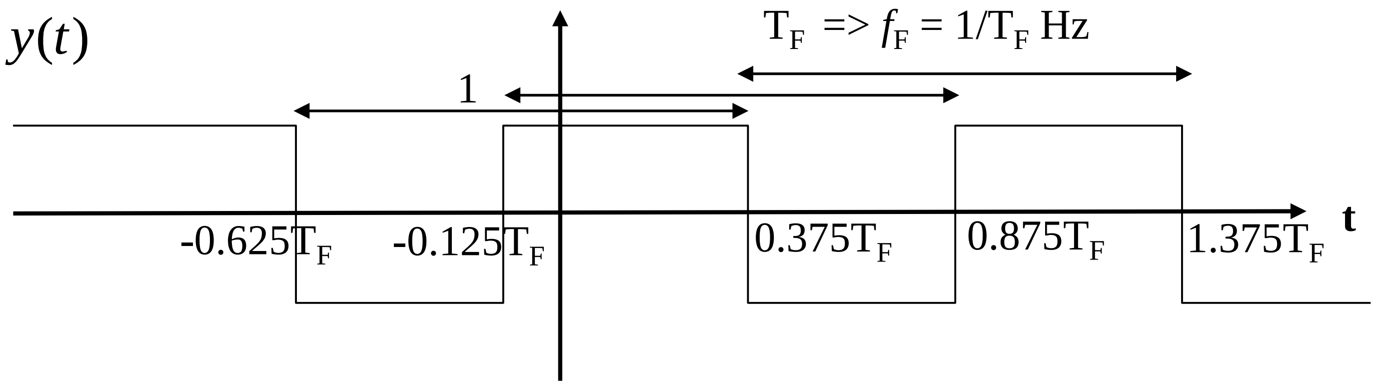
$$y(t) = \frac{4}{\pi}\cos(2\pi f_F t - \frac{\pi}{2}) - \frac{4}{3\pi}\cos(2\pi(3f_F)t - \frac{\pi 3}{2})$$
$$+\frac{4}{5\pi}\cos(2\pi(5f_F)t - \frac{\pi 5}{2}) - \frac{4}{7\pi}\cos(2\pi(7f_F)t - \frac{\pi 7}{2}) + \ldots$$

$$\Rightarrow y(t) = \frac{4}{\pi}\sin(2\pi f_F t) + \frac{4}{3\pi}\sin(2\pi(3f_F)t)$$
$$+\frac{4}{5\pi}\sin(2\pi(5f_F)t) + \frac{4}{7\pi}\sin(2\pi(7f_F)t) + \ldots$$

$$|X_c[k]| \rightarrow |Y_s[k]|$$

Please note that you have also found
M[k] and θ[k]!

Example 4



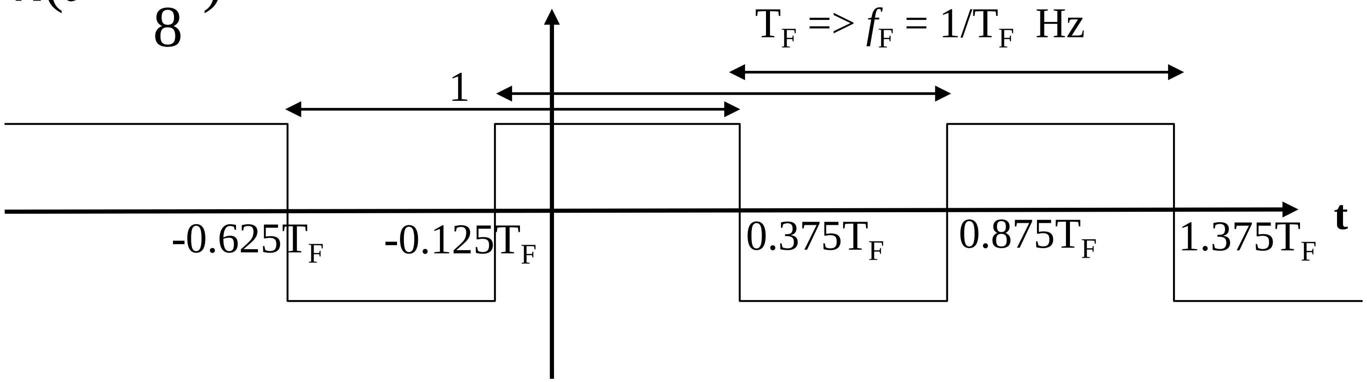
$$y(t) = Y[0] + \sum_{k=1}^{k=\infty} (Y_c[k]\cos(2\pi k f_F t) + Y_s[k]\sin(2\pi k f_F t))$$

$$Y[0] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} y(t)dt = 0$$

$$y(t) = x(t - \frac{T_F}{8})$$

Example 4 … Cont.

$$y(t) = x(t - \frac{T_F}{8})$$



$$x(t) = \frac{4}{\pi}\cos(2\pi f_F t) - \frac{4}{3\pi}\cos(2\pi(3f_F)t) + \frac{4}{5\pi}\cos(2\pi(5f_F)t)$$

$$-\frac{4}{7\pi}\cos(2\pi(7f_F)t) + \ldots$$

$$\Rightarrow y(t) = \frac{4}{\pi}\cos(2\pi f_F(t - \frac{T_F}{8})) - \frac{4}{3\pi}\cos(2\pi(3f_F)(t - \frac{T_F}{8}))$$

$$+\frac{4}{5\pi}\cos(2\pi(5f_F)(t - \frac{T_F}{8})) - \frac{4}{7\pi}\cos(2\pi(7f_F)(t - \frac{T_F}{8})) + \ldots$$

$$y(t)=\frac{4}{\pi}\cos(2\pi f_F(t-\frac{T_F}{8}))-\frac{4}{3\pi}\cos(2\pi(3f_F)(t-\frac{T_F}{8}))$$
$$+\frac{4}{5\pi}\cos(2\pi(5f_F)(t-\frac{T_F}{8}))-\frac{4}{7\pi}\cos(2\pi(7f_F)(t-\frac{T_F}{8}))+\ldots$$

$$y(t)=\frac{4}{\pi}\cos(2\pi f_F t-\frac{\pi f_F T_F}{4})-\frac{4}{3\pi}\cos(2\pi(3f_F)t-\frac{\pi 3 f_F T_F}{4})$$
$$+\frac{4}{5\pi}\cos(2\pi(5f_F)t-\frac{\pi 5 f_F T_F}{4})-\frac{4}{7\pi}\cos(2\pi(7f_F)t-\frac{\pi 7 f_F T_F}{4})+\ldots$$

$$y(t)=\frac{4}{\pi}\cos(2\pi f_F t-\frac{\pi}{4})-\frac{4}{3\pi}\cos(2\pi(3f_F)t-\frac{\pi 3}{4})$$
$$+\frac{4}{5\pi}\cos(2\pi(5f_F)t-\frac{\pi 5}{4})-\frac{4}{7\pi}\cos(2\pi(7f_F)t-\frac{\pi 7}{4})+\ldots$$

$$y(t)=\frac{4}{\pi}\cos(2\pi f_F t-\frac{\pi}{4})-\frac{4}{3\pi}\cos(2\pi(3f_F)t-\frac{\pi 3}{4})$$

$$+\frac{4}{5\pi}\cos(2\pi(5f_F)t-\frac{\pi 5}{4})-\frac{4}{7\pi}\cos(2\pi(7f_F)t-\frac{\pi 7}{4})+\ldots$$

You have found M[k] and θ[k]!

or

$$\Rightarrow X_c[k]\rightarrow \quad non-zero \quad Y_c[k] \quad and \quad Y_s[k]$$

Hint -> $\cos(a\pm b)=\cos a\cos b\mp\sin a\sin b$

Changing from f (Hz) to ω (radians/sec)

Fourier Series Pair in f

$$\begin{cases} x(t) = \sum_{k=-\infty}^{k=\infty} X[k] e^{j2\pi(kf_F)t} \\ X[k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t) e^{-j2\pi(kf_F)t} dt \end{cases}$$

$$\omega = 2\pi f$$

$$\Downarrow$$

Fourier Series Pair in ω

$$\begin{cases} x(t) = \sum_{k=-\infty}^{k=\infty} X[k] e^{jk\omega_F t} \\ X[k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t) e^{-jk\omega_F t} dt \end{cases}$$