From Fourier Transform to Laplace Transform

Fourier Transform of a Signal x(t)

$$X(\omega) = F[x(t)]$$

$$X(f) = F[x(t)]$$

OR

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

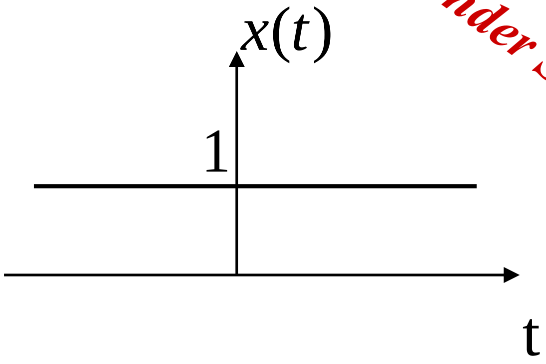
Fourier Transform of a Constant Function

Reminder Slide

Fourier Transform of a Singal x(t)

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega t} dt \quad = \left| \frac{e^{-j\omega t}}{-j\omega} \right|_{-\infty}^{\infty} = 0 + \infty = \infty$$

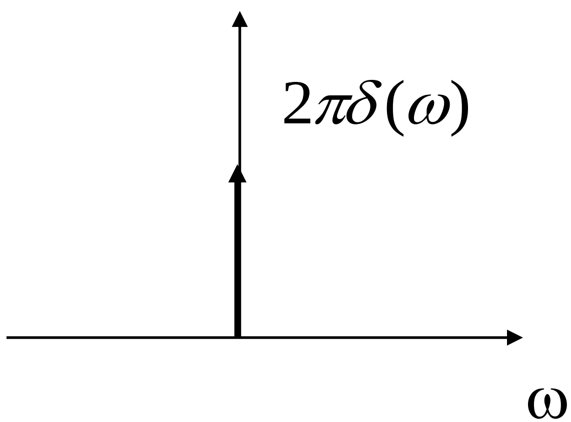


Let's try indirectly – let's find the Inverse Fourier Transform of δ(ω)

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\Rightarrow F^{-1}[\delta(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{0} = \frac{1}{2\pi}$$

$$\Rightarrow F[\frac{1}{2\pi}] = \delta(\omega) \qquad \Rightarrow F[1] = 2\pi\delta(\omega)$$

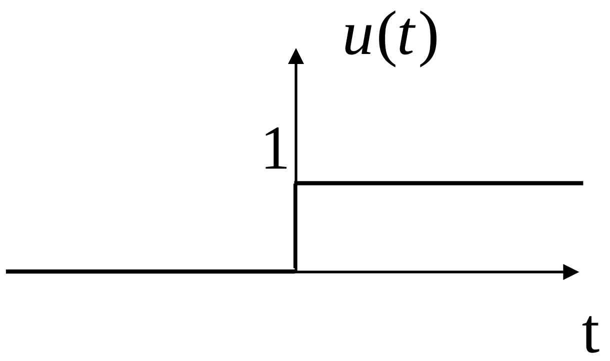


What about Fourier Transform of Unit Step Function

$$F[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-j\omega t} dt$$

$$= \left| \frac{e^{-j\omega t}}{-j\omega} \right|_{0}^{\infty} =$$ ***Does not Converge***



What about Fourier Transform of Unit Step Function

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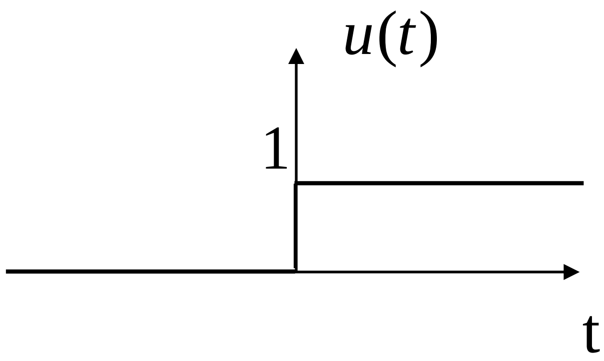


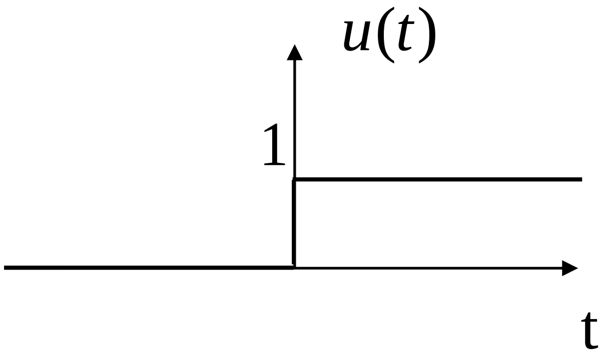
How about if $j\omega \rightarrow \sigma + j\omega = s$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad X(s) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$F[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-j\omega t} dt$$

$$= \left| \frac{e^{-j\omega t}}{-j\omega} \right|_{0}^{\infty} =$$ ***Does not Converge***



$$NewTransform[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-st} dt = \left| \frac{e^{-st}}{-s} \right|_{0}^{\infty} = \frac{1}{s} \qquad \sigma > 0$$

How Convergence Occurs

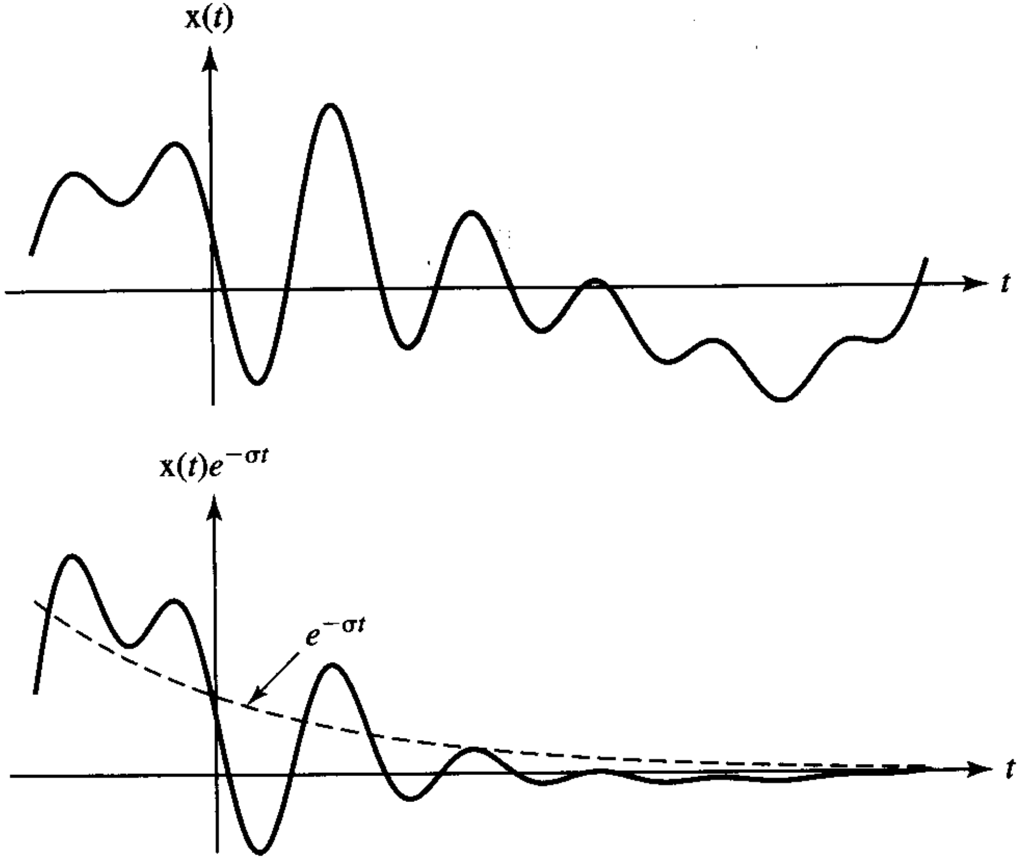


Figure 9.1
The effect of the decaying-exponential convergence factor on the original function.

Laplace Transform

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-(\sigma + j\omega)t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

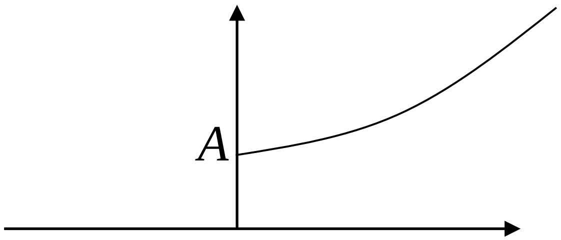
$$= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt$$

$$= F[x(t)e^{-\sigma t}]$$

Laplace Transform - Example

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) = Ae^{\alpha t}u(t)$$



Let's look at another example

$$x(t) = Ae^{\alpha t}u(t)$$

x(t) does not have Fourier transform but Laplace transform exists

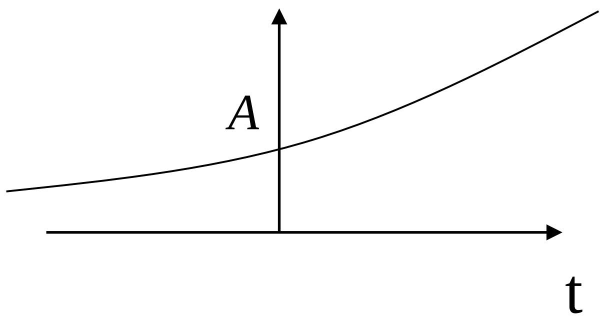
$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} Ae^{\alpha t}u(t)e^{-st}dt$$

$$= \int_{0}^{\infty} Ae^{-(s-\alpha)t}dt = \left|\frac{Ae^{-(s-\alpha)t}}{-(s-\alpha)}\right|_{0}^{\infty} = \frac{A}{s-\alpha} \qquad \sigma > \alpha$$

Now Let's look at another example

$$x(t) = Ae^{\alpha t}$$

$x(t) = Ae^{\alpha t}$



Does Laplace Transform exists?

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} Ae^{\alpha t} e^{-st} dt$$

$$= \int_{-\infty}^{\infty} Ae^{-(s-\alpha)t} dt = A\int_{-\infty}^{\infty} e^{-(\sigma-\alpha)t} e^{-j\omega t} dt$$

This integral does not converge

Therefore, the defined Laplace transform does not exist for this function

Bilateral vs. Unilateral Laplace Transform

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Bilateral Laplace Transform

To avoid non-convergence Laplace transform is redefined for causal signals

$$L[x(t)] = X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$

Unilateral Laplace Transform

(applies to causal signals only)

Laplace Transform - Example

$$x(t) = e^{-\alpha t}\cos(\omega_0 t)u(t)$$

$$L[x(t)] = X(s) = \int_0^\infty x(t)e^{-st}dt$$

$$= \int_0^\infty e^{-\alpha t}\cos(\omega_0 t)u(t)e^{-st}dt$$

$$= \int_0^\infty e^{-\alpha t}\cos(\omega_0 t)e^{-st}dt$$

$$= \int_0^\infty e^{-\alpha t}\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}e^{-st}dt$$

$$L[x(t)] = X(s) = \int_0^\infty e^{-\alpha t} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} e^{-st} dt$$

$$= \frac{1}{2} \int_0^\infty (e^{(j\omega_0 - s - \alpha)t} + e^{(-j\omega_0 - s - \alpha)t}) dt$$

$$= \frac{1}{2} \left[\frac{e^{(j\omega_0 - s - \alpha)t}}{j\omega_0 - (s + \alpha)} + \frac{e^{-(j\omega_0 + s + \alpha)t}}{-j\omega_0 - (s + \alpha)} \right]_0^\infty$$

$$= \frac{1}{2} \left[\frac{-1}{j\omega_0 - (s + \alpha)} + \frac{-1}{-j\omega_0 - (s + \alpha)} \right]$$

$$= \frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2} \qquad \sigma > \alpha$$

$$L[e^{-\alpha t}\cos(\omega_0 t)u(t)] = \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$$

Similarly

$$L[e^{-\alpha t}\sin(\omega_0 t)u(t)] = \frac{\omega_0^2}{(s+\alpha)^2+\omega_0^2}$$

$$\Rightarrow L[\cos(\omega_0 t)u(t)] = \frac{s}{s^2+\omega_0^2}$$

and

$$L[\sin(\omega_0 t)u(t)] = \frac{\omega_0^2}{s^2+\omega_0^2}$$

Also

$$L[e^{-\alpha t}u(t)] = \frac{1}{s+\alpha}$$

Properties of Laplace Transform

Linearity

$$k_1 x_1(t) + k_2 x_2(t) \Leftrightarrow k_1 X_1(\omega) + k_2 X_2(\omega)$$ **FT**

$$k_1 x_1(t) + k_2 x_2(t) \Leftrightarrow k_1 X_1(s) + k_2 X_2(s)$$ **LT**

Time Shifting

$$x(t - t_0) \Leftrightarrow X(\omega) e^{-j\omega t_0}$$ **FT**

$$x(t - t_0) \Leftrightarrow X(s) e^{-s t_0}$$ **LT**

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(\omega - \omega_0)$$ **FT**

$$x(t)e^{s_0 t} \Leftrightarrow G(s - s_0)$$ **LT**

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(\frac{\omega}{a})$$ **FT**

$$x(at) \Leftrightarrow \frac{1}{a} X(\frac{s}{a}), \qquad a > 0$$ **LT**

Convolution

$$x(t)*y(t) \Leftrightarrow X(\omega)Y(\omega) \qquad \textbf{\textit{FT}}$$

$$x(t)*y(t) \Leftrightarrow X(s)Y(s) \qquad \textbf{\textit{LT}}$$

Similarly

$$y(t) = x(t)*h(t)$$

$$\Rightarrow Y(s) = X(s)H(s)$$

Time Differentiation – once

$$L[x(t)] = X(s) = \int_0^{\infty} x(t)e^{-st}dt$$

We know that Integration by parts rule

$$\int u dv = uv - \int v du$$

If we let

$$u = x(t) \qquad \textbf{and} \qquad dv = e^{-st}dt$$

$$\Rightarrow du = \frac{d}{dt}(x(t))dt \qquad \Rightarrow v = -\frac{e^{-st}}{s}$$

$$\Rightarrow \int_0^{\infty} x(t)e^{-st}dt = x(t)(-\frac{e^{-st}}{s})\Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-st}}{s}\frac{d}{dt}(x(t))dt$$

$$\Rightarrow \int_{-\infty}^{\infty} x(t)e^{-st}dt = x(t)(-\frac{e^{-st}}{s})\Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-st}}{s}\frac{d}{dt}(x(t))dt$$

$$X(s) = \frac{x(0)}{s} + \frac{1}{s}\int_0^{\infty} \frac{d}{dt}(x(t))e^{-st}dt$$

$$X(s) = \frac{x(0)}{s} + \frac{1}{s}L[\frac{d}{dt}(x(t))]$$

$$sX(s) = x(0) + L[\frac{d}{dt}(x(t))]$$

$$\Rightarrow L[\frac{d}{dt}(x(t))] = sX(s) - x(0)$$

Time Differentiation – twice

$$\frac{d^2}{dt^2}x(t) = \frac{d}{dt}\left(\frac{d}{dt}x(t)\right)$$

we already know

$$L\left[\frac{d}{dt}(x(t))\right] = sX(s) - x(0)$$

Therefore

$$L\left[\frac{d^2}{dt^2}x(t)\right] = sL\left\{\frac{d}{dt}x(t)\right\} - \left.\frac{d}{dt}x(t)\right|_{t=0}$$

$$L\left[\frac{d^2}{dt^2}x(t)\right] = s[sX(s) - x(0)] - \left.\frac{d}{dt}x(t)\right|_{t=0}$$

$$L\left[\frac{d^2}{dt^2}x(t)\right] = s^2X(s) - sx(0) - \left.\frac{d}{dt}x(t)\right|_{t=0}$$

Initial Value Theorem

$$x(0) = \lim_{s\to\infty} sX(s)$$

We know

$$L[\frac{d}{dt}(x(t))] = sX(s) - x(0)$$

Also

$$L[\frac{d}{dt}x(t)] = \int_0^\infty \frac{d}{dt}x(t)e^{-st}dt$$

$$\Rightarrow \lim_{s\to\infty}\int_0^\infty \frac{d}{dt}x(t)e^{-st}dt = \lim_{s\to\infty} sX(s) - x(0)$$

$$0 = \lim_{s\to\infty} sX(s) - x(0) \Rightarrow \lim_{s\to\infty} sX(s) = x(0)$$

Final Value Theorem

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

We know

$$L[\frac{d}{dt}(x(t))] = sX(s) - x(0)$$

Also

$$L[\frac{d}{dt}x(t)] = \int_0^\infty \frac{d}{dt}x(t)e^{-st}dt$$

$$\Rightarrow \lim_{s \to 0} \int_0^\infty \frac{d}{dt}x(t)e^{-st}dt = \lim_{s \to 0} sX(s) - x(0)$$

$$\int_0^\infty \lim_{s \to 0} \left\{ \frac{d}{dt}x(t)e^{-st} \right\} dt = \lim_{s \to 0} sX(s) - x(0)$$

$$\int_0^\infty \lim_{s\to 0}\left\{\frac{d}{dt}x(t)e^{-st}\right\}dt = \lim_{s\to 0} sX(s) - x(0)$$

$$\int_0^\infty \frac{d}{dt}x(t)dt = \lim_{s\to 0} sX(s) - x(0)$$

$$x(t)\Big|_0^\infty = \lim_{s\to 0} sX(s) - x(0)$$

$$\lim_{t\to\infty} x(t) - x(0) = \lim_{s\to 0} sX(s) - x(0)$$

$$\Rightarrow \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Example (1)

$$H(s) = \frac{s+3}{s^2+5s+6}$$

What is the final value of the impulse response?

What is the final value of the unit step response?

What is the impulse response in time domain?

What is the unit step response in time domain?

z Transform

Discrete Time Fourier Transform

$$X(F)=\sum_{n=-\infty}^{n=\infty} x[n]e^{-j2\pi Fn}$$

OR

$$X(\Omega)=\sum_{n=-\infty}^{n=\infty} x[n]e^{-j\Omega n}$$

Discrete Time Inverse Fourier Transform

$$x[n]=\int_{1} X(F)e^{j2\pi Fn}dF$$

OR

$$x[n]=\frac{1}{2\pi}\int_{2\pi} X(F)e^{j\Omega n}d\Omega$$

Where $2\pi F=\Omega$

$$X(S)=\sum_{n=-\infty}^{n=\infty} x[n]e^{-(\Sigma+j\Omega)n}=\sum_{n=-\infty}^{n=\infty} x[n]e^{-Sn}$$

$$X(S)=\sum_{n=-\infty}^{n=\infty}x[n]e^{-(\Sigma+j\Omega)n}=\sum_{n=-\infty}^{n=\infty}x[n]e^{-Sn}$$

$$X(S)=\sum_{n=-\infty}^{n=\infty}(x[n]e^{-\Sigma n})e^{-j\Omega n}$$

$$=DTFT(x[n]e^{-\Sigma n})$$

Also

$$X(S)=\sum_{n=-\infty}^{n=\infty}x[n]e^{-(\Sigma+j\Omega)n}=\sum_{n=-\infty}^{n=\infty}x[n]e^{-Sn}$$

$$\Rightarrow X(z)=\sum_{n=-\infty}^{n=\infty}x[n]z^{-n}$$ ***Where*** $e^{S}=z$

z Transform of u[n]

First DTFT

$$X(\Omega) = \sum_{n=-\infty}^{n=\infty} x[n] e^{-j\Omega n}$$

$$\Rightarrow X(\Omega) = \sum_{n=-\infty}^{n=\infty} u[n] e^{-j\Omega n} = \sum_{n=0}^{n=\infty} e^{-j\Omega n}$$

Does not converge

Then z-transform

$$X(z) = \sum_{n=-\infty}^{n=\infty} u[n] z^{-n} = \sum_{n=0}^{n=\infty} z^{-n}$$

$$= \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

ROC

$$|z| > 1$$

Unilateral z Transform

For similar reasoning as in Laplace Transform, unilateral z-transform is used

$$X(z) = \sum_{n=0}^{n=\infty} x[n] z^{-n}$$

Applies to only causal signals