28 Frieze Groups and Crystallographic Groups

Symmetry and group theory have an uncanny way of directing physicists to the right path.

Mario Livio, *The Equation That Could Not Be Solved*

Group theory is the bread and butter of crystallography.

Mario Livio, *The Equation That Could Not Be Solved*

The Frieze Groups

In this chapter, we discuss an interesting collection of infinite symmetry groups that arise from periodic designs in a plane. There are two types of such groups. The *discrete frieze groups* are the plane symmetry groups of patterns whose subgroups of translations are isomorphic to Z. These kinds of designs are the ones used for decorative strips and for patterns on jewelry, as illustrated in Figure 28.1. In mathematics, familiar examples include the graphs of $y = \sin x$, $y = \tan x$, $y = |\sin x|$, and $|y| = \sin x$. After we analyze the discrete frieze groups, we examine the discrete symmetry groups of plane patterns whose subgroups of translations are isomorphic to $Z \oplus Z$.

In previous chapters, it was our custom to view two isomorphic groups as the same group, since we could not distinguish between them algebraically. In the case of the frieze groups, we will soon see that, although some of them are isomorphic as groups (that is, algebraically the same), geometrically they are quite different. To emphasize this difference, we will treat them separately. In each of the following cases, the given pattern extends infinitely far in both directions. A proof that there are exactly seven types of frieze patterns is given in the appendix to [6].

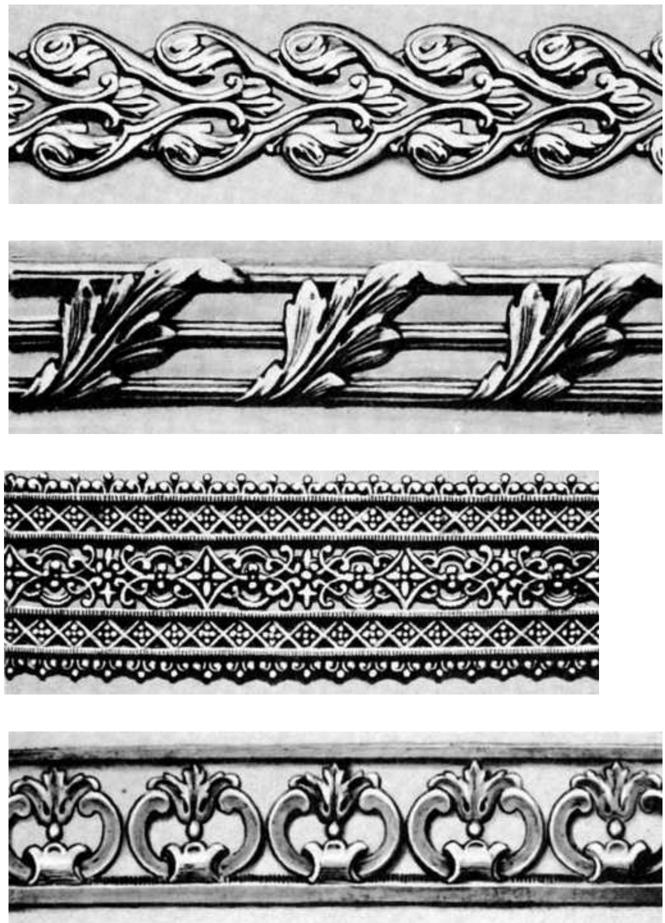


Figure 28.1 Frieze patterns

The symmetry group of pattern I (Figure 28.2) consists of translations only. Letting x denote a translation to the right of one unit (that is, the distance between two consecutive R's), we may write the symmetry group of pattern I as

$$F_1 = \{x^n \mid n \in Z\}.$$

R R R R

Figure 28.2 Pattern I

The group for pattern II (Figure 28.3), like that of pattern I, is infinitely cyclic. Letting x denote a glide-reflection, we may write the symmetry group of pattern II as

$$F_2 = \{x^n \mid n \in Z\}.$$

Figure 28.3 Pattern II

Notice that the translation subgroup of pattern II is just $\langle x^2 \rangle$.

The symmetry group for pattern III (Figure 28.4) is generated by a translation x and a reflection y across the dashed vertical line. (There are infinitely many axes of reflective symmetry, including those midway between consecutive pairs of opposite-facing R's. Any one will do.) The entire group (the operation is function composition) is

$$F_3 = \{x^n y^m \mid n \in Z, m = 0 \text{ or } 1\}.$$

Figure 28.4 Pattern III

Note that the two elements xy and y have order 2, they generate F_3, and their product $(xy)y = x$ has infinite order. Thus, by Theorem 26.5, F_3 is the infinite dihedral group. A geometric fact about pattern III worth mentioning is that the distance between consecutive pairs of vertical reflection axes is half the length of the smallest translation vector.

In pattern IV (Figure 28.5), the symmetry group F_4 is generated by a translation x and a rotation y of 180° about a point p midway between consecutive R's (such a rotation is often called a *half-turn*). This group, like F_3, is also infinite dihedral. (Another rotation point lies between a top and bottom R. As in pattern III, the distance between consecutive points of rotational symmetry is half the length of the smallest translation vector.) Therefore,

$$F_4 = \{x^n y^m \mid n \in Z, m = 0 \text{ or } m = 1\}.$$

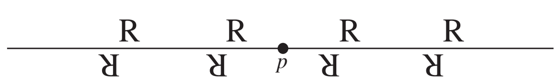


Figure 28.5 Pattern IV

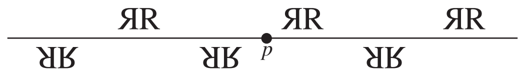


Figure 28.6 Pattern V

The symmetry group F_5 for pattern V (Figure 28.6) is yet another infinite dihedral group generated by a glide-reflection x and a rotation y of 180° about the point p. Notice that pattern V has vertical reflection symmetry xy. The rotation points are midway between the vertical reflection axes. Thus,

$$F_5 = \{x^n y^m \mid n \in Z, m = 0 \text{ or } m = 1\}.$$

The symmetry group F_6 for pattern VI (Figure 28.7) is generated by a translation x and a horizontal reflection y. The group is

$$F_6 = \{x^n y^m \mid n \in Z, m = 0 \text{ or } m = 1\}.$$

Note that, since x and y commute, F_6 is not infinite dihedral. In fact, F_6 is isomorphic to $Z \oplus Z_2$. Pattern VI is invariant under a glide-reflection also, but in this case the glide-reflection is called *trivial*, since the axis of the glide-reflection is also an axis of reflection. (Conversely, a glide-reflection is *nontrivial* if its glide-axis is not an axis of reflective symmetry for the pattern.)

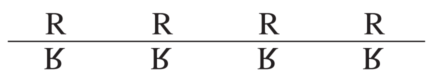


Figure 28.7 Pattern VI

The symmetry group F_7 of pattern VII (Figure 28.8) is generated by a translation x, a horizontal reflection y, and a vertical reflection z. It is isomorphic to the direct product of the infinite dihedral group and Z_2. The product of y and z is a 180° rotation. Therefore,

$$F_7 = \{x^n y^m z^k \mid n \in Z, m = 0 \text{ or } m = 1, k = 0 \text{ or } k = 1\}.$$

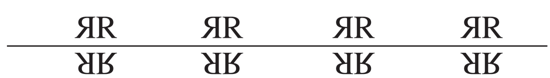


Figure 28.8 Pattern VII

The preceding discussion is summarized in Figure 28.9. Figure 28.10 provides an identification algorithm for the frieze patterns.

In describing the seven frieze groups, we have not explicitly said how multiplication is done algebraically. However, each group element corresponds to some isometry, so multiplication is the same as function

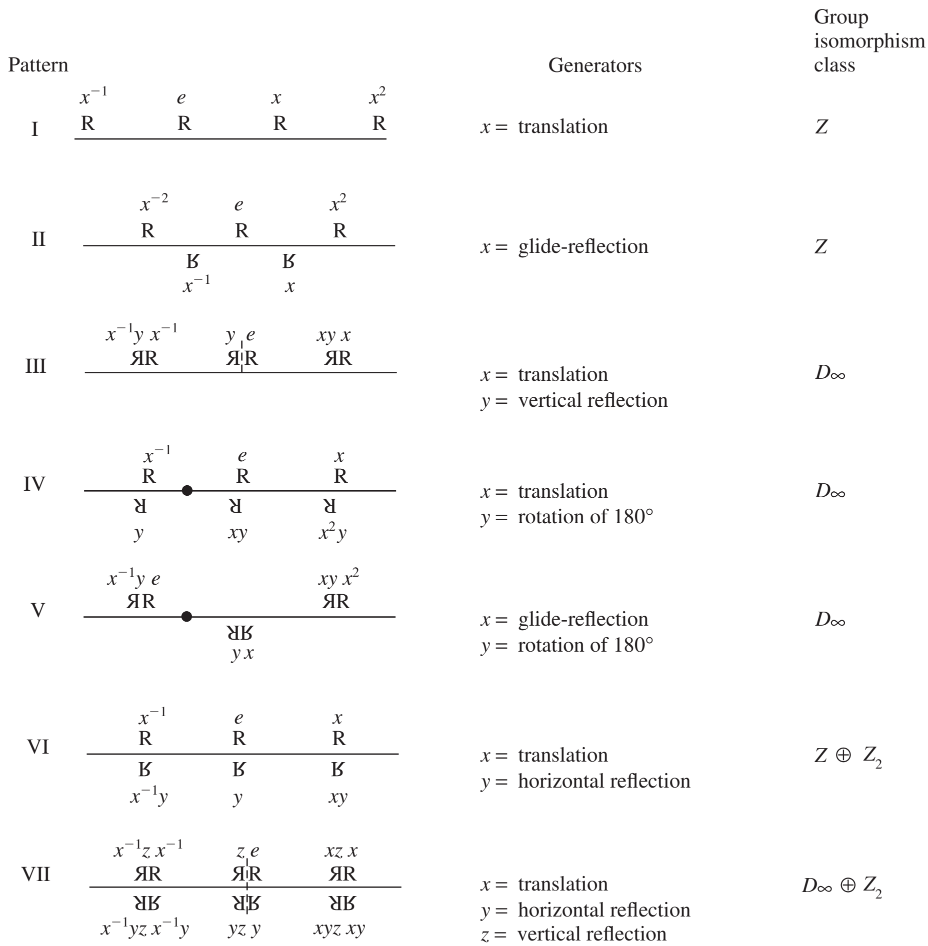


Figure 28.9 The seven frieze patterns and their groups of symmetries

composition. Thus, we can always use the geometry to determine the product of any particular string of elements.

For example, we know that every element of F_7 can be written in the form $x^n y^m z^k$. So, just for fun, let's determine the appropriate values for n, m, and k for the element $g = x^{-1}yzxz$. We may do this simply by looking at the effect that g has on pattern VII. For convenience, we will pick out a particular R in the pattern and trace the action of g one step at a time. To distinguish this R, we enclose it in a shaded box. Also, we draw the axis of the vertical reflection z as a dashed line segment. See Figure 28.11.

Now, comparing the starting position of the shaded R with its final position, we see that $x^{-1}yzxz = x^{-2}y$. Exercise 7 suggests how one may arrive at the same result through purely algebraic manipulation.

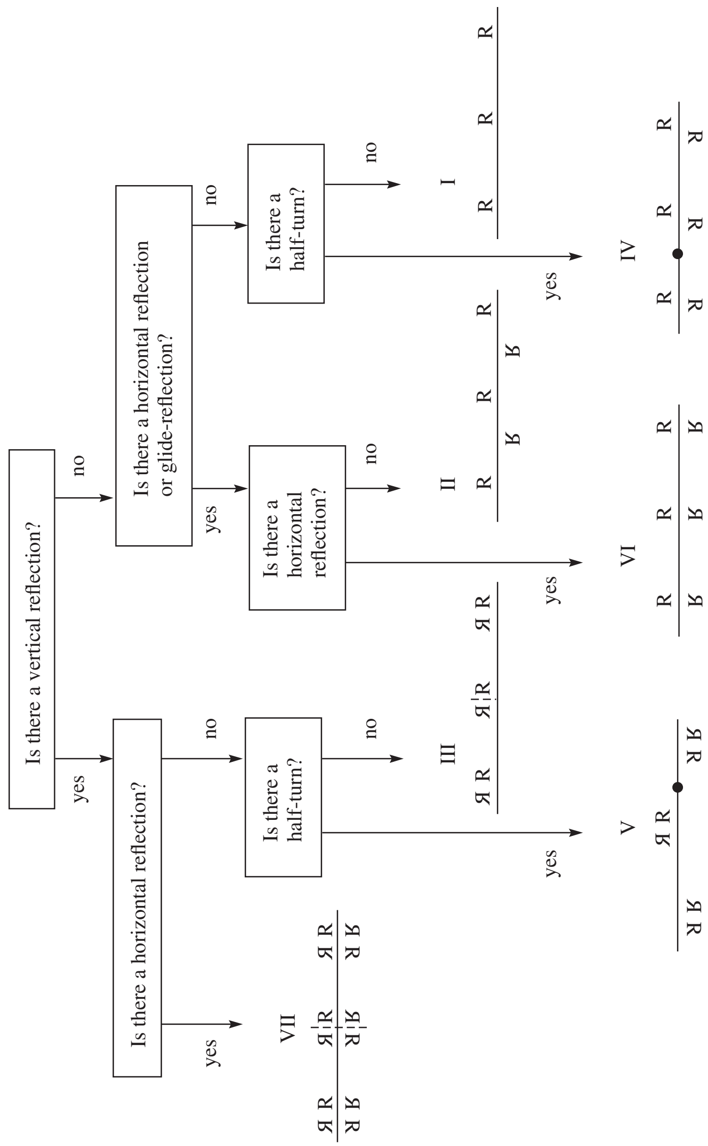


Figure 28.10 Recognition chart for frieze patterns. Adapted from [6, p. 83].*

*Adaptation of figure from Dorothy K. Washburn and Donald W. Crowe. *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*. Copyright © 1988 by the University of Washington Press. Used by permission.

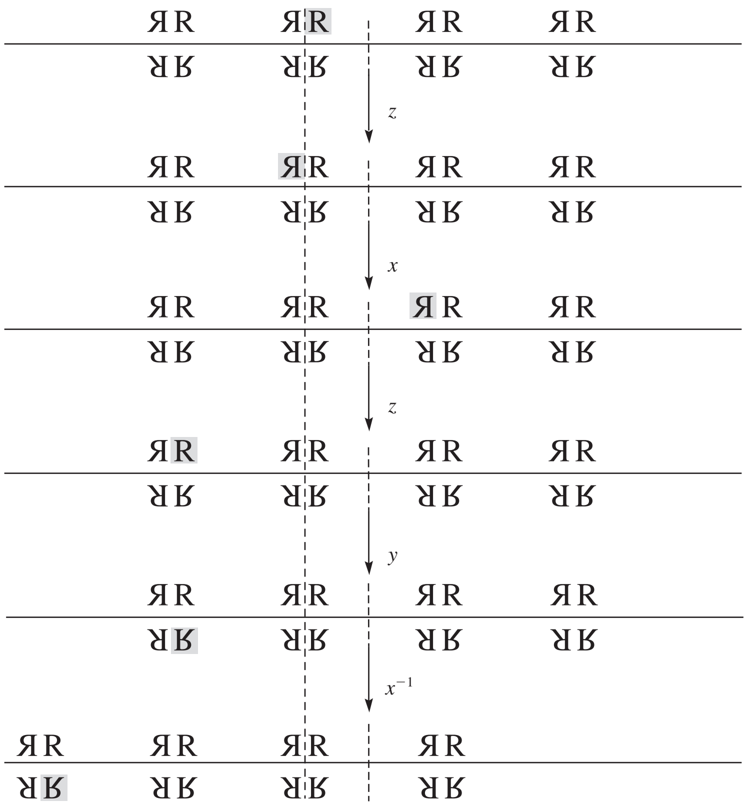


Figure 28.11

The Crystallographic Groups

The seven frieze groups catalog all symmetry groups that leave a design invariant under all multiples of just one translation. However, there are 17 additional kinds of discrete plane symmetry groups that arise from infinitely repeating designs in a plane. These groups are the symmetry groups of plane patterns whose subgroups of translations are isomorphic to $Z \oplus Z$. Consequently, the patterns are invariant under linear combinations of two linearly independent translations. These 17 groups were first studied by 19th-century crystallographers and are often called the *plane crystallographic groups.* Another term occasionally used for these groups is *wallpaper groups.*

Our approach to the crystallographic groups will be geometric. It is adapted from the excellent article by Schattschneider [5] and the monograph by Crowe [1]. Our goal is to enable the reader to determine which of the 17 plane symmetry groups corresponds to a given periodic pattern. We begin with some examples.

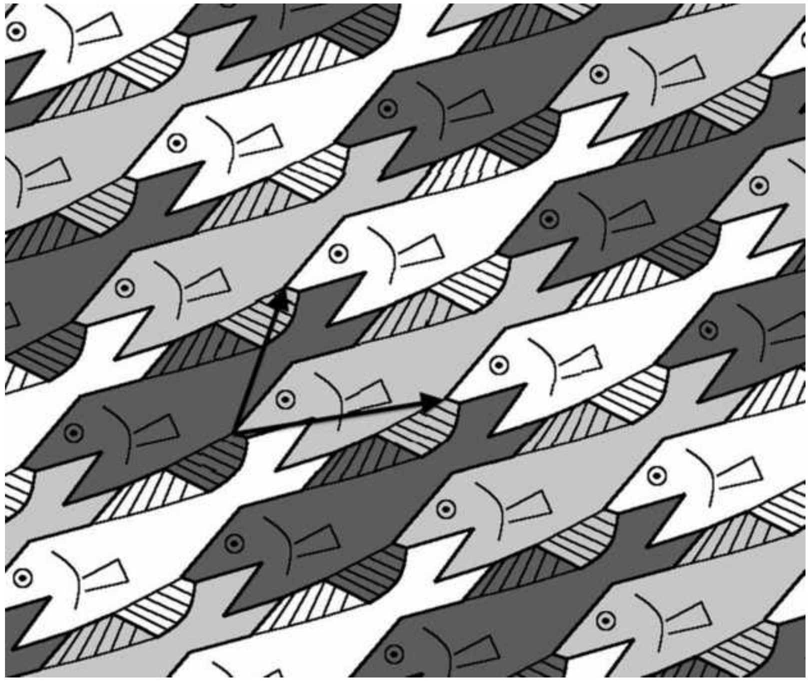


Figure 28.12 Fish3 by Makoto Nakamura, adapted by Kevin Lee. Design with symmetry group *p*1 (disregarding shading). The inserted arrows are translation vectors.

The simplest of the 17 crystallographic groups contains translations only. In Figure 28.12, we present an illustration of a representative pattern for this group (imagine the pattern repeated to fill the entire plane). The crystallographic notation for it is $p1$. (This notation is explained in [5].)

The symmetry group of the pattern in Figure 28.13 contains translations and glide-reflections. This group has no (nonzero) rotational or reflective symmetry. The crystallographic notation for it is pg.

Figure 28.14 has translational symmetry and threefold rotational symmetry (that is, the figure can be rotated 120° about certain points and be brought into coincidence with itself). The notation for this group is $p3$.

Representative patterns for all 17 plane crystallographic groups, together with their notations, are given in Figures 28.15 and 28.16. Figure 28.17 uses a triangle motif to illustrate the 17 classes of symmetry patterns.

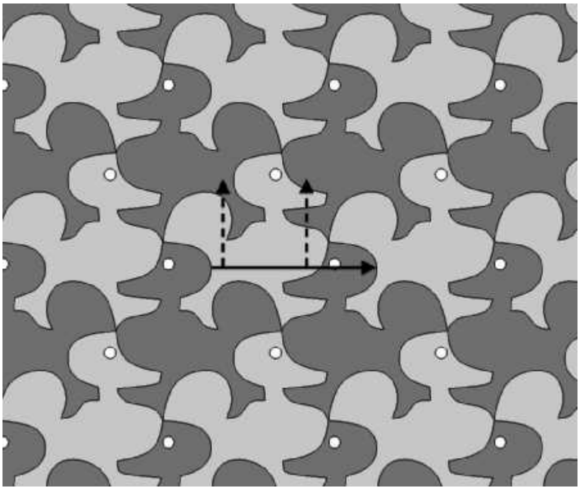


Figure 28.13 Fish5 by Makoto Nakamura, adapted by Kevin Lee. Design with symmetry group *pg* (disregarding shading). The solid arrow is the translation vector. The dashed arrows are the glide-reflection vectors.

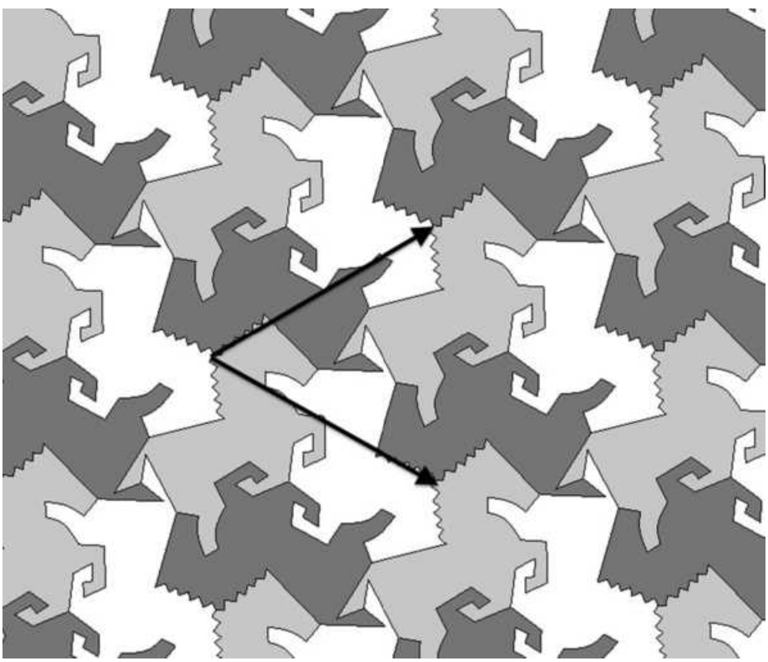


Figure 28.14 Horses1 by Makoto Nakamura, adapted by Kevin Lee. Design with symmetry group *p3* (disregarding shading). The inserted arrows are translation vectors.

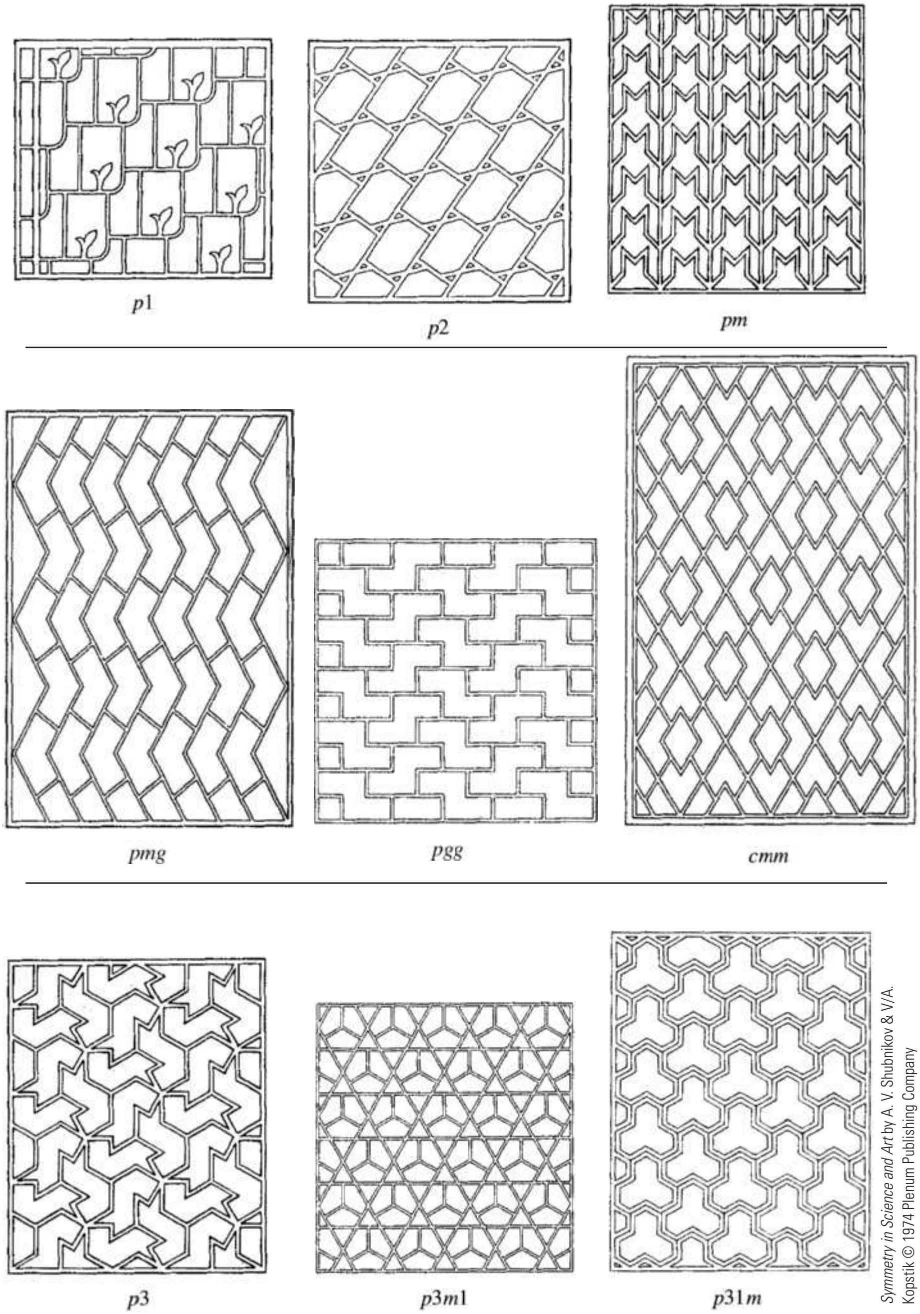


Figure 28.15 The plane symmetry groups

All designs in Figures 28.15 and 28.16 except *pm*, *p*3, and *pg* are found in [2]. The designs for *p*3 and *pg* are based on elements of Chinese lattice designs found in [2]; the design for *pm* is based on a weaving pattern from Hawaii, found in [3].

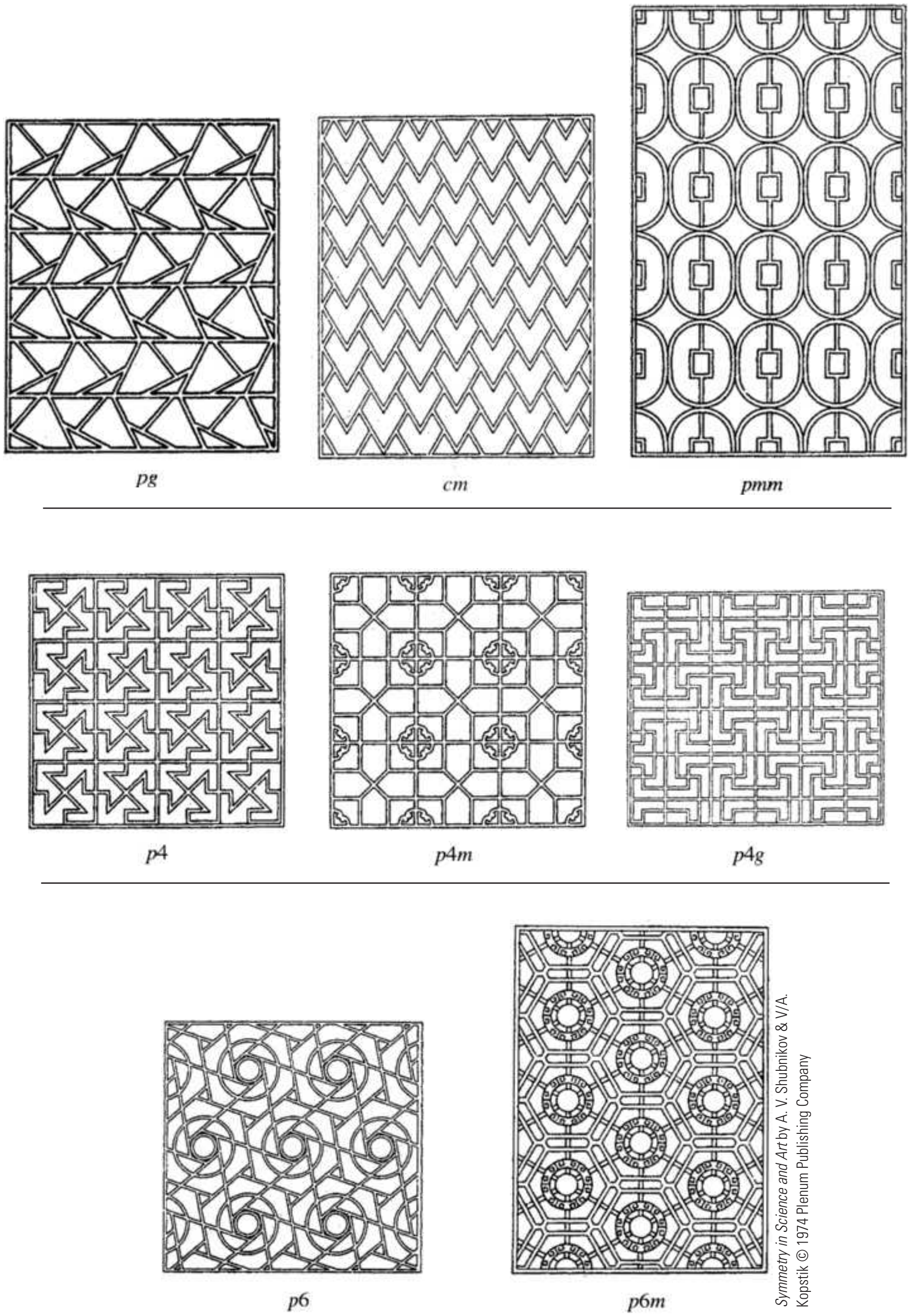


Figure 28.16 The plane symmetry groups

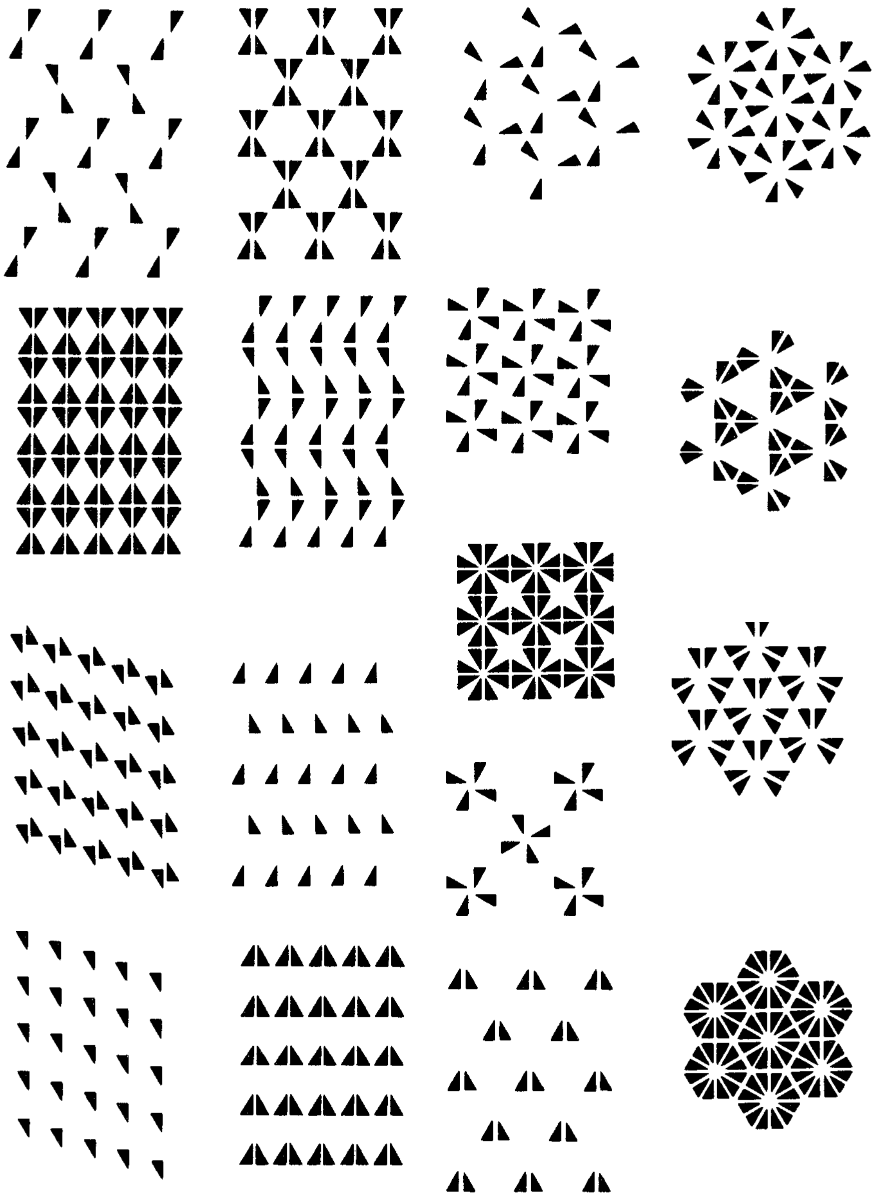


Figure 28.17 The 17 plane periodic patterns formed using a triangle motif

Identification of Plane Periodic Patterns

To decide which of the 17 classes any particular plane periodic pattern belongs to, we may use the flowchart presented in Figure 28.18. This is done by determining the rotational symmetry and whether or not the pattern has reflection symmetry or nontrivial glide-reflection symmetry. These three pieces of information will narrow the list of candidates to at most two. The final test, if necessary, is to determine the locations of the centers of rotation.

For example, consider the two patterns in Figure 28.19 generated in a hockey stick motif. Both patterns have a smallest positive rotational symmetry of 120°; both have reflectional and nontrivial glide-reflectional symmetry. Now, according to Figure 28.18, these patterns must be of type $p3m1$ or $p31m$. But notice that the pattern on the left has all its threefold centers of rotation on the reflection axis, whereas in the pattern on the right the points where the three blades meet are not on a reflection axis. Thus, the left pattern is $p3m1$, and the right pattern is $p31m$.

Table 28.1 (reproduced from [5, p. 443]) can also be used to determine the type of periodic pattern and contains two other features that are often useful. A *lattice of points* of a pattern is a set of images of any particular point acted on by the translation group of the pattern. A *lattice unit* of a pattern whose translation subgroup is generated by u and v is a parallelogram formed by a point of the pattern and its image under u, v, and $u + v$. The possible lattices for periodic patterns in a plane, together with lattice units, are shown in Figure 28.20. A *generating region* (or *fundamental region*) of a periodic pattern is the smallest portion of the lattice unit whose images under the full symmetry group of the pattern cover the plane. Examples of generating regions for the patterns represented in Figures 28.12, 28.13, and 28.14 are given in Figure 28.21. In Figure 28.21, the portion of the lattice unit with vertical bars is the generating region. The only symmetry pattern in which the lattice unit and the generating region coincide is the $p1$ pattern illustrated in Figure 28.12. Table 28.1 tells what proportion of the lattice unit constitutes the generating region of each plane periodic pattern.

Notice that Table 28.1 reveals that the only possible n-fold rotational symmetries occur when $n = 1, 2, 3, 4$, and 6. This fact is commonly called the *crystallographic restriction.* The first proof of this was given by the Englishman W. Barlow over 100 years ago. The information in Table 28.1 can also be used in reverse to create patterns with a specific symmetry group. The patterns in Figure 28.19 were made in this way.

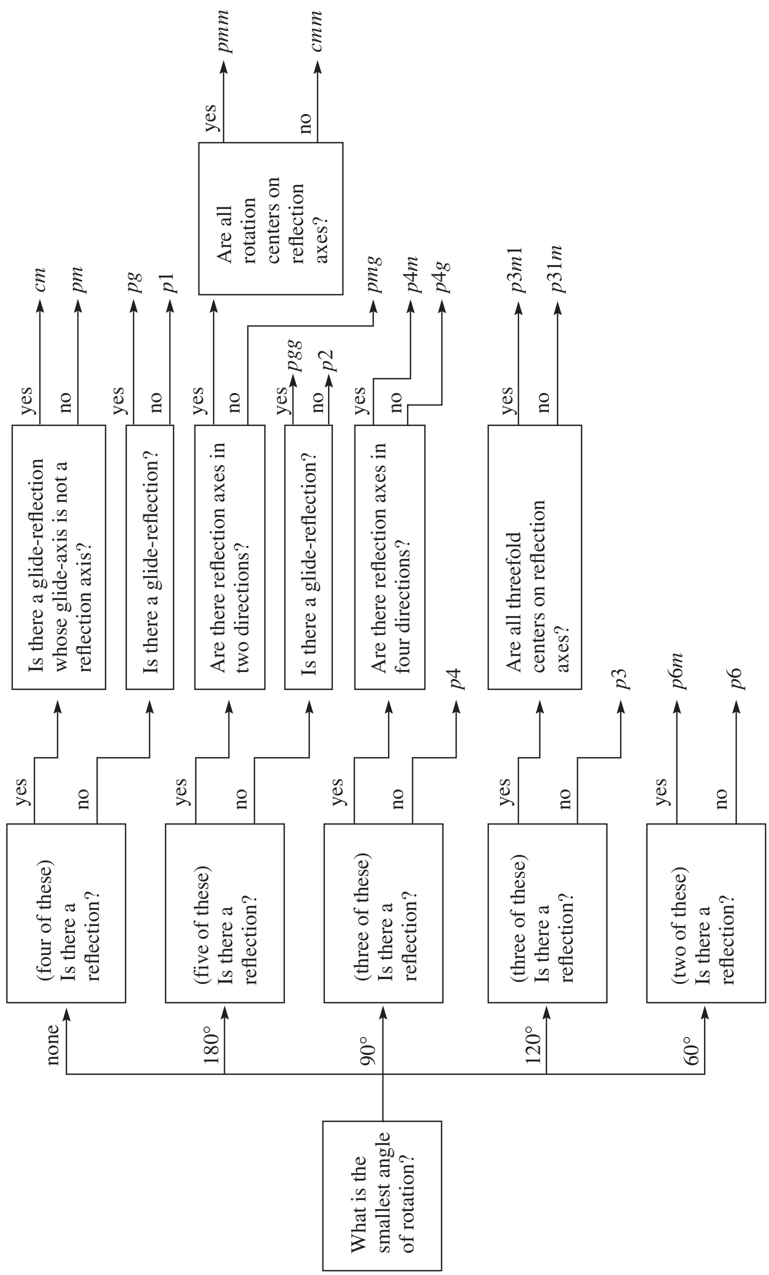


Figure 28.18 Identification flowchart for symmetries of plane periodic patterns

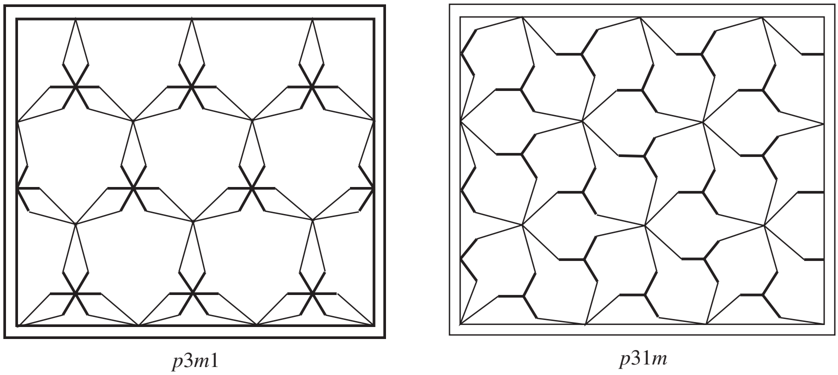


Figure 28.19 Patterns generated in a hockey stick motif

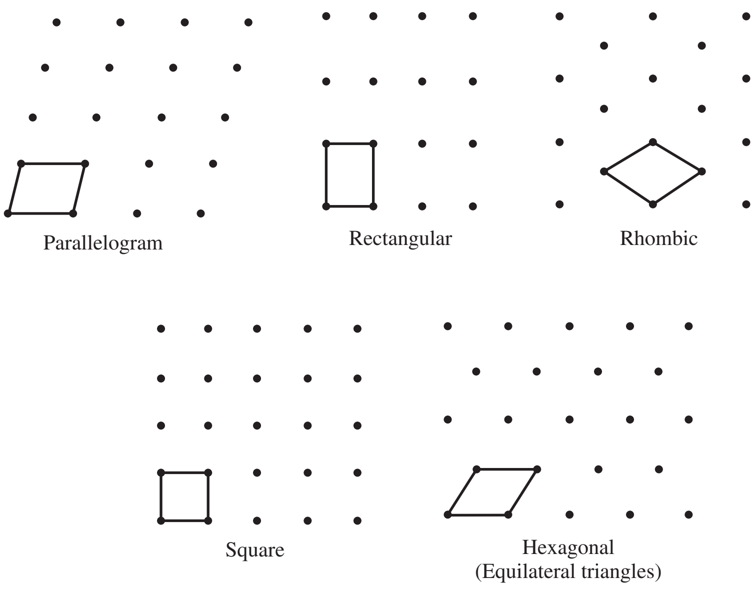


Figure 28.20 Possible lattices for plane periodic patterns

In sharp contrast to the situation for finite symmetry groups, the transition from two-dimensional crystallographic groups to three-dimensional crystallographic groups introduces a great many more possibilities, since the motif is repeated indefinitely by three independent translations. Indeed, there are 230 three-dimensional crystallographic groups (often called *space groups*). These were independently determined by Fedorov, Schönflies, and Barlow in the 1890s. David Hilbert, one of the leading mathematicians of the 20th century, focused attention on the crystallographic groups in his

Table 28.1 Identification Chart for Plane Periodic Patterns[a]

Type	Lattice	Highest Order of Rotation	Reflections	Nontrivial Glide-Reflections	Generating Region	Helpful Distinguishing Properties
*p*1	Parallelogram	1	No	No	1 unit	
*p*2	Parallelogram	2	No	No	$\frac{1}{2}$ unit	
pm	Rectangular	1	Yes	No	$\frac{1}{2}$ unit	
pg	Rectangular	1	No	Yes	$\frac{1}{2}$ unit	
cm	Rhombic	1	Yes	Yes	$\frac{1}{2}$ unit	
pmm	Rectangular	2	Yes	No	$\frac{1}{4}$ unit	
pmg	Rectangular	2	Yes	Yes	$\frac{1}{4}$ unit	Parallel reflection axes
pgg	Rectangular	2	No	Yes	$\frac{1}{4}$ unit	
cmm	Rhombic	2	Yes	Yes	$\frac{1}{4}$ unit	Perpendicular reflection axes
*p*4	Square	4	No	No	$\frac{1}{4}$ unit	
*p*4*m*	Square	4	Yes	Yes	$\frac{1}{8}$ unit	Fourfold centers on reflection axes
*p*4*g*	Square	4	Yes	Yes	$\frac{1}{8}$ unit	Fourfold centers not on reflection axes
*p*3	Hexagonal	3	No	No	$\frac{1}{3}$ unit	
*p*3*m*1	Hexagonal	3	Yes	Yes	$\frac{1}{6}$ unit	All threefold centers on reflection axes
*p*31*m*	Hexagonal	3	Yes	Yes	$\frac{1}{6}$ unit	Not all threefold centers on reflection axes
*p*6	Hexagonal	6	No	No	$\frac{1}{6}$ unit	
*p*6*m*	Hexagonal	6	Yes	Yes	$\frac{1}{12}$ unit	

[a]A rotation through an angle of 360°/n is said to have order n. A glide-reflection is nontrivial if its glide-axis is not an axis of reflective symmetry for the pattern.

famous lecture in 1900 at the International Congress of Mathematicians in Paris. One of 23 problems he posed was whether or not the number of crystallographic groups in n dimensions is always finite. This was answered affirmatively by L. Bieberbach in 1910. We mention in passing that in four dimensions, there are 4783 symmetry groups for infinitely repeating patterns.

As one might expect, the crystallographic groups are fundamentally important in the study of crystals. In fact, a crystal is defined as a rigid body in which the component particles are arranged in a pattern that repeats in three directions (the repetition is caused by the chemical

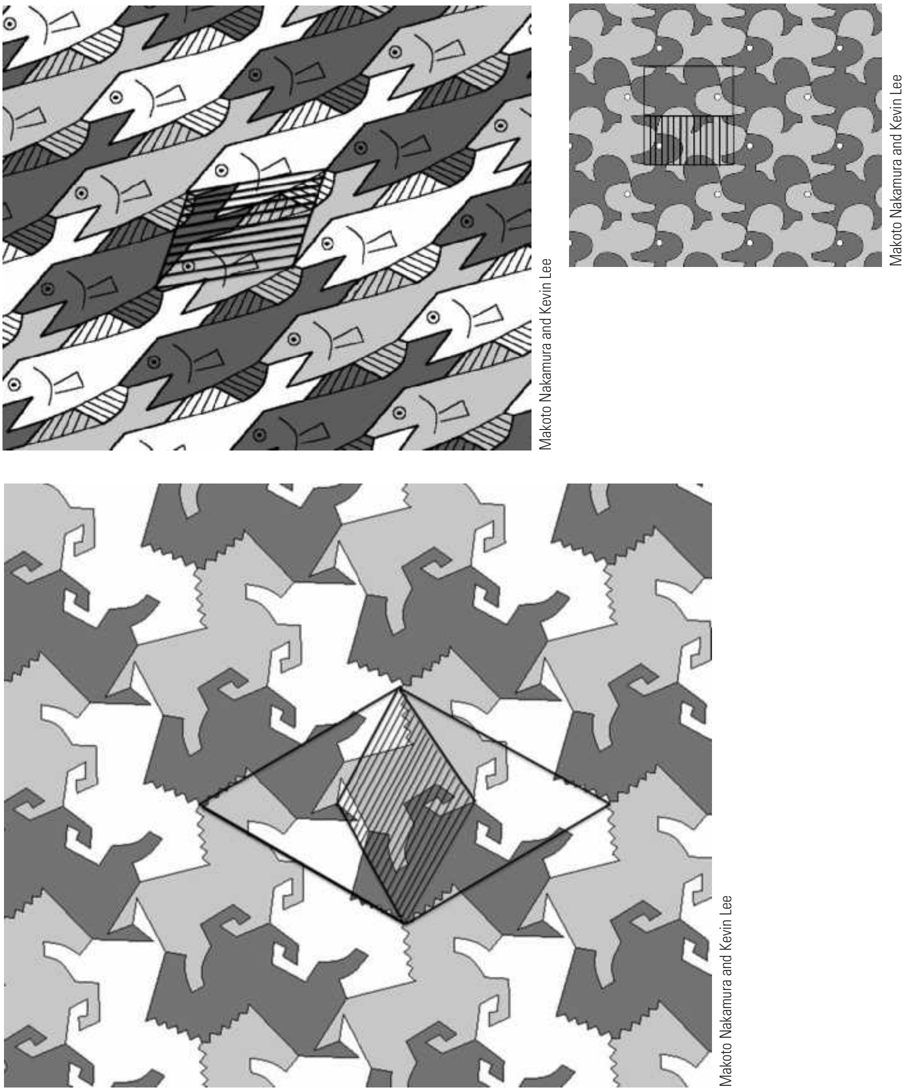


Figure 28.21 A lattice unit and generating region for the patterns in Figures 28.12, 28.13, and 28.14. Generating regions are shaded with bars.

bonding). A grain of salt and a grain of sugar are two examples of common crystals. In crystalline materials, the motif units are atoms, ions, ionic groups, clusters of ions, or molecules.

Perhaps it is fitting to conclude this chapter by recounting two episodes in the history of science in which an understanding of symmetry groups was crucial to a great discovery. In 1912, Max von Laue, a young German physicist, hypothesized that a narrow beam of x-rays directed onto a crystal with a photographic film behind it would be deflected (the technical term is "diffracted") by the unit cell (made up of atoms or ions) and would show up on the film as spots. (See Figure 1.3.) Shortly thereafter, two British scientists, Sir William Henry Bragg and his 22-year-old son William Lawrence Bragg, who was a student, noted that von Laue's diffraction spots, together with the known information about crystallographic space groups, could be used to calculate the shape of the internal array of atoms. This discovery marked the birth of modern mineralogy. From the first crystal structures deduced by the Braggs to the present, x-ray diffraction has been the means by which the internal structures of crystals are determined. Von Laue was awarded the Nobel Prize in physics in 1914, and the Braggs were jointly awarded the Nobel Prize in physics in 1915.

Our second episode took place in the early 1950s, when a handful of scientists were attempting to learn the structure of the DNA molecule—the basic genetic material. One of these was a graduate student named Francis Crick; another was an x-ray crystallographer, Rosalind Franklin. On one occasion, Crick was shown one of Franklin's research reports and an x-ray diffraction photograph of DNA. At this point, we let Horace Judson [4, pp. 165–166], our source, continue the story.

> Crick saw in Franklin's words and numbers something just as important, indeed eventually just as visualizable. There was drama, too: Crick's insight began with an extraordinary coincidence. Crystallographers distinguish 230 different space groups, of which the face-centered monoclinic cell with its curious properties of symmetry is only one—though in biological substances a fairly common one. The principal experimental subject of Crick's dissertation, however, was the x-ray diffraction of the crystals of a protein that was of exactly the same space group as DNA. So Crick saw at once the symmetry that neither Franklin nor Wilkins had comprehended, that Perutz, for that matter, hadn't noticed, that had escaped the theoretical crystallographer in Wilkins' lab, Alexander Stokes—namely, that the molecule of DNA, rotated a half turn, came back to congruence with itself. The structure was dyadic, one half matching the other half in reverse.

This was a crucial fact. Shortly thereafter, James Watson and Crick built an accurate model of DNA. In 1962, Watson, Crick, and Maurice Wilkins received the Nobel Prize in medicine and physiology for their discovery. The opinion has been expressed that, had Franklin correctly recognized the symmetry of the DNA molecule, she might have been the one to unravel the mystery and receive the Nobel Prize [4, p. 172].

Exercises

You can see a lot just by looking.

Yogi Berra

1. Show that the frieze group F_6 is isomorphic to $Z \oplus Z_2$.
2. How many nonisomorphic frieze groups are there?
3. In the frieze group F_7, write x^2yzxz in the form $x^ny^mz^k$.
4. In the frieze group F_7, write $x^{-3}zxyz$ in the form $x^ny^mz^k$.
5. In the frieze group F_7, show that $yz = zy$ and $xy = yx$.
6. In the frieze group F_7, show that $zxz = x^{-1}$.
7. Use the results of Exercises 5 and 6 to do Exercises 3 and 4 through symbol manipulation only (that is, without referring to the pattern). (This exercise is referred to in this chapter.)
8. Prove that in F_7 the cyclic subgroup generated by x is a normal subgroup.
9. Quote a previous result that tells why the subgroups $\langle x, y\rangle$ and $\langle x, z\rangle$ must be normal in F_7.
10. Look up the word *frieze* in an ordinary dictionary. Explain why the frieze groups are appropriately named.
11. Determine which of the seven frieze groups is the symmetry group of each of the following patterns.

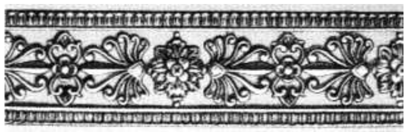
a.

b.

c.

d.

e.

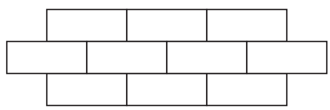


f.

12. Determine the frieze group corresponding to each of the following patterns.

a. $y = \sin x$
b. $y = |\sin x|$
c. $|y| = \sin x$
d. $y = \tan x$
e. $y = \csc x$

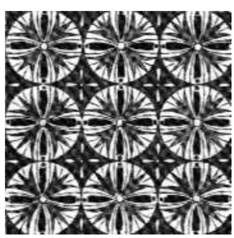
13. Determine the symmetry group of the tessellation of the plane exemplified by the brickwork shown.



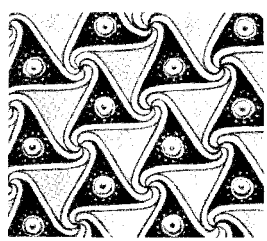
14. Determine the plane symmetry group for each of the patterns in Figure 28.17.

15. Determine which of the 17 crystallographic groups is the symmetry group of each of the following patterns.

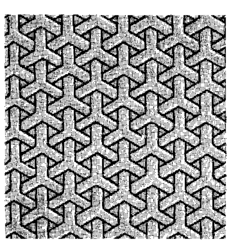
a.



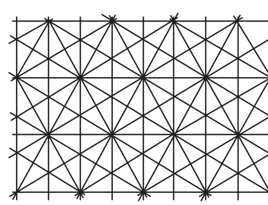
b.



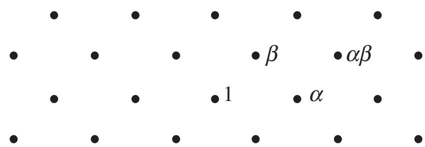
c.



d.



16. In the following figure, there is a point labeled 1. Let α be the translation of the plane that carries the point labeled 1 to the point labeled α, and let β be the translation of the plane that carries the point labeled 1 to the point labeled β. The image of 1 under the composition of α and β is labeled $\alpha\beta$. In the corresponding fashion, label the remaining points in the figure in the form $\alpha^i\beta^j$.



17. The patterns made by automobile tire treads in the snow are frieze patterns. An extensive study of automobile tires revealed that only five of the seven frieze patterns occur. Speculate on which two patterns do not occur and give a possible reason why they do not.

18. Locate a nontrivial glide-reflection axis of symmetry in the *cm* pattern in Figure 28.16.

19. Determine which of the frieze groups is the symmetry group of each of the following patterns.

a. · · · D D D D · · ·
b. · · · V Λ V Λ · · ·
c. · · · L L L L · · ·
d. · · · V V V V · · ·
e. · · · N N N N · · ·
f. · · · H H H H · · ·
g. · · · L ⅂ L ⅂ · · ·

20. Locate a nontrivial glide-reflection axis of symmetry in the pattern third from the left in the bottom row in Figure 28.17.

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4. Horace Freeland Judson, *The Eighth Day of Creation,* New York: Simon and Schuster, 1979.

5. D. Schattschneider, "The Plane Symmetry Groups: Their Recognition and Notation," *The American Mathematical Monthly* 85 (1978): 439–450.

6. D. K. Washburn and D. W. Crowe, *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis,* Seattle: University of Washington Press, 1988.

Suggested Readings

S. Garfunkel et al., *For All Practical Purposes,* 9th ed., New York: W. H. Freeman, 2012.

This book has a well-written, richly illustrated chapter on symmetry in art and nature.

W. G. Jackson, "Symmetry in Automobile Tires and the Left-Right Problem," *Journal of Chemical Education* 69 (1992): 624–626.

This article uses automobile tires as a tool for introducing and explaining the symmetry terms and concepts important in chemistry.

C. MacGillivray, *Fantasy and Symmetry—The Periodic Drawings of M. C. Escher*, New York: Harry N. Abrams, 1976.

This is a collection of Escher's periodic drawings together with a mathematical discussion of each one.

D. Schattschneider, *Visions of Symmetry,* New York: Harry Abrams, 2004.

A loving, lavish, encyclopedic book on the drawings of M. C. Escher.

H. von Baeyer, "Impossible Crystals," *Discover* 11 (2) (1990): 69–78.

This article tells how the discovery of nonperiodic tilings of the plane led to the discovery of quasicrystals. The x-ray diffraction patterns of quasicrystals exhibit fivefold symmetry—something that had been thought to be impossible.

CHAPTER 28 9TH EDITION
Solutions for the odd numbered exercises for the chapter on Frieze Groups and Crystallographic Groups

1. The mapping $\phi(x^m y^n) = (m, n)$ is an isomorphism. Onto is by observation. If $\phi(x^m y^n) = \phi(x^i y^j)$, then $(m, n) = (i, j)$ and therefore, $m = n$ and $i = j$. Also, $\phi((x^m y^n)(x^i y^j)) = \phi(x^{m+i} y^{n+j}) = (m + i, n + j) = (m, n)(i, j) = \phi(x^m y^n)\phi(x^i y^j)$.

3. Using Figure 28.9 we obtain $x^2yzxz = xy$.

5. Use Figure 28.9.

7. $x^2yzxz = x^2yx^{-1} = x^2x^{-1}y = xy$
   $x^{-3}zxzy = x^{-3}x^{-1}y = x^{-4}y$

9. A subgroup of index 2 is normal.

11. a. V, b. I, c. II, d. VI, e. VII, and f. III.

13. *cmm*

15. a. $p4m$, b. $p3$, c. $p31m$, and d. $p6m$

17. The principle purpose of tire tread design is to carry water away from the tire. Patterns I and III do not have horizontal reflective symmetry. Thus these designs would not carry water away equally on both halves of the tire.

19. a. VI, b. V, c. I, d. III, e. IV, f. VII, g. IV