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The Search for Finite Simple Groups

The eighty year quest for the building blocks of group theory reflects sporadic growth spurts whenever new basic techniques were discovered.

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At present, simple group theory is the most active and glamorous area of research in the theory of groups and it seems certain that this will remain the case for many years to come. Roughly speaking, the central problem is to find some reasonable description of all finite simple groups. A number of expository papers [36], [42], [45], [47], [49], [79] and books [21], [46], [67] detailing progress on this problem have been written for professional group theorists, but very little has appeared which is accessible to undergraduates. (Only Goldschmidt's proof of the Brauer-Suzuki-Wall theorem [44] comes to mind.) This paper is intended as a historical account of the search for simple groups for readers who are not experts in the subject. It is the hope of the author that the paper may profitably be read by one who is conversant with the contents of Herstein's algebra book [55]. A complete discussion of all important contributions to simple group theory is beyond the scope of this paper.

What are simple groups and why are they important? Évariste Galois (1811–1832) called a group simple if its only normal subgroups were the identity subgroup and the group itself. The Abelian simple groups are the group of order 1 and the cyclic groups of prime order, while the nonabelian simple groups generally have very complicated structures. These groups are important because they play a role in group theory somewhat analogous to that which the primes play in number theory or the elements do in chemistry; that is, they serve as the "building blocks" for all groups. These "building blocks" are called the composition factors of the group and may be determined in the following way. Given a finite group G, choose a maximal normal subgroup G_1 of $G = G_0$. Then the factor group G_0/G_1 is simple, and we next choose a maximal normal subgroup G_2 of G_1. Then G_1/G_2 is also simple, and we continue in this fashion until we arrive at $G_n = \{e\}$. The simple groups $G_0/G_1, G_1/G_2, \ldots, G_{n-1}/G_n$ are the composition factors of G and by the Jordan-Hölder theorem these groups are independent of the choices of the normal subgroups made in the process described. In a certain sense, a group can be reconstructed from its composition factors and many of the properties of a group are determined by the nature of its composition factors. This, and the fact that many questions about finite groups can be reduced (by induction) to questions about simple groups, make clear the importance of determining all nonabelian finite simple groups.

The narrative which follows is divided into 16 sections which appear in more or less chronological order according to theme. Within a particular section however, we usually include a number of results which are related to the theme without regard to time. Thus, for example, the section on the odd order problem appears early in the paper but includes results ranging from 1895 to 1963. In this way we hope to emphasize two points: (1) the problems of one generation very often have deep roots in the work of previous generations and (2) there is frequently a large temporal gap between certain results and their subsequent improvements.

Throughout the remainder of this paper we use the term simple group to mean a finite nonabelian simple group.

1. The alternating groups and the classical linear groups

Although Galois had formulated the definition of a simple group and had observed that the alternating group (of even permutations) on 5 symbols was simple, the first major results in the theory were due to Camille Jordan (1838–1922). In 1870, Jordan published *Traité des Substitutions*, the first book ever written on group theory **[58]**. In this book he established the existence of five infinite families of finite simple groups. One of these families, which we denote by A_n, consists of the alternating permutation groups on $n > 4$ symbols. Jordan formed the other four families by using matrices with entries from finite fields. One of these may be described as follows. For $m > 1$, the special linear group $SL(m, p^n)$ is the multiplicative group of $m \times m$ matrices of determinant 1 with entries from the field with p^n elements and the projective special linear group $PSL(m, p^n)$ is the factor group $SL(m, p^n)/Z(SL(m, p^n))$ where $Z(SL(m, p^n))$, the center of $SL(m, p^n)$, is the subgroup of $SL(m, p^n)$ consisting of all scalar matrices with determinant 1. Jordan proved that $PSL(m, p)$ is simple when (m, p) is not $(2, 2)$ or $(2, 3)$. The other three families have been given the names orthogonal, unitary and symplectic groups and, following Hermann Weyl, mathematicians refer to these four families collectively as the classical simple groups.

The last three types mentioned above are most easily defined as certain groups of invertible linear transformations of a finite dimensional vector space V over a finite field modulo the center of the group and in each case the group is obtained by considering those transformations T which leave a nondegenerate form f of V invariant (i.e., f is a certain function from $V \times V$ into the field and $f(Tx, Ty) = f(x, y)$ for all x, y in V). A symmetric bilinear form (i.e., a dot product) gives an orthogonal group; hermitian, a unitary group; and skew symmetric bilinear, a symplectic group. The precise definitions of these groups are not needed here but the reader can find them in [2] and **[6]**. Jordan introduced these three families as groups of matrices instead of groups of linear transformations and proved they are simple when the field has prime order (except for a few trivial cases).

2. Range problem 1–660

A different approach was taken by Otto Hölder (1859–1937) when in 1892 he initiated what we will call the range problem; namely the complete determination of all simple groups whose orders are in a given range. Here both the existence and the uniqueness questions must be considered; that is, it must be determined which integers in the range are the orders of simple groups and, for each such integer, all possible simple groups of that order must also be determined (up to isomorphism). Hölder **[56]** proved that the only two simple groups whose orders lie between 1 and 200 are A_5 of order 60 and $PSL(2, 7)$ of order 168. F. N. Cole (1861–1927), the first American-born mathematician to publish in group theory, followed Hölder's lead in 1892 **[23]** when he examined the integers between 201 and 500 for simple groups. He was not totally successful for he was unable to prove that A_6 was the unique simple group of order 360; nor was he able to show 432 was not the order of a simple group. He overcame these difficulties **[24]** a year later, however, when he completed the determination of all the simple groups with orders in the range 1 to 660. In addition to the ones in this range already found by Jordan, Cole discovered one more, $PSL(2, 8)$, having order 504. This provided the first example of a simple group not known to Jordan and the first proof of the simplicity of one of the groups $PSL(m, q)$ with q not prime.

Date	Integers	Individual
1892	1–200	Hölder [**56**]
1892–93	201–660	Cole [**23**, **24**]
1895	661–1092	Burnside [**15**]
1900	1093–2000	Ling and Miller [**60**]
1912	2001–3640 (except 2520)	Siceloff [**71**]
1922	2520	Miller [**65**]
1924	3641–6232 (except 5616 and 6048)	Cole [**26**]
1942	5616 and 6048	Brauer [**7**]
1963	6233–20,000	Michaels [**62**]
1972	1–1,000,000 (21 exceptions)	Hall [**49**, **50**]
1975	Hall's exceptions	Beisiegel and Stingl [**3**]

CHRONOLOGY OF THE RANGE PROBLEM: The search for all simple groups of specified orders reveals sporadic progress as new methods made possible sudden bursts of successful analysis on groups of increasingly large order. At present the range problem is completed through groups of order 1,000,000: all simple groups of order less than 1,000,000 are known, but only some of those beyond that order have been discovered.

The methods of Hölder and Cole are of interest. The three Sylow theorems rule out 596 of the first 660 integers as possible orders of simple groups. (In fact, they rule out 9431 of the first 10,000 [**74**].) If G is assumed to be simple and H is a proper subgroup of G, then G is isomorphic to a subgroup of the symmetric group on the cosets of H in G (compare with the proof of Theorem 2.92 in [**55**]). Thus G with order $|G|$ is represented as a group of permutations on $|G|/|H|$ symbols and it follows that $(|G|/|H|)!$ is a multiple of $|G|$. This last fact is called the index theorem and it further reduces the list of integers to be examined to 47. Finally, a combination of the Sylow theorems, the index theorem, and other elementary techniques such as counting elements reduces to 33 the list of those integers from 1 to 660 which require *ad hoc* arguments. Since the theory of permutation groups was much further developed than the theory of abstract groups at that time, these remaining 33 integers were handled with permutation group techniques. An example of a permutation-type argument will be given later.

It is noteworthy that while the proofs of the non-simplicity of a group of order 144 or 180 occupied more than 10 pages of Hölder's paper, the author has had undergraduates [**59**] who have done this in less than 2 pages using only the results found in Herstein [**55**]. Similarly, using a bit more machinery, three undergraduates from the University of Wisconsin [**27**] covered all the integers up to 1000 with the exceptions of 720 and the uniqueness question. Their proofs for the cases 144 and 180 require only 12 lines.

3. $PSL(m, p^n)$

Cole's discovery of the simplicity of $PSL(2,8)$ had far-reaching consequences because that same year E. H. Moore (1862–1932), the first mathematics department chairman of the University of Chicago, used it for the starting point of his investigations which resulted [**66**] in a proof that the family of groups $PSL(2, p^n)$ are all simple except when $p^n = 2$ or 3. William Burnside (1852–1927) also obtained this result [**13**] shortly after Moore. Moore's paper, in turn, led his first Ph.D. student, Leonard E. Dickson (1874–1954), to the complete generalization of Jordan's original result when in 1897 he proved [**29**] that the family of groups $PSL(m, p^n)$ $(m > 1)$ consisted of simple groups except when $p^n = 2$ or 3. Dickson called this family a triply infinite system since each of p, n, and m may take on infinitely many values. Moore's paper also contains many interesting results on finite fields, the most important of which is that for each prime power p^k there exists a unique field of order p^k (Galois had proved such fields exist in 1830 [**41**]). In the opinion of E.T. Bell [**4**, p.10] these results on finite fields clearly mark the beginning of abstract algebra in America.

4. Range problem to 1092

In 1895 Burnside [**14, 15**] obtained several powerful arithmetic tests for simple groups. By far the most important of these is the fact that a simple group of even order must be divisible by one of 12, 16 or 56. In proving this result, he showed that an even order simple group cannot have a cyclic Sylow 2-subgroup. This theorem appears to be the first nonsimplicity criterion which is based on the structure of the Sylow 2-subgroups. In the past two decades much of the research in simple group theory has dealt with the problem of classifying all simple groups whose Sylow 2-subgroups have a specified structure. (All the Sylow 2-subgroups of a group are isomorphic.) For example, John Walter [**77**], in a long (110 pages!) and difficult proof, obtained a broad generalization of Burnside's result when he determined all simple groups with Abelian Sylow 2-subgroups. Similarly, all simple groups whose Sylow 2-subgroups are dihedral (that is, are groups of symmetries of some regular n-sided polygon) have been determined by Daniel Gorenstein and Walter [**48**]. The proof of this important result appears in three papers and runs 160 pages! Commenting on this proof in *Mathematical Reviews* (32 #7634), John Thompson wrote "The techniques of these papers cover the spectrum of finite group theory more thoroughly than any single paper known to the reviewer."

By 1893 the range problem had been completed as far as 660. Since 1092 was the next integer known to be the order of a simple group, Burnside [**15**] decided to examine the integers between these two. The arithmetical tests of his previous paper disposed of all but 17 of the 432 integers in this range and the Sylow theorems ruled out six more. The remaining 11 integers were considered individually although his proof for the hardest integer in the range, 720, was erroneous and he inadvertently omitted 1008. The efficiency of Burnside's nonsimplicity tests is further evidenced by the fact that they dispose of all odd integers up to 2025 and all but 14 of the odd integers less than 9000. As a rule, even integers are much harder to eliminate than odd integers but Burnside's "12, 16, 56 theorem" alone rules out 3691 of the first 5000 even integers [**74**].

5. Permutation representations and character theory

In obtaining their results Burnside, Cole, and Hölder utilized permutation representations of groups. Certain permutation groups—transitive, doubly transitive, and primitive—play an especially important role in simple group theory. (A permutation group on a set S is called transitive on S if for each pair a, b of letters of S there is an element in G which sends a to b; G is called doubly transitive on S if for each two ordered pairs of distinct letters of S, (a, a') and (b, b'), there is an element in G which sends a to b and a' to b'; see [**78**, p. 15] for the definition of a primitive group.) The reasons for the importance of these groups are that the representation of a group as permutations of the cosets of a subgroup is transitive and that many of the known simple groups can be represented as a doubly transitive (and therefore primitive) permutation group. Thus, a common technique when dealing with a simple group G is to represent it as a transitive, doubly transitive or primitive group and then utilize the theory of these groups to obtain important information about G.

Much effort was devoted in the late 1800's and early 1900's to classifying the transitive and primitive permutation groups of low degree. These results often prove useful in simple group theory. To illustrate, let us consider Siceloff's proof [**71**] that there is no simple group of order $1188 = 2^2 \cdot 3^3 \cdot 11$. If G were a simple group of order 1188, Sylow's theorem implies G has 12 subgroups of order 11 which are conjugate in G. If for each element g in G we define T_g to be the mapping which sends the Sylow 11-subgroup S to the Sylow 11-subgroup $g^{-1}Sg$, we see that G may be viewed as a transitive permutation group on the set of Sylow 11-subgroups of G (cf. proof of 2.92 in [**55**]). Then letting H denote a subgroup of G which consists of all permutations which have some Sylow 11-subgroup fixed, it follows that $|H| = 3^2 \cdot 11$ (see [**78**, p. 51]) and H is a permutation group on the other 11 Sylow 11-subgroups. By Sylow's theorem H has an element of order 11 and so this element is an 11-cycle. Thus H is a transitive permutation group of degree 11 and order 99. But Cole [**25**] has shown no such group exists.

A homomorphism from a group into a group of matrices with entries from some field is called a representation of the group. If T is a representation of G, the character of this representation is the function X from G to the field defined by $X(g) = \text{trace}(T(g))$ for all g in G. There exist numerous arithmetical relations on the characters of a group G which are intimately related to the structure of G. Thus a knowledge of the characters of a group reveals much information about the group itself. The theory of group characters has profoundly influenced the search for simple groups. This theory was developed by Georg Frobenius (1849–1917) in a series of papers beginning in 1896. (The historical background to Frobenius' creation of group characters is detailed in [**51, 52**].) Around the turn of the century Issai Schur (1875–1941) and especially Burnside simplified the theory and found many important applications of it. In recent times, character theory has been further developed and refined by Brauer, Suzuki, and Feit.

6. Odd order problem

During the period 1895–1901 much attention was focused, particularly by Burnside, on the possibility of the existence of a simple group of odd order. In his 1895 paper [**15**], Burnside had shown that there is no simple group of odd order less than 2025. He later extended this to 9000 [**16**] and then to 40,000 [**18**]. Numerous arithmetical theorems obtained by Burnside in this period reduced the list of possible odd orders less than 40,000 to 7; these were then eliminated by elementary considerations.

In 1901 Burnside [**17**] used character theory to prove that a nonsolvable transitive permutation group of prime degree is doubly transitive. Since a simple group which has a subgroup of prime index can be represented as a transitive permutation group on the cosets of this subgroup it must be doubly transitive. But the order of a doubly transitive group of degree n is divisible by $n(n-1)$ [**78**, p. 20] so Burnside's result shows there is no odd order simple group which has a subgroup of prime index. Burnside [**18**] also proved in 1901 that if a simple group has odd order n and p is the smallest prime divisor of n then n is divisible either by p^4 or by both p^3 and a prime factor of p^2+p+1.

Burnside's efforts convinced him that there were no simple groups of odd order and that the eventual proof of this would involve the use of character theory. In fact, he wrote [**20**, p. 503] "The contrast that these results shew between groups of odd and of even order suggests inevitably that simple groups of odd order do not exist." He further wrote [**17**] "The results obtained in this paper, partial as they necessarily are, appear to me to indicate that an answer to the interesting question as to the existence or non-existence of simple groups of odd order may be arrived at by further study of the theory of group characters."

The next important step in this direction however, did not come for more than 50 years. In 1957, Michio Suzuki [**72**] used character theory to prove that a simple group in which the centralizer of any nonidentity element is Abelian must have even order. (The centralizer of an element x in a group G is the subgroup $C(x) = \{g \in G \mid gx = xg\}$.) Three years later, in a major work [**37**], Walter Feit, Marshall Hall, Jr., and John Thompson obtained a broad generalization of Suzuki's result by showing that "Abelian" could be replaced by the much weaker condition "nilpotent." (A group is nilpotent if all of its Sylow subgroups are normal.) Their proof was similar to Suzuki's and character theory played an important role in it.

Burnside's prophecy was at last fulfilled in 1963 when Feit and Thompson expanded on the ideas of the two papers mentioned above and proved [**38**] that groups of odd order are solvable. (A finite group is solvable if all of its composition factors have prime order; thus, solvable groups are not simple.) The difficulty of this proof and the significance of both the theorem and the methods employed cannot be exaggerated. Concerning one portion of the proof, Suzuki wrote in *Mathematical Reviews* (29 #3538) "... [This 50 page portion] represents one of the highest points ever achieved in the theory of finite groups."

The proof of the "Odd Order Theorem" occupies an entire 255 page issue of the *Pacific Journal of Mathematics.* It proceeds by assuming that there is a group G of minimal odd order which is not solvable. Then every proper subgroup of G is solvable and therefore Philip Hall's extensive work on solvable groups could be brought to bear on the subgroups. Ultimately, they were able to derive a contradiction. For their achievement, Feit and Thompson were awarded the Frank Nelson Cole Prize in Algebra by the American Mathematical Society in 1965. (The Cole Prize is named after the same Cole who had determined the simple groups with orders between 201 and 660 and was established in his honor in recognition of his many years of service to the Society.)

7. Dickson's simple groups

In the period from 1897 to 1905 Dickson made many fundamental contributions to the theory of simple groups. In a series of papers appearing from 1897 to 1899 he extended Jordan's results on the simplicity of the orthogonal, unitary and symplectic groups over fields of prime order to arbitrary finite fields. Much of this work emanated from his Ph.D. dissertation, the first one ever done in mathematics at the University of Chicago. Whether there exist two nonisomorphic simple groups of the same order had been a long-standing problem by 1899. But Dickson's proof in 1897 that $PSL(3,4)$ is simple provided a possible answer to this question since it and the simple group A_8 both have order 20,160. It was quickly suspected that these two were not isomorphic since A_8 contains elements of orders 6 and 15 while no such elements were known to be in $PSL(3,4)$. At Moore's suggestion, Ida Schottenfels investigated these two groups and proved [70] they were not isomorphic. Shortly thereafter, Dickson showed [30] that there are infinitely many such examples. Since these examples were given by Dickson no others have been found and there is no known triple of nonisomorphic simple groups of equal order. After 20,160 the next known integer for which there is a pair of nonisomorphic simple groups of equal order is 4,585,351,680, and it wasn't until the mid 1960's that 20,160 was shown to be the smallest possible integer for which this can happen.

In his classic book *Linear Groups* Dickson listed all the isomorphisms between the simple groups he knew. For example, A_5, $PSL(2,4)$, and $PSL(2,5)$ are defined differently but are isomorphic. The question of whether Dickson's list of isomorphisms contained all which were possible among the simple groups known to him was not answered until 50 years later when Jean Dieudonné proved [35]

A Chronological Collection of ...

1870	Jordan	Established simplicity of alternating groups and linear groups over fields of prime order.
1892	Hölder	Began range problem.
1895–1900	Cole, Miller	Proved simplicity of Mathieu groups.
1896–1901	Frobenius-Burnside	Developed character theory.
1897–1905	Dickson	Established simplicity of linear groups over arbitrary finite fields. Discovered a family of simple groups of Lie type.
1904	Burnside	Proved $p^a q^b$ theorem.
1954	Brauer	Began the program of characterizing simple groups in terms of centralizers of involutions.
1955	Chevalley	Discovered new approach to simple groups. Discovered new families of simple groups of Lie type.
1958–1961	Steinberg, Tits, Hertzig, Ree	Extended Chevalley's methods and discovered new infinite families of simple groups of Lie type.

that Dickson's list was complete. Dickson also listed all the coincidences in the orders of the simple groups known to him but whether this list was complete was not determined until 1955 when Emil Artin (1898–1962) proved [1] with an elegant number-theoretical study that it was.

Without going into detail we mention that the classical linear groups over the field of complex numbers are Lie groups (because, roughly, they possess a smooth geometric structure) and Wilhelm Killing (1847–1923) and Elie Cartan (1869–1951) proved (1888–1894) that besides the simple Lie groups corresponding to the classical groups there are only five additional simple ones called exceptional Lie groups. In two papers in 1901 and 1905 Dickson discovered a new infinite family of finite simple groups by defining analogs over finite fields of one of these exceptional Lie groups [31, 32]. It is remarkable that no additional new finite simple groups were found until Claude Chevalley and others, 50 years later, were able to show (among other results) that the remaining four exceptional Lie groups also had finite analogs.

8. The Mathieu groups

In 1861 E. Mathieu discovered a family of five transitive permutation groups. This remarkable family has become very important in both the theory of simple groups and coding theory as well as in permutation group theory. In 1895, while determining all transitive permutation groups on 10 or 11 symbols, Cole observed [25] that the smallest Mathieu group (order 7920) is simple and by 1900 G. A. Miller (1863–1951) had shown [63, 64] the other four are also simple. Three of these groups have order less than 1,000,000 and this brought to 53 the number of such simple groups known in 1900. This number would not be enlarged until 1960.

Among all the simple groups known by 1905 the Mathieu groups had the peculiar distinction of being the only ones which were not part of an infinite family of simple groups (such as A_n or $PSL(m, p^n)$). To this date they (and 21 or so other simple groups) still have not been shown to be members of any infinite family of simple groups in a natural way.

... Highlights in the Theory of Simple Groups

1960	Suzuki	Discovered new infinite family of simple groups (only simple groups with orders not divisible by 3).
1963	Feit-Thompson	Proved simple groups have even order.
1965	Gorenstein-Walter	Classified all simple groups with dihedral Sylow 2-subgroups.
1966–1975	Janko, Hall, Higman, Sims, McKay, McLaughlin, Suzuki, Held, Conway, Thompson, Fischer, Lyons, Rudvalis, Wales, O'Nan, Smith	Discovered new sporadic simple groups.
1968	Thompson	Proved N-theorem. Classified all minimal simple groups.
1969	Walter	Classified all simple groups with Abelian Sylow 2-subgroups.
1971	Thompson	Proved Suzuki groups are the only simple groups with orders not divisible by 3.
1972	Wales	Classified all simple groups with orders of the form $p^a q^b r$.
1975	Hall, Beisiegel-Stingl	Completed range problem to 1,000,000.

9. Range problem to 6232

The determination of all simple groups of order up to 2000 was completed in 1900 when G. Ling and Miller proved [**60**] that there is no simple group whose order is between 1093 and 2000. It is interesting to note that although there are 908 integers in this range only 28 required any special treatment. For example, Burnside's result on odd order simple groups and his "even, but not divisible by 12, 16 or 56" theorem eliminated all but 118 possibilities. Then the Sylow theorems, the index theorem, and several results of Frobenius on groups with orders of the form $p^a q^b$ reduced the possibilities to 28.

In 1902 Frobenius determined [**40**] all transitive permutation groups of degree $p+1$ and order $p(p^2-1)/2$ where p is prime. Since any simple group of order $p(p^2-1)/2$ can be represented as a transitive permutation group on the $p+1$ conjugate Sylow p-subgroups, it follows from Frobenius' theorem that $PSL(2,p)$ is the only simple group of order $p(p^2-1)/2$. This appears to be the first arithmetical characterization of an infinite family of simple groups. Fifty-six years later this uniqueness theorem was generalized in answer to a question of Artin. Artin had observed that the simple groups $PSL(2,p)$ $(p>3)$ and $PSL(2,2^n)$, where 2^n+1 is prime, have orders which are divisible by a prime whose cube exceeds the order of the group. He conjectured that these were the only such simple groups and Brauer and Reynolds [**11**] used modular character theory (the character values lie in a field of prime characteristic) to prove this conjecture.

Twelve years after Ling and Miller completed the range problem up to 2000, L. P. Siceloff, at the suggestion of Cole, proved [**71**] that the only integers between 2001 and 3640 which are orders of simple groups are 2448, 2520, and 3420. All of these were on Dickson's list in 1901. Since 2448 and 3420 have the form $p(p^2-1)/2$ the uniqueness question concerning these integers had been answered affirmatively by Frobenius ten years earlier and it was Miller who ten years afterward showed [**65**] that A_7 is the unique simple group of order 2520. Thus by 1922 all simple groups of order up to 3640 had been determined.

Again there was a twelve year hiatus before the exhaustive enumeration of integers in a certain range was continued. In 1924, Cole returned to the problem again and showed [**26**] that the only integers between 3641 and 6232 which are orders of simple groups were the four on Dickson's 1901 list. Unfortunately Cole's paper was so lacking in detail that its value was diminished. The uniqueness question for $PSL(2,23)$ (order 6072) had previously been settled affirmatively by Frobenius, and Cole did the same for $PSL(2,16)$ (order 4080). Eighteen years after Cole's paper Brauer [**7**] showed, with the use of character theory, that the remaining two integers also corresponded to unique simple groups. So by 1942 all simple groups of order as far as 6232 had been determined.

10. Burnside's $p^a q^b$ theorem

In 1904, Burnside used character theory to prove [**19**] that every group of order $p^a q^b$ where p and q are primes is solvable. This theorem represented the final generalization of a large number of special cases which had been established by Sylow, Frobenius, Burnside, Jordan, and Cole; it has become the classic example of the power of character theory. With character theory a simple proof was possible 70 years ago, but a character-free proof of Burnside's theorem, although long sought, has appeared only in the past few years. Thompson had indicated in his fundamental paper [**75**] on minimal simple groups (see section 14) that a character-free proof of the Burnside theorem could be extracted from that paper and the "odd order paper" [**38**]. David Goldschmidt [**43**], in 1970, gave a short character-free proof of the theorem when p and q are odd and Helmut Bender [**5**] two years later proved the general result without character theory. By combining the arguments of Bender in the odd order case and of Hiroshi Matsuyama [**61**] in the even order case it is now possible to obtain a short and attractive character-free proof of the $p^a q^b$ theorem.

Despite the early outstanding achievements with character theory by Burnside and Frobenius, others seemed to ignore it as a tool in simple group theory until Brauer brought it to the forefront in the 1940's and 50's. This is partly explained by the fact that interest in simple groups began to wane around 1905.

11. The Chevalley groups

In 1948 Dieudonné [34] classified all the known simple groups according to their method of construction. This classification scheme modernized the one Dickson had devised in 1901. Whereas Dickson obtained his results by means of complicated matrix calculations which often obscured the underlying ideas, Dieudonné utilized the geometric properties of linear transformations and vector space theory to simplify and clarify Dickson's work. Although Dieudonné's approach to the classical linear groups is much more elegant than Dickson's, his methods still required that each family of simple groups be treated individually and he found no new simple groups. Thus, simple group theory was revitalized in 1955 when Chevalley, in his celebrated paper [22], introduced a new approach which provided a uniform method for investigating three kinds of the classical linear groups and Dickson's simple groups of Lie type as well. In addition to encompassing most of the then-known simple groups, his method also yielded several new infinite families of finite simple groups.

This was accomplished in the following way. With every pair (L, K) where L is a Lie algebra over the complex numbers (i.e., an algebra whose associative law is replaced by the Jacobi identity and also satisfies the condition $x^2 = 0$ for all x) and K is a field, a new Lie algebra L_K is constructed. Chevalley was able to associate with every such pair a certain subgroup of the automorphism group of L_K which is simple. With the appropriate choice of L and K these simple groups are those investigated by Jordan, Dickson, and Dieudonné. With other choices of L and K Chevalley obtained his new simple groups (the smallest of these has order $2^{24}3^6 5^2 7^2 13 \cdot 17$). These groups were the first new simple groups found in more than 50 years. That they were indeed new was established by comparing their orders with the orders of the simple groups which had been known. The formulas for the orders of the Chevalley groups over finite fields were derived by using topological properties of the Lie group of the same type. Artin [1] developed a new classification scheme for the known simple groups which included the Chevalley groups. He used fewer classes than did Dickson and Dieudonné and his method considerably improved theirs (see also [2]).

12. Groups of Lie type

During the period 1958–1959 Chevalley's methods were extended and modified by Robert Steinberg, Jacques Tits, and D. Hertzig (see [21]) to obtain additional new infinite families of simple groups and the classical groups not handled by Chevalley. Shortly thereafter, Suzuki [73], while in the process of classifying a certain type of doubly transitive permutation groups, also discovered another new infinite family. Analyzing the Suzuki groups, Rimhak Ree noticed that when interpreted from a Lie-theoretical point of view, they were closely related to a certain family of Chevalley groups. He then showed that the method of Steinberg could be used to construct the Suzuki groups. This in turn led him to investigate two other similar situations and eventually discover his two families of simple groups [68, 69]. The Suzuki and Ree groups together with those of Chevalley and Steinberg are collectively referred to as the simple groups of Lie type. These, together with the alternating groups A_n $(n \geqq 5)$ account for all but 26 or so of the finite simple groups known to date.

The Suzuki groups are noteworthy for another reason. They provided the first examples of simple groups whose orders are not divisible by 3, and Thompson, in a major classification theorem, has recently shown that these are the only possible such groups. The elements of the Suzuki groups are certain 4×4 matrices with entries from the Galois fields of order 2^{2n+1}, again illustrating the extremely important role that matrix groups over finite fields play in simple group theory.

13. Sporadic simple groups

A simple group which no one has yet been able to fit into an infinite class of simple groups in a natural way is called a sporadic simple group. For example, A_n and $PSL(n, q)$ are infinite families of

The Known Finite Simple Groups ...

Date and Discoverer	Type: *Name*	Group Notation	Order (p is a prime, $q = p^n$)
1870 Jordan	— *alternating group of degree m*	$A_m\ (m > 4)$	$m!/2$
1870 Jordan	classical linear *projective special linear*	$L_m(p)$ or $PSL(m,p)\ (m > 1)$	$d^{-1}p^{m(m-1)/2}\prod_{i=2}^{m}(p^i - 1)$; $d = (m, p-1)$
1870 Jordan	classical linear *symplectic*	$S_{2m}(p)\ (m > 1)$	$d^{-1}p^{m^2}\prod_{i=1}^{m}(p^{2i} - 1)$; $d = (2, p-1)$
1870 Jordan	classical linear *orthogonal*	$O_{2m}(\varepsilon, p),\ \varepsilon = \pm 1,\ (m > 3)$	$d^{-1}p^{m(m-1)}(p^m - \varepsilon)\cdot\prod_{i=1}^{m-1}(p^{2i} - 1)$; $d = (4, p^m - \varepsilon)$
1870 Jordan	classical linear *unitary*	$U_m(p)\ (m > 2)$	$d^{-1}p^{m(m-1)/2}\prod_{i=2}^{m}(p^i - (-1)^i)$ $d = (m, p+1)$
1893 Cole	classical linear $L_2(8)$	$L_2(8)$ or $PSL(2, 8)$	$504 = 2^3\cdot 3^2\cdot 7$
1893 Moore	classical linear *projective special linear*	$L_2(q)$ or $PSL(2, q)\ (q > 3)$	$d^{-1}q(q^2 - 1)$; $d = (2, q-1)$
1895 Mathieu-Cole	sporadic *Mathieu* 11	M_{11}	$7920 = 2^4\cdot 3^2\cdot 5\cdot 11$
1897 Dickson	classical linear *projective special linear*	$L_m(q)$ or $PSL(m,q)(m > 1)$	$d^{-1}q^{m(m-1)/2}\prod_{i=2}^{m}(q^i - 1)$ $d = (m, q-1)$
1897 Dickson	classical linear *symplectic*	$S_{2m}(q)\ (m > 1)$	$d^{-1}q^{m^2}\prod_{i=1}^{m}(q^{2i} - 1)$; $d = (2, q-1)$
1898 Dickson	classical linear *orthogonal*	$O_{2m}(\varepsilon, q),\ \varepsilon = \pm 1,\ (m > 3)$	$d^{-1}q^{m(m-1)}(q^m - \varepsilon)\cdot\prod_{i=1}^{m-1}(q^{2i} - 1)$; $d = (4, q^m - \varepsilon)$
1898 Dickson	classical linear *unitary*	$U_m(q)\ (m > 2)$	$d^{-1}q^{m(m-1)/2}\cdot\prod_{i=2}^{m}(q^i - (-1)^i)$ $d = (m, q+1)$
1899 Mathieu-Miller	sporadic *Mathieu* 12	M_{12}	$95{,}040 = 2^6\cdot 3^3\cdot 5\cdot 11$
1900 Mathieu-Miller	sporadic *Mathieu* 22	M_{22}	$443{,}520 = 2^7\cdot 3^2\cdot 5\cdot 7\cdot 11$
1900 Mathieu-Miller	sporadic *Mathieu* 23	M_{23}	$10{,}200{,}960 = 2^7\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$

simple groups while the five Mathieu groups are sporadic. In the classification schemes of Dickson, Dieudonné, and Artin only the five Mathieu groups are sporadic and no new ones were discovered until Zvonimir Janko found one of order 175,560 in 1966 [**57**]. Remarkably, the discoveries of Mathieu and Janko are separated by more than one hundred years. Using other results of Janko, M. Hall proved [**49**] the existence of a sporadic simple group of order 604, 800 in 1967. This brought to 56 the number of known simple groups of order less than 1,000,000 and to 7 the number of sporadic simple groups. Paraphrasing Gorenstein [**47**, p. 14] we recount the following anecdote in connection with Hall's simple group:

> Shortly after Hall constructed his group he gave a lecture on it at Oxford. Donald Higman and Charles Sims were in the audience and they were both struck by the fact that one of the Mathieu groups had certain permutation properties analogous to those of a group which Hall had used in his construction. That very same night this observation led them to the construction of a new sporadic simple group!

Hall's method has also led to the discovery of two others by analogous methods. By now there are 26 or so sporadic simple groups and it has been proved that there are exactly 56 simple groups of order less than 1,000,000. (Of late, new sporadic simple groups are being discovered so frequently that it is difficult for one to be sure of their precise number.)

Some of the sporadic simple groups have been discovered in the course of solving certain problems in permutation group theory while others have turned up as the automorphism group of a distance-transitive graph (see Chapter 4 and the Appendix in [**6**]). Quite often, two or more sporadic groups are related in some way. Indeed, the Conway .1 simple group contains 12 sporadic simple groups as subgroups! The existence of many of these recently-discovered groups has been verified by means of a permutation representation of the group and extensive use of computers.

An important technique which has led to the discovery of a number of sporadic simple groups involves the notion of the centralizer of an involution (i.e., of an element of order 2). This method is employed in the following way. Choose H to be the centralizer of an involution from some known simple group G. Next, assume G^* is any simple group which contains an involution x such that $C(x)$ is isomorphic to H. (By a theorem of Brauer and Fowler [**10**] only a finite number of such groups can exist, so H "almost" determines G.) Then a great deal of information about G^* can be obtained. With this information it is often possible to show that G^* is isomorphic to G or to some other known simple group. For example Dieter Held [**53**] has proved such a theorem when $G = A_8$ or A_9. If it cannot be shown that G^* must be isomorphic to some known simple group, the information may be adequate to suggest a method of constructing a new simple group. Held [**54**] has also been instrumental in accomplishing this. He began with the observation that $PSL(5,2)$ and the largest Mathieu group possess involutions with isomorphic centralizers and no other known simple group has this property. Choosing this for H, he was led to three possible configurations for G^*. Ultimately, enough properties of this third group were derived so that Graham Higman and John McKay were able to construct it with the use of a computer.

Similarly, one may proceed by assuming G is a simple group which contains an involution whose centralizer closely resembles the form of a centralizer of an involution from a known simple group. In this case, if the information about G is not self-contradictory it suggests the possible existence of a new simple group and may be sufficient to lead to an actual construction of the group. This is how Janko discovered his simple group of order 175,560. Each member of a family of simple groups discovered by Ree has a centralizer isomorphic to $Z_2 \times PSL(2,3^n)$ and has its Sylow 2-subgroups isomorphic to $Z_2 \times Z_2 \times Z_2$. Janko set out to determine all simple groups which have a centralizer isomorphic to $Z_2 \times PSL(2,p^n)$, p odd, and with Sylow 2-subgroups isomorphic to $Z_2 \times Z_2 \times Z_2$. Eventually he was able to show that either $p = 3$ and the group is of Ree type or $p^n = 5$. The information he obtained about this latter case led him to write down a pair of 7×7 matrices with entries in the field of order 11 which generated a new simple group.

Date and Discoverer	Type: *Name*	Group Notation	Order p is a prime, $q = p^n$
1900 Mathieu-Miller	sporadic *Mathieu 24*	M_{24}	$244{,}823{,}040 = 2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$
1901 and 1905 Dickson	Lie *groups of type G_2*	$G_2(q)$	$q^6(q^6-1)(q^2-1)$
1955 Chevalley	Lie *Chevalley groups of type E_4*	$E_4(q)$	$q^{24}(q^{12}-1)(q^8-1)\cdot (q^6-1)(q^2-1)$
1955 Chevalley	Lie *Chevalley groups of type E_6*	$E_6(q)$	$d^{-1}q^{36}(q^{12}-1)(q^9-1)\cdot (q^8-1)(q^6-1)(q^5-1)\cdot (q^2-1);\ d=(3,q-1)$
1955 Chevalley	Lie *Chevalley groups of type E_7*	$E_7(q)$	$d^{-1}q^{63}(q^{18}-1)(q^{14}-1)\cdot (q^{12}-1)(q^{10}-1)(q^8-1)\cdot (q^6-1)(q^2-1);\ d=(2,q-1)$
1955 Chevalley	Lie *Chevalley groups of type E_8*	$E_8(q)$	$q^{120}(q^{30}-1)(q^{24}-1)\cdot (q^{20}-1)(q^{18}-1)(q^{14}-1)\cdot (q^{12}-1)(q^8-1)(q^2-1)$
1959 Steinberg-Tits-Hertzig	Lie *twisted groups of type E_6*	${}^2E_6(q^2)$	$d^{-1}q^{36}(q^2-1)(q^5+1)\cdot (q^6-1)(q^8-1)(q^9+1)\cdot (q^{12}-1);\ d=(3,q+1)$
1959 Steinberg-Tits-Hertzig	Lie *twisted groups of type D_4*	${}^3D_4(q^3)$	$q^{12}(q^2-1)(q^6-1)\cdot (q^8+q^4+1)$
1960 Suzuki	Lie *Suzuki groups*	$Sz(q)$ or ${}^2B_2(q)$, $q=2^{2m+1}$	$q^2(q^2+1)(q-1)$
1961 Ree	Lie *Ree groups of type G_2*	${}^2G_2(q)$, $q=3^{2m+1}$	$q^3(q^3+1)(q-1)$
1961 Ree	Lie *Ree groups of type F_4*	${}^2F_4(q)$, $q=2^{2m+1}$	$q^{12}(q^6+1)(q^4-1)\cdot (q^3+1)(q-1)$
1966 Janko	sporadic *Janko*	Ja	$175{,}560 = 2^3\cdot 3\cdot 5\cdot 7\cdot 11\cdot 19$
1967 Hall-Janko	sporadic *Hall-Janko*	HaJ	$604{,}800 = 2^7\cdot 3^3\cdot 5^2\cdot 7$
1968 Higman-Sims	sporadic *Higman-Sims*	HiS	$44{,}352{,}000 = 2^9\cdot 3^2\cdot 5^3\cdot 7\cdot 11$
1969 Hall-Janko-McKay	sporadic *Hall-Janko-McKay*	HJM	$50{,}232{,}960 = 2^7\cdot 3^5\cdot 5\cdot 17\cdot 19$
1969 McLaughlin	sporadic *McLaughlin*	McL	$898{,}128{,}000 = 2^7\cdot 3^6\cdot 5^3\cdot 7\cdot 11$

This method was also recently used independently by Bernd Fischer and Thompson and by Robert Griess to discover a possible new simple group. This object, called the "Monster" M, is not defined in terms of generators and relations. In fact, it has not been defined at all! What Fischer, Thompson and Griess did was to assume that there exists a finite simple group M that satisfies certain hypotheses. They showed, under this assumption, that M would be in fact a new simple group (i.e., not isomorphic to any existing simple group). They also obtained a great deal of other information about M, such as its order, properties of certain subgroups, and a portion of its character table. Thompson has computed the order of M to be

$$808{,}017{,}424{,}794{,}512{,}875{,}886{,}459{,}904{,}961{,}710{,}757{,}005{,}754{,}368{,}000{,}000{,}000$$
$$= 2^{46}3^{20}5^{9}7^{6}11^{2}13^{3}17\cdot19\cdot23\cdot29\cdot31\cdot41\cdot59\cdot71$$

(hence the name). If there is a simple group satisfying the stipulated hypothesis we should be able to deduce what it must look like and once we know what it looks like, we should be able to define it. For the Monster M this last step has not yet been accomplished.

A certain section of M (i.e., a group of the form H/K where H and K are subgroups of M and K is normal in H), the "Baby Monster" B, is also a possible new simple group. Fischer has computed the order of B to be $4{,}154{,}781{,}481{,}226{,}426{,}191{,}177{,}580{,}544{,}000{,}000 = 2^{41}3^{13}5^{6}7^{2}11\cdot13\cdot17\cdot19\cdot23\cdot41\cdot47$. Of course, neither of the groups M or B has yet been shown to exist.

In addition to the role the centralizers of involutions play in the discovery of new simple groups, these subgroups are important for another reason. There is presently underway a systematic attempt to use the centralizers of involutions as a means for classifying all the known finite simple groups. This program began in 1954 with Brauer's characterization of $PSL(3,q)$, q odd, and is now almost complete. According to Gorenstein [**47**, p. 21], "Probably more individuals have been involved in this effort than in any other single area of simple group theory." We refer the reader to section 4.4 of [**36**] for a survey of results of this type.

14. Thompson's *N*-paper

Most simple groups contain other simple groups as subgroups. For example, $A_5 \subset A_6 \subset A_7 \ldots$. On the other hand, a minimal simple group is one, all of whose proper subgroups are solvable. It follows then that every simple group has a minimal simple group as a section. Minimal simple groups are therefore basic and the complete determination of all such groups would clearly be of great value. In the early 1960's Thompson set out to do just this. Such an endeavor was a natural successor to the Odd Order Theorem since the minimal counterexample G in that proof was a minimal simple group. Actually, Thompson decided to tackle a more general classification problem. The normalizer of a nonidentity solvable subgroup of a group G is called a local subgroup of G and an N-group is one in which all local subgroups are solvable. Evidently, every minimal simple group is also an N-group.

As early as 1963, Thompson had concluded that with only finitely many exceptions the simple N-groups were $PSL(2,q)$ $(q>3)$ and the Suzuki groups. The complete classification of all nonsolvable N-groups however, did not come until several years later. The 407 page proof (!!) [**75**] of this remarkable theorem is spread out over six journal issues during the seven year period 1968–1974. Describing his approach, Thompson writes [**75**, p. 383]:

> In a broad way, this paper may be thought of as a successful transformation of the theory of solvable groups to the theory of simple groups. By this is meant that a substantial structure is constructed which makes it possible to exploit properties of solvable groups to obtain delicate information about the structure and embedding of many solvable subgroups of the simple group under consideration. In this way, routine results about solvable groups acquire great power.

(An essay which outlines the organization of the proof and discusses some of the arguments used is given in [**46**, pp. 473–480].)

Date and Discoverer	Type: *Name*	Group Notation	Order p is a prime, $q = p^n$
1969 Suzuki	sporadic *Suzuki*	*Suz*	$448{,}345{,}497{,}600 = 2^{13}\cdot 3^7\cdot 5^2\cdot 7\cdot 11\cdot 13$
1969 Held-Higman-McKay	sporadic *Held-Higman-McKay*	*HHM*	$4{,}030{,}387{,}200 = 2^{10}\cdot 3^3\cdot 5^2\cdot 7^3\cdot 17$
1969 Conway-Thompson	sporadic *Conway's .1 group*	Co_1	$4{,}157{,}776{,}806{,}543{,}360{,}000 = 2^{21}\cdot 3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 23$
1969 Conway-Thompson	sporadic *Conway's .2 group*	Co_2	$42{,}305{,}421{,}312{,}000 = 2^{18}\cdot 3^6\cdot 5^3\cdot 7\cdot 11\cdot 23$
1969 Conway-Thompson	sporadic *Conway's .3 group*	Co_3	$495{,}766{,}656{,}000 = 2^{10}\cdot 3^7\cdot 5^3\cdot 7\cdot 11\cdot 23$
1969 Fischer	sporadic *Fischer 22*	Fi_{22}	$64{,}561{,}751{,}654{,}400 = 2^{17}\cdot 3^9\cdot 5^2\cdot 7\cdot 11\cdot 13$
1969 Fischer	sporadic *Fischer 23*	Fi_{23}	$4{,}089{,}460{,}473{,}293{,}004{,}800 = 2^{18}\cdot 3^{13}\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 17\cdot 23$
1969 Fischer	sporadic *Fischer 24*	Fi_{24}	$1{,}255{,}205{,}709{,}190{,}661{,}721{,}292{,}800 = 2^{21}\cdot 3^{16}\cdot 5^2\cdot 7^3\cdot 11\cdot 13\cdot 17\cdot 23\cdot 29$
1971 Lyons-Sims	sporadic *Lyons-Sims*	*LyS*	$51{,}765{,}179{,}004{,}000{,}000 = 2^8\cdot 3^7\cdot 5^6\cdot 7\cdot 11\cdot 31\cdot 37\cdot 67$
1972 Rudvalis-Conway-Wales	sporadic *Rudvalis*	*Rud*	$145{,}926{,}144{,}000 = 2^{14}\cdot 3^3\cdot 5^3\cdot 7\cdot 13\cdot 29$
1973 O'Nan-Sims	sporadic *O'Nan*	*O'N*	$460{,}815{,}505{,}920 = 2^9\cdot 3^4\cdot 5\cdot 7^3\cdot 11\cdot 19\cdot 31$
1974 Fischer	sporadic *Monster*	*M* or F_1 (possible new simple group)	$808{,}017{,}424{,}794{,}512{,}875{,}886{,}459{,}904{,}961{,}710{,}757{,}005{,}754{,}368{,}000{,}000{,}000 = 2^{46}\cdot 3^{20}\cdot 5^9\cdot 7^6\cdot 11^2\cdot 13^3\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71$
1974 Fischer	sporadic *Baby Monster*	*B* or F_2 (possible new simple group)	$4{,}154{,}781{,}481{,}226{,}426{,}191{,}177{,}580{,}544{,}000{,}000 = 2^{41}\cdot 3^{13}\cdot 5^6\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 41\cdot 47$
1974 Fischer-Smith-Thompson	sporadic *Fischer 3 or Thompson group*	F_3 or E	$90{,}745{,}943{,}887{,}872{,}000 = 2^{15}\cdot 3^{10}\cdot 5^3\cdot 7^2\cdot 13\cdot 19\cdot 31$
1974 Fischer-Smith	sporadic *Fischer 5 or Harada group*	F_5 or F	$273{,}030{,}912{,}000{,}000 = 2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11\cdot 19$
1975 Janko	sporadic *Janko 4*	J_4 (possible new simple group)	$86{,}775{,}571{,}046{,}077{,}562{,}880 = 2^{21}\cdot 3^3\cdot 5\cdot 7\cdot 11^3\cdot 23\cdot 29\cdot 31\cdot 37\cdot 43$

This result has a number of important corollaries, the most important of which is a classification of all minimal simple groups. A few consequences of this corollary are mentioned in the next two sections. For his many profound contributions to simple group theory, Thompson was awarded the Fields Medal — the mathematical equivalent of the Nobel Prize — by the International Congress of Mathematicians in 1970.

15. The $p^aq^br^c$ problem

One of the corollaries of Thompson's result classifying the minimal simple groups seems to have put the solution of a very difficult, well-known problem within reach. This problem concerns the natural generalization of Burnside's p^aq^b theorem to three primes. Since there are eight known simple groups which have orders divisible by exactly three distinct primes the logical extension of Burnside's result would be the complete determination of all simple groups with orders of the form $p^aq^br^c$. The first steps in this direction were taken by Burnside, Frobenius and E. Maillet. For example, Burnside [**14**] showed that there is no simple group whose order has the form pq^br where $p<q<r$. Fifty years later Brauer and Hsio Tuan [**12**] used character theory to show that except for the groups $PSL(2,5)$ and $PSL(2,7)$, the restriction "$p<q<r$" was unnecessary. In 1962 and again in 1968 Brauer [**8, 9**] returned to this problem and determined all simple groups whose orders have the form p^aq^br where $a=1$ or 2 or the form 2^a3^b5.

In spite of the fact that Brauer and his predecessors had solved the "three-prime problem" in numerous special cases, the complete solution was far from sight until Thompson proved his result. Specifically, he proved that a simple group whose order is divisible by exactly three distinct primes must have one of $PSL(2,4)$, $PSL(2,7)$, $PSL(2,8)$, $PSL(2,17)$ or $PSL(3,3)$ as a section. From this it follows that such a group must have order of the form $2^a3^bp^c$ where p is 5, 7, 13 or 17. Then, since character theory is a natural tool for analyzing groups whose orders have a prime to the first power only, David Wales [**76**] used it in conjunction with the N-paper to determine all simple groups (8 of them) whose orders have the form 2^a3^bp. Finally, Kenneth Klinger and Geoffrey Mason are presently in the midst of showing (they hope) that there are no simple groups with orders of the form $2^a3^bp^c$ with $c>1$. Of course, the completion of this work will finish the $p^aq^br^c$ problem.

16. The range problem to 1,000,000

At the suggestion of Brauer, Sister Michaels, in her 1963 Ph.D. dissertation [**62**], showed that there was no known simple group in the range 6232 to 20,000. Her work was superseded during the late 60's and early 70's when the range problem was taken up by M. Hall [**49, 50**]. Using a wide assortment of methods from elementary to advanced as well as a computer he succeeded in eliminating all but 21 of the first 1,000,000 integers as possible orders for new simple groups. For the integers not eliminated by elementary considerations, the theory of modular characters and Thompson's result on minimal simple groups played an important role. The character theory yields integer equations which certain parameters of the group must satisfy and a computer was used to make the verifications. Every simple group must have a section which is a minimal simple group so the order of any simple group is divisible by the order of a minimal simple group.

Hall was able to show that from among Thompson's list of minimal simple groups only $PSL(2,5)$, $PSL(2,7)$, $PSL(2,8)$, $PSL(2,13)$, $PSL(2,17)$, $PSL(3,3)$, $PSL(2,23)$ and $PSL(2,27)$ could occur as a section of an unknown simple group of order less than 1,000,000. From a result of Gorenstein it then follows that such a group has order divisible by 840 or 2184. Eventually the list was pared down to 1146 integers which required individual consideration. The first paper [**49**] eliminates all but about 100 of these and the second paper [**50**] reduces the list to 21.

Then Paul Fong [**39**] classified all simple groups whose Sylow 2-subgroups have order at most 2^6 and this reduced Hall's list to 13 integers as possible orders for new simple groups of order less than 1,000,000. Finally, the range problem to 1,000,000 was recently finished when two students of Held, Bert Beisiegel and Volker Stingl [3], eliminated the remaining integers on Hall's list by extending Fong's work as far as the case 2^{10}.

In conclusion we mention that even though the range problem is not the central one in simple group theory, this achievement is a dramatic illustration of how far the theory has progressed in the 84 years since Hölder determined the simple groups of order up to 200.

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References

[1] E. Artin, The orders of the classical simple groups, Comm. Pure Appl. Math., 8 (1955) 455–472.

[2] ———, Geometric Algebra, Interscience, New York, 1957.

[3] B. Beisiegel and V. Stingl, The finite simple groups with Sylow 2-subgroups of order at most 2^{10}, to appear in Comm. Alg.

[4] E. T. Bell, Fifty years of algebra in America, 1888–1938, in Amer. Math. Soc. Semicentennial Publications. Vol. II, 1–34, New York, 1938.

[5] H. Bender, A group theoretic proof of Burnside's p^aq^b theorem, Math. Z., 126 (1972) 327–338.

[6] N. Biggs, Finite Groups of Automorphisms, Cambridge University Press, Cambridge, 1971.

[7] R. Brauer, On groups whose order contains a prime number to the first power I, Amer. J. Math., 64 (1942) 401–420.

[8] ———, On some conjectures concerning finite simple groups, in Studies in Mathematical Analysis and Related Topics, 56–61, Stanford U. Press, Palo Alto, 1962.

[9] ———, On simple groups of order $5\ 3^a2^b$, Bull. Amer. Math. Soc., 74 (1968) 900–903.

[10] R. Brauer and K. A. Fowler, Groups of even order, Ann. of Math., 62 (1955) 565–583.

[11] R. Brauer and W. F. Reynolds, On a problem of E. Artin, Ann. of Math., 68 (1958) 713–720.

[12] R. Brauer and H. Tuan, On simple groups of finite order, I, Bull. Amer. Math. Soc., 51 (1945) 756–766.

[13] W. Burnside, On a class of groups defined by congruences, Proc. London Math. Soc., 25 (1894) 113–139.

[14] ———, Notes on the theory of groups of finite order, Proc. London Math. Soc., 26 (1895) 191–214.

[15] ———, Notes on the theory of groups of finite order (continued), Proc. London Math. Soc., 26 (1895) 325–338.

[16] ———, On transitive groups of degree n and class $n-1$, Proc. London Math. Soc., 32 (1900) 240–246.

[17] ———, On some properties of groups of odd order, Proc. London Math. Soc., 33 (1901) 162–185.

[18] ———, On some properties of groups of odd order (second paper), Proc. London Math. Soc., 33 (1901) 257–268.

[19] ———, On groups of order $p^\alpha q^\beta$, Proc. London Math. Soc., 2 (1904) 388–392.

[20] ———, Theory of Groups of Finite Order, 2nd ed., Dover, New York, 1955.

[21] R. Carter, Simple Groups of Lie Type, Wiley, London, 1972.

[22] C. Chevalley, Sur certains groupes simples, Tohoku Math. J., 7 (1955) 14–66.

[23] F. Cole, Simple groups from order 201 to order 500, Amer. J. Math., 14 (1892) 378–388.

[24] ———, Simple groups as far as order 660, Amer. J. Math., 15 (1893) 303–315.

[25] ———, List of the transitive substitution groups of ten and of eleven letters, Quart. J. Pure Appl. Math., 27 (1895) 39–50.

[26] ———, On simple groups of low order, Bull. Amer. Math. Soc., 30 (1924) 489–492.

[27] G. Cornell, N. Pelc, M. Wage, Simple groups of order less than 1000, J. of Undergraduate Mathematics, 5 (1973) 77–86.

[28] C. W. Curtis, The classical groups as a source of algebraic problems, Amer. Math. Monthly, 74 (1967) 80–91.

[29] L. E. Dickson, The analytic representation of substitutions on a power of a prime number of letters with a discussion of the linear group, Ann. of Math., 11 (1897) 161–183.

[30] ———, Proof of the non-isomorphism of the simple Abelian group on $2m$ indices and the orthogonal group on $2m+1$ indices for $m>2$, Quart. J. Pure Appl. Math., 32 (1900) 42–63.

[31] ———, Theory of linear groups in an arbitrary field, Trans. Amer. Math. Soc., 2 (1901) 363–394.

[32] ———, A new system of simple groups, Math. Ann., 60 (1905) 137–150.

[33] ———, Linear Groups with an Exposition of the Galois Field Theory, Dover, New York, 1958.

[34] J. Dieudonné, Sur les Groupes Classiques, Actualités Sci. Indust., No. 1040, Hermann, Paris, 1948.

[35] ———, On the automorphisms of the classical groups, Mem. Amer. Math. Soc., 2 (1951).

[36] W. Feit, The current situations in the theory of finite simple groups, Proc. Internat. Congr. Mathematicians (Nice, 1970), Gauthier-Villars, Paris, 1971.

[37] W. Feit, M. Hall, Jr., J. G. Thompson, Finite groups in which the centralizer of any non-identity element is nilpotent, Math. Z., 74 (1960) 1–17.
[38] W. Feit and J. G. Thompson, Solvability of groups of odd order, Pacific J. Math., 13 (1963) 775–1029.
[39] P. Fong, private communication.
[40] G. Frobenius, Über Gruppen des Grades p oder $p+1$, Berliner Sitzgsb., (1902) 351–369.
[41] E. Galois, Sur la théorie des nombres, J. Math. Pures Appl., 11 (1846) 398–407.
[42] G. Glauberman, Subgroups of finite groups, Bull. Amer. Math. Soc., 73 (1967) 1–12.
[43] D. Goldschmidt, A group theoretic proof of the $p^a q^b$ theorem for odd primes, Math. Z., 113 (1970) 373–375.
[44] ———, Elements of order two in finite groups, Delta, 4 (1974) 45–58.
[45] D. Gorenstein, Some topics in the theory of finite groups, Rend. Mat. e Appl., (5) 23 (1964) 298–315.
[46] ———, Finite Groups, Harper and Row, New York, 1968.
[47] ———, Finite simple groups and their classification, Israel J. Math., 19 (1974) 5–66.
[48] D. Gorenstein and J. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups, J. Algebra, 2 (1965) 85–151, 218–270, 354–393.
[49] M. Hall, A search for simple groups of order less than one million, Computational Problems in Abstract Algebra (John Leech, Ed.), 137–168, Pergamon Press, New York, 1969.
[50] ———, Simple groups of order less than one million, J. Algebra, 20 (1972) 98–102.
[51] T. Hawkins, The origins of the theory of group characters, Archive Hist. Exact Sci., 7 (1971) 142–170.
[52] ———, New light on Frobenius' creation of the theory of group characters, Archive Hist. Exact Sci., 12 (1974) 215–243.
[53] D. Held, A characterization of the alternating groups of degrees eight and nine, J. Algebra, 7 (1967) 218–237.
[54] ———, The simple group related to M_{24}, J. Algebra, 13 (1969), 253–296.
[55] I. N. Herstein, Topics in Algebra, 2nd ed., Xerox College, Lexington, Mass., 1975.
[56] O. Hölder, Die einfachen Gruppen im ersten und zweiten Hundert der Ordnungszahlen. Math. Ann., 40 (1892) 55–88.
[57] Z. Janko, A new finite simple group with Abelian Sylow 2-subgroups and its characterization, J. Algebra, 3 (1966) 147–186.
[58] C. Jordan, Traité des Substitutions, Gauthier-Villars, Paris, 1870.
[59] R. Lindberg and J. Robinson, A project in simple group theory, Senior paper, U. of Minn., Duluth, 1973.
[60] G. Ling and G. A. Miller, Proof that there is no simple group whose order lies between 1092 and 2001, Amer. J. Math., 22 (1900) 13–26.
[61] H. Matsuyama, Solvability of groups of order $2^a p^b$, Osaka J. Math., 10 (1973) 375–378.
[62] E. Michaels, A study of simple groups of even order, Ph.D. dissertation, U. of Notre Dame, 1963.
[63] G. A. Miller, On the simple groups which can be represented as substitution groups that contain cyclical substitutions of a prime degree, Amer. Math. Monthly, 6 (1899) 102–103.
[64] ———, Sur plusieurs groupes simples, Bull. Soc. Math. France, 28 (1900) 266–267.
[65] ———, The simple group of order 2520, Bull. Amer. Math. Soc., 22 (1922) 98–102.
[66] E. H. Moore, A doubly-infinite system of simple groups, Bull. New York Math. Soc., 3 (1893) 73–78.
[67] M. B. Powell and G. Higman, editors, Finite Simple Groups, Academic Press, London, 1971.
[68] R. Ree, A family of simple groups associated with the simple Lie algebra of type (F_4), Amer. J. Math., 83 (1961) 401–420.
[69] ———, A family of simple groups associated with the simple Lie algebra of type (G_2), Amer. J. Math., 83 (1961) 432–462.
[70] I. M. Schottenfels, Two non-isomorphic simple groups of the same order 20,160, Ann. of Math., 2nd series, 1 (1900) 147–152.
[71] L. Siceloff, Simple groups from order 2001 to order 3640, Amer. J. Math., 34 (1912) 361–372.
[72] M. Suzuki, The nonexistence of a certain type of simple groups of odd order, Proc. Amer. Math. Soc., 8 (1957) 686–695.
[73] ———, A new type of simple groups of finite order, Proc. Nat. Acad. Sci. U.S.A., 46 (1960) 868–870.
[74] P. Telega, A computer project in simple group theory, Senior paper, U. of Minn., Duluth, 1975.
[75] J. G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable, Bull. Amer. Math. Soc., 74 (1968) 383–437; Pacific J. Math., 33 (1970) 451–537; Pacific J. Math., 39 (1971) 483–534; Pacific J. Math., 48 (1973) 511–592; Pacific J. Math., 50 (1974) 215–297; Pacific J. Math., 51 (1974) 573–630.
[76] D. Wales, Simple groups of order $13 \cdot 3^a 2^b$, J. Algebra, 20 (1972) 124–143.
[77] J. H. Walter, The characterization of finite groups with abelian Sylow 2-subgroups, Ann. of Math., 89 (1969) 405–514.
[78] H. Wielandt, Finite Permutation Groups, Academic Press, New York, 1964.
[79] W. J. Wong, Recent work on finite simple groups, Math. Chronicle, 1 (1969) 5–12.