1. The English alphabet contains 21 consonants and five vowels. How many strings of four lowercase letters of the English alphabet contain:

   (a) A vowel in position 2?

   Solution: $5 \cdot 26^3$

   (b) Vowels in positions 2 and 3?

   Solution: $5^2 \cdot 26^2$

   (c) At least one consonant?

   Solution: Count the complement: 5^4 strings are made of only vowels so we have $26^4 - 5^4$.

   (d) No vowels in consecutive positions? ('have' is OK, 'beat' is not)

   Solution: I tend to do this by cases, based on how many vowels we have. If no vowels, 21^4. If one vowel, $4 \cdot 5 \cdot 21^3$, and if two vowels, they can be in positions 1, 3 or 1, 4 or 2, 4 so we get $3 \cdot 5^2 \cdot 21^2$.

2. A bitstring question.

   (a) How many bitstrings of length 10 are there?

   Solution: 2^{10}

   (b) How many bitstrings of length 10 contain exactly 3 ones?

   Solution: $\binom{10}{3}$

   (c) How many bitstrings of length 10 contain exactly 3 ones if exactly one of those ones is in an even position?

   Solution: $\binom{5}{3}$

   (d) How many bitstrings of length 10 contain exactly 3 ones if none of the ones are in consecutive positions?

   Solution: View the problem as putting 7 balls (the 0's in the bitstring) in 4 boxes (the positions relative to the 3 1's). First, we put a ball between each of the 1's, which leaves 5 balls for the 4 positions giving $\binom{5+4-1}{4-1} = \binom{8}{3}$.

3. Arrangements of the word **RECURRENCE**:

(a) How many arrangements are there?

Solution:
$$\frac{10!}{3!3!2!}$$

(b) How many arrangements have the three R's in consecutive positions?

Solution: A common trick is to view RRR as a single symbol, which leaves 8 symbols (RRR, E, E, E, C, C, U, N) which can be distributed in $\dfrac{8!}{3!2!}$ ways.

(c) How many arrangements have no consecutive R's?

Solution: View the R's as separating the other 7 letters into 4 boxes (as with the bit string problem). We want (at least) a letter between each pair of R's, and this can be done in $\binom{8}{3}$ ways. But we still have to arrange the 7 letters. They still have repeats (three E's and two C's) so we get $\dbinom{8}{3}\dfrac{7!}{3!2!}$.

(d) How many arrangements have exactly two consecutive R's?

Solution: This is a "count the complement" problem. The cases where there are not exactly two consecutive R's are (1) all three R's together, or all three R's separate, and these are disjoint. Thus, the answer is

$$\frac{10!}{3!3!2!} - \frac{8!}{3!2!} - \binom{8}{3}\frac{7!}{3!2!}$$

4. Some binomial theorem questions:

(a) Find the coefficient of x^{17} in $(2x-3)^{40}$.

Solution:
$$(2x-3)^{40} = \sum_{k=0}^{40} \binom{40}{k} (2x)^{40-k}(-3)^k.$$

We want the coefficient of x^{17} so we need $40-k=17$, or $k=23$. The coefficient will be $\dbinom{40}{23}2^{17}(-3)^{23} = -\dbinom{40}{23}2^{17}3^{23}$.

(b) Evaluate $\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} 5^k$.

> **Solution:** This is a straight binomial theorem, we get $(2+5)^n = 7^n$.

(c) Evaluate $\sum_{k=0}^{n} \binom{n}{k} k$.

> **Solution:** If we differentiate $(1+x)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k$ we have $n(1+x)^{n-1} = \displaystyle\sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$. Plugging in $x = 1$ gives $\displaystyle\sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}$.

(d) Evaluate $\sum_{k=0}^{n} \binom{n}{k} k 2^k$.

> **Solution:** This is essentially the same as the previous problem. If we plug in $x = 2$ to the differentiated binomial theorem, we have $\displaystyle\sum_{k=0}^{n} \binom{n}{k} k 2^{k-1} = n3^{n-1}$. To get the form we want, we multiply by 2, so $\displaystyle\sum_{k=0}^{n} \binom{n}{k} k 2^k = 2n3^{n-1}$.

(e) Evaluate $\sum_{k=0}^{n} \binom{n}{k} k(-2)^k$.

> **Solution:** Plugging $x = -2$ into the differentiated binomial theorem gives $\displaystyle\sum_{k=0}^{n} \binom{n}{k} k(-2)^{k-1} = n(-1)^{n-1}$ so $\displaystyle\sum_{k=0}^{n} \binom{n}{k} k(-2)^k = 2n(-1)^n$.

5. Some pigeonhole problems:

(a) Show that any subset of $n+1$ integers between 2 and $2n$ (where $n \geq 2$) always has a pair of integers with no common divisors.

> **Solution:** This is just a slightly tricky version of something we have already done. One of the problems I assigned but did not collect was to show that given $n = 1$ integers between 1 and $2n$, there are two consecutive integers. This was almost identical to one I did collect. If you partition the numbers from 1 to $2n$ into n sets $1, 2,\ 3, 4,\ \ldots, 2n-1, 2n$, then given $n+1$ numbers, there must be two in the same partition, and those numbers are consecutive. If we have fewer numbers to partition, from 2 to $2n$ instead, then obviously, there are still two consecutive numbers. Finally, two consecutive integers don't have any common divisors (if d divides m and $m+1$, then it must divide their difference, and the only divisor of 1 is 1.)

(b) In a party of 20 people, suppose that there are exactly 48 pairs of people who know each other. Show that someone has 4 or fewer acquaintances.

> **Solution:** This requires one fact: The total number of pairs of people who know each other is half of the sum of the number of acquaintances each person has. This is sometimes called the "handshaking lemma," that if a bunch of people shake hands, the total number of handshakes is half the sum of the number each person had. The reason: each handshake involves two people.
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> With this, if everyone at the party has 5 or more acquaintances, then the number of pairs of people who know each other is at least $\frac{1}{2}(5)(20) = 50 > 48$.

(c) Show that any set of n integers with $n \geq 3$ has a pair of integers whose difference is divisible by $n - 1$.

> **Solution:** Given n integers, if we divide by $n - 1$ we have n remainders, but only $n-1$ possible distinct remainders. Thus, two numbers must have the same remainder when divided by $n - 1$, so the difference of these two numbers is divisible by $n - 1$.

(d) If the numbers 1 through 10 are arranged randomly in a circle, show that there are three in a row that sum to at least 17.

> **Solution:** I don't have an eligent solution to this problem. **I will give extra credit to anyone ho gives me what I consider a really nice proof.** Let's try to avoid a sum of 17. Consider the possible things that might be grouped with 10. Since things are arranged in a circle, there the the two things on each side of it: $x_1, x_2, 10, x_3, x_4$. If and of the x's is 6, 7, 8, 9, then there will be a sum of at least 17 (the 10, the 6, and the third number which must be at least 1.) To avoid a sum of 17, none of the x's is 6, 7, 8 or 9. Then these four numbers must be among the other five in a row opposite the 10. If any three of them are in a row, the sum is at least 17, so we consider the configuration L, L, S, L, L, where L means large (one of 6, 7, 8, 9) and S is something small. If 9 is grouped with 7 or 8, we get a sum of 17 (with the third number) so 9 must be in one of the two end positions, with 7 and 8 on the other end. This leads to configurations such as 9, 6, S, 8, 7. Finally, if we take, say the 8 and 7, then there are things on both sides of them: $y, 8, 7, x$ leading to two triples $y + 8 + 7$ and $8 + 7 + x$. But at most one of x and y can be 1 so the other must be at least 2 giving a sum of at least $2 + 7 + 8 = 17$.

(e) If there are 100 people at a party and each person knows an even number of people (possibly zero) show that there are three people with the same number of acquaintances.

> **Solution:** This is similar to a homework problem. There are 50 possibilities for the number of acquaintances a person can have: 0, 2, 4, ..., 98. If three people know no one, we are done. Suppose that exactly two people know no one. Any other person can't know these two people or themselves, so no one can know 98 people. This leaves 98 people with possible number of acquaintances 2, 4, 6, ..., 96. That is, there are 48 possible number of acquaintances for the 98 people. Since $98 > 2 \cdot 48$ there must be three people with the same number of acquaintances.
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> Finally, if there is no one with 0 acquaintances, or exactly one person with 0 acquaintances, then there are at least 99 people, with 49 possible numbers of acquaintances (evens 2 through 98), and again there must be at least $\left\lfloor \dfrac{99}{49} \right\rfloor = 3$ people with some number of acquaintances.

6. Show that the sequence $\binom{n}{0}, 5\binom{n}{1}, 5^2\binom{n}{2}, \ldots$ is unimodal and find the largest term.

> **Solution:** I was unhappy with my treatment of this in class. In class, if the sequence was $a_0, a_1, \ldots$, I looked at the ratio $\dfrac{a_{k+1}}{a_k}$. When this is larger than 1, $a_{k+1} > a_k$. The problem with this approach is that if you want the largest a_k, it occurs one spot after the ratio becomes less than 1. That is, if $\frac{a_{k+1}}{a_k} > 1$ but $\frac{a_{k+2}}{a_{k+1}} < 1$, then the maximum will occur at a_{k+1}. This can be confusing. To make things less confusing, let's use the ratio $\frac{a_k}{a_{k-1}}$ instead.
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> For this problem, $a_k = 5^k \dbinom{n}{k}$. We have
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> $$\frac{a_k}{a_{k-1}} = \frac{5^k\binom{n}{k}}{5^{k-1}\binom{n}{k-1}} = \frac{5^k \frac{n!}{k!(n-k)!}}{5^{k-1}\frac{n!}{(k-1)!(n-k+1)!}} = \frac{5(k-1)!(n-k+1)!}{k!(n-k)!} = \frac{5(n-k+1)}{k}.$$
>
> We compare $\frac{5(n-k+1)}{k}$ to 1. When it is larger, $a_k > a_{k-1}$, and when it is smaller than 1, $a_k < a_{k-1}$. Now $\frac{5(n-k+1)}{k} - 1 = \frac{5n-6k+5}{k}$, so terms increase while $5n - 6k + 5 > 0$ or $k < \frac{5n+5}{6}$, and the terms decrease when $k > \frac{5n+5}{6}$. This shows that the sequence is unimodal and the largest term is the largest k with $k < \frac{5n+5}{6}$ so the largest term will have $k = \left\lfloor \dfrac{5n+5}{6} \right\rfloor$.

7. Give a combinatorial proof for each of the following.

(a) $n^3 = (n-1)^3 + 3(n-1)^2 + 3(n-1) + 1$ Hint: Count strings xyz where each of x, y, z is an integer between 1 and n. For the right hand side, break into cases based on how many 1's there are in the string.

> **Solution:** Following the hint, if we count strings xyz where each term is between 1 and n, we get $n \cdot n \cdot n = n^3$ such strings.
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> Next, we count these in a different way, based on the number of 1's in the string. If there are no 1's, then there are only $n-1$ choices each for x, y, z, giving $(n-1)^3$ strings of this type. If there is one 1, it can be placed in any of three positions and there are $n-1$ choices for the other variables for a count of $3(n-1)^2$. If there are two 1's, they can still be placed in 3 ways, with the third variable between 2 and n for $n-1$ strings. Finally if x, y, z are all 1, there is only one such string (111). Adding the cases gives $(n-1)^3 + 3(n-1)^2 + 3(n-1) + 1$, and this has to equal the simpler count so $n^3 = (n-1)^3 + 3(n-1)^2 + 3(n-1) + 1$.

(b) $\binom{3n}{2} = \binom{n}{2} + 2n^2 + \binom{2n}{2}$ Hint: Pick two numbers between 1 and $3n$. For the right hand side, use cases based on how many of the numbers are divisible by 3. If you like committee models, you could count how many committees of size 2 could be formed from n cats and $2n$ dogs.

> **Solution:** I will use the committee approach. We want a committee of two animals, selecting them from a total of n cats and $2n$ dogs. If we ignore species, there are $3n$ animals and we want 2 for a count of $\binom{3n}{2}$.
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> A second count uses cases based on the species. There are three cases: no dogs, 1 dog, or two dogs. In the first case, both animals are cats, which can be selected in $\binom{n}{2}$ ways. If we have a cat and a dog, we have n choices for the cat and $2n$ choices for the dog for $2n^2$ committees. Finally, if both are dogs, they can be selected in $\binom{2n}{2}$ ways. The total for this approach is $\binom{n}{2} + 2n^2 + \binom{2n}{2}$, and this gives the same answer as the simpler approach so $\binom{3n}{2} = \binom{n}{2} + 2n^2 + \binom{2n}{2}$.

(c) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$. One way: count strings of length n made of a's, b's, c's in two ways. One way: based on how many a's the string contains.

> **Solution:** I will follow the hint. First, there are 3^n sequences of length n made up of a's, b's and c's. We count these sequences in another way based on how many a's there are. The number of a's can be 0 or 1 or 2 or …or n. I will let the generic case be $n-k$ a's so the count comes out nicest. In this case, select $n-k$ places to put the a's, which can be done in $\binom{n}{n-k} = \binom{n}{k}$ ways. Next, fill

in the k remaining spaces using b's or c's. Each such space can be filled in 2 ways, giving 2^k ways to place the b's, c's. All totaled, this case contributes $\binom{n}{k}2^k$ strings. Summing over all k gives $\displaystyle\sum_{k=0}^{n}\binom{n}{k}2^k = 3^n$.

A better approach might have been to count how many letters are **not** a's. That is, suppose that k of the positions don't have a's in them. Select which k positions don't have a's, and then for each position, determine whether it will be a b or a c.

(d) $\displaystyle\sum_{k=0}^{n} k\binom{n}{k}2^k = 2n3^{n-1}$. One way: count strings of length n made of a's, b's, c's in two ways. One way: based on how many a's the string contains.

Solution: Sorry I gave exactly the same hint as in part (c). This has to be a little different since it is a different identity. Here is one way to handle the difference: we want the string to contain a capital B or a capital C. That is, exactly one of the b's or c's in the string is capitalized. Here is the simple count: Pick one of n positions to stick the capital letter, decide whether it should be a b or a c, and then fill the remaining $n-1$ positions with a's, b's and c's. This gives the right hand count of $2n3^{n-1}$.

For the more complicated count, I'll use the alternative approach above. Suppose that exactly k positions in the string are not a's. Select these positions in $\binom{n}{k}$ ways and place a's in all the **other** positions. For these k positions, one of them contains a capital letter, and that letter is either B or C for a count of $2k$. The remaining $k-1$ positions get either a b or a c. Thus the case of k spots for b's and c's has a total count of $\binom{n}{k}2k2^{k-1} = k\binom{n}{k}2^k$. Summing over all possible values of k gives the identity.

8. Use binomial coefficient identities to solve the following problems.

(a) Use binomial coefficients to find a formula for $1\cdot 2 + 2\cdot 3 + \cdots + n(n+1)$.

Solution: Since $k(k+1) = 2\binom{k+1}{2}$, our sum is

$$2\left(\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n+1}{2}\right) = 2\binom{n+2}{3}.$$

This follows from a formula I gave in class:

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Thus, the sum is $2\dfrac{(n+2)(n+1)n}{6} = \dfrac{n(n+1)(n+2)}{3}$.

(b) Use binomial coefficients to find a formula for $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + n(n+1)(n+2)$.

Solution: This is the same kind of problem as part (a). Here, $k(k+1)(k+2) = 3!\binom{k+2}{3}$. Using the formula from class, with $k = 3$ and replacing n by $n+2$, we get a sum of $6\dbinom{n+3}{4} = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

(c) Write $\binom{10}{0}\binom{5}{0} + \binom{10}{1}\binom{5}{1} + \binom{10}{2}\binom{5}{2} + \binom{10}{3}\binom{5}{3} + \binom{10}{4}\binom{5}{4} + \binom{10}{5}\binom{5}{5}$ as a single binomial coefficient. Generalize for $\binom{2n}{0}\binom{n}{0} + \cdots + \binom{2n}{n}\binom{n}{n}$.

Solution: In general,

$$\begin{aligned}\binom{2n}{0}\binom{n}{0} &+ \binom{2n}{1}\binom{n}{1} + \cdots + \binom{2n}{n}\binom{n}{n} \\ &= \binom{2n}{0}\binom{n}{n} + \binom{2n}{1}\binom{n}{n-1} + \cdots + \binom{2n}{n}\binom{n}{0},\end{aligned}$$

and by the Chu-Vandermonde convolution, this sums to $\dbinom{3n}{n}$. The original sum evaluates to $\binom{15}{5}$.

(d) Evaluate $\displaystyle\sum_{k=0}^{n} k^2 \binom{n}{k}$.

Solution: I'm not sure why I put this here and not as part of problem 4. Here is a problem 4-style solution. Start with the binomial theorem and differentiate it to get $n(1+x)^{n-1} = \displaystyle\sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$. One way to get another factor of k is to multiply by x, so $\displaystyle\sum_{k=0}^{n} \binom{n}{k} k x^{k} = nx(1+x)^{n-1}$. Now we differentiate again to get $\displaystyle\sum_{k=0}^{n} \binom{n}{k} k^2 x^{k-1} = n(1+x)^{n-1} + n(n-1)x(x+1)^{n-2}$. Finally, setting $n = 1$, $\displaystyle\sum_{k=0}^{n} \binom{n}{k} k^2 = n2^{n-1} + n(n-1)2^{n-2} = n(n+1)2^{n-2}$.