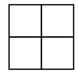
Math 5366
Fall 2017

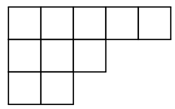
Some notes on rook polynomials

Rook Polynomials

A rook polynomial is the generating function for the number of ways to put non attacking rooks on a generalized board. Here, board tends to be used very broadly. As an example, suppose we have the board



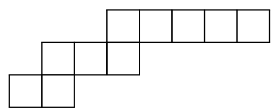
We say we can put no rooks on a board in 1 way. In this case, one rook can be put any where, and there are exactly two ways to place two rooks on the board. The rook polynomial is $1+4x+2x^2$. We let r_0 be the number of ways to place two rooks. This number is always 1. We let r_k be the number of ways to place k rooks. The rook polynomial is $r_0 + xr_1 + x^2r_2 + \cdots$. This looks like an infinite series but it only has finitely many terms, making it a polynomial, since there can't be more rooks than rows or columns in the board. We don't need a square board. For example, if the board looks like



$r_0 = 1,\ r_1 = 10$. For two rooks, use cases. If there is a rook in the bottom row, there are 6 ways to place the second rook, for 12 total. If there is no rook in the bottom row, there are $3 \cdot 4 = 12$ ways to place the two rooks, so $r_2 = 24$. For three rooks we have $r_3 = 2 \cdot 2 \cdot 3 = 12$ so our polynomial is $1 + 10x + 24x^2 + 12x^3$.

Boards can be more exotic, like

B :



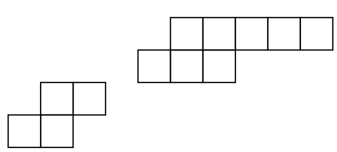
(1)

This still has 10 squares with rows of length 5, 3, 2. But now the rook polynomial is $1+10x^2+29x^2+23x^3$. Here is how I got this: If there are two rooks, there are $2\cdot 8+3\cdot 5 = 31$ ways to place them in different rows. Of these 31, two are not allowed because two rooks would be in the same column. For three rooks, I counted based on the rook in the middle row: $5 + 10 + 8 = 23$, with, for example, 8 ways to place the upper and lower rooks if the middle rook is in the right most square. In general, if B is a board we denote the rook polynomial for B as $R(x, B)$.

There are some recursive schemes for getting rook polynomials for a large board from smaller boards. First, if the large board is the union of smaller boards which do not share any rows or columns, then the rook polynomial for the large board is the product of the smaller polynomials. Symbolically, $R(x, B_1 \cup B_2) = R(x, B_1)R(x, B_2)$, when B_1 and B_2 do not share a row or column. For example, with

B_1

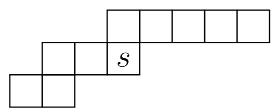
B_2



the rook polynomial is $R(x, B_1 \cup B_2) = R(x, B_1)R(x, B_2) = (1 + 4x + 3x^2)(1 + 8x + 13x^2) = 1 + 12x + 48x^2 + 76x^3 + 39x^4$.

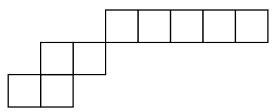
Another trick is the following: given a board B, and some square s on that board, there either is a rook on that square or there is not. If there is not, we need the rook polynomial for the board with s deleted. If there is a rook on s, then there can not be another rook on the same row or column as s. Let B_s be the board where s is deleted and B_s^*, the board were s's row and column are removed from B. Then $R(x, B) = R(x, B_s) + xR(x, B_s^*)$. The x in front of $R(x, B_s^*)$ is because we have to count the rook in square s. For example, going back to the board in (1), let s be the rightmost square in the second row:

B :

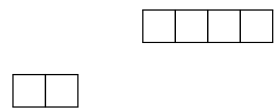


Then

$B_s =$



$B_s^* =$



So

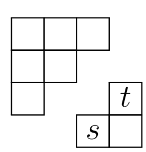
$$
\begin{aligned}
R(x, B) = R(x, B_s) + xR(x, B_s^*) &= (1 + 5x)(1 + 4x + 3x^2) + x(1 + 4x)(1 + 2x) \\
&= 1 + 9x + 23x^2 + 15x^3 + \quad x + 6x^2 + 8x^3 \\
&= 1 + 10x + 29x^2 + 23x^3,
\end{aligned}
$$

as before.

Applications to permutations with forbidden positions

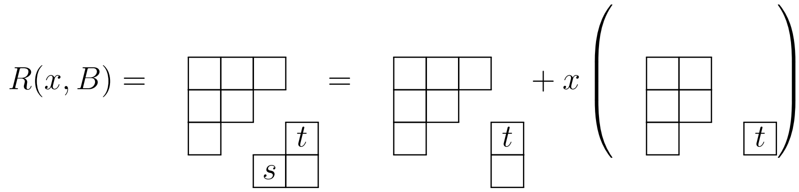
Suppose we want to know now many permutations of 1, 2, 3, 4 there are, subject to some conditions: 1 is not in position 4, 2 is not in positions 3 or 4, 3 is not in positions 2 or 3, and 4 is not in positions 1 or 2. Permutations are like non attacking rooks, so this is the same as the number of ways to place 4 rooks on a 4×4 board, but with certain positions not allowed. The board would look like this:

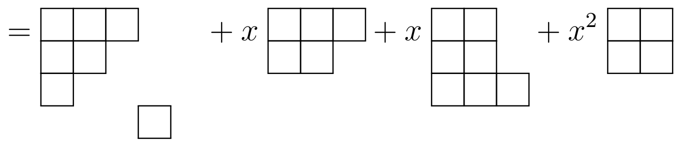
B :



(2)

We can find the rook polynomial of B by using squares s and t: By an abuse of notation we have

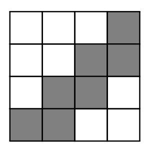




$$
\begin{aligned}
&= (1+x)(1+6x+7x^2+x^3) + 2x(1+5x+4x^2) + x^2(1+4x+2x^2) \\
&= 1 + 9x + 24x^2 + 20x^3 + 3x^4.
\end{aligned}
$$

The answer to the question is $r_4 = 3$.

Rather than depicting the board as in (2), we usually shade in the squares a rook is not allowed to go on, squares called forbidden positions. In this case, this would look like



Some comments on such problems: First, there is a lot of symmetry. The diagram can be rotated 90, or 180, or the rows or columns put in different orders, and the rook polynomial remains unchanged. Often this is done so as to get the nicest looking diagram to look at and analyze. Second, we only want the largest coefficient of the rook polynomial in these problems, and it turns out there is a second approach to getting this coefficient.

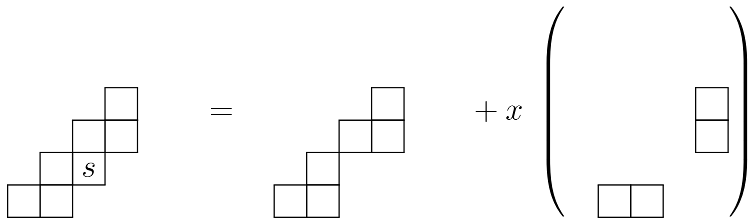
Theorem 1 *The number of ways to place n non attacking rooks on an $n \times n$ board with forbidden positions is*

$$n! - r_1(n-1)! + r_2(n-2)! - + \cdots + (-1)^n r_n 0!,$$

where the numbers r_k are the coefficients of the rook polynomial for ***the forbidden positions****.*

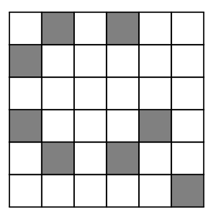
Proof: Let A_1 be the number of ways to place a rook in a forbidden square in the first row, A_2 the number of ways to place a rook in a forbidden square in the second row, and so on. Then we want $|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = |U| - S_1 + S_2 - + \cdots + (-1)^n S_n$, where S_k is the sum of the sizes of the k-fold intersections of the conditions A_i. Now $|U| = n!$, and for S_k, we need the number of ways to place k rooks in forbidden positions, with no restrictions on the remaining $n-k$ rooks, except that they can't be in the same row or column as any other rook. This means that $S_k = r_k(n-k)!$, where r_k is the number of ways to put k non attacking rooks on the forbidden squares. ■

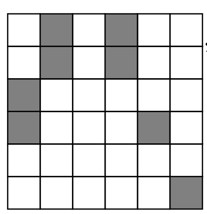
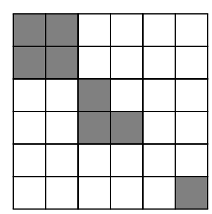
I will give two examples. First, for the example in (2), rather than finding the rook polynomial for the good squares as we originally did, we find the polynomial for the forbidden squares, and use the theorem. We use



which gives $R(x,B) = (1+3x+x^2)^2 + x(1+2x)^2 = 1+7x+15x^2+10x^3+x^4$. By the theorem, the number we want is $4! - 7 \cdot 3! + 15 \cdot 2! - 10 \cdot 1! + 1 \cdot 0! = 3$. The theorem is advantageous when the board of forbidden positions is small compared to the board of allowed positions.

For a second example, consider permutations of 6 where 1 is not in positions 2, 4; 2 is not in position 1; 4 is not in positions 1 or 5, 5 is not in positions 2 or 4; and 6 is not in position 6. The board for this is



Rearrange this to , and by rearranging columns to 

The rook polynomial for the forbidden squares is $(1+4x+2x^2)(1+3x+x^2)(1+x) = 1+8x+22x^2+25x^3+12x^4+2x^5$ so the number of allowed permutations is

$$6! - 8 \cdot 5! + 22 \cdot 4! - 25 \cdot 3! + 12 \cdot 2! - 2 \cdot 1! + 0 \cdot 0! = 160.$$