Math 4326
Fall 2017

Extending an independent set to a basis: examples

I thought I would give some examples of extending independent sets to a basis. For a first example, suppose we wish to extend $\{t^3+t+1, t^4+2t^2+t\}$ to a basis for P_4. The usual method on a problem like this: Extend the set to a spanning set by appending the vectors from a known spanning set, and then remove dependent vectors, making sure to keep the ones in the original set. In a case like this, it is simplest to append the vectors for the standard basis for P_4 to get the spanning set $\{t^3+t+1, t^4+2t^2+t,\ 1,\ t,\ t^2,\ t^3,\ t^4\}$. We may also use the fact that every basis for P_4 consists of five vectors, so any 5-vector spanning set must be a basis. Thus, we must find two dependence relations to remove vectors. The great thing about a standard basis, is that it is easy to take any vector and write it as a combination of standard basis vectors. We make use of that fact.

Let's label the vectors in our spanning set $v_1, v_2, v_3, v_4, v_5, v_6$ and v_7. One dependence relation is $v_1 = v_3 + v_4 + v_6$. We want to keep v_1, but that's ok. We can view any vector in a dependence relation as depending on the others. So we can pick any of v_3, v_4, v_6 to remove. That is, we can rewrite things, say, $v_4 = v_1 - v_3 - v_6$ or $v_6 = v_1 - v_3 - v_4$, etc. I usually just select the largest index, so I will remove v_6 from our set. Next, $v_2 = v_4 + 2v_5 + v_7$, and so v_7 can also be removed. Our basis is $\{v_1, v_2, v_3, v_4, v_5\} = \{t^3+t+1, t^4+2t^2+t,\ 1,\ t,\ t^2\}$.

It is worth noting here that which vector you remove can make a difference. For example, suppose we remove v_4 at the first stage. Then v_4 can't be used in a linear dependence relation at the next stage. But $\{v_1, v_2, v_3, v_5, v_6, v_7\} = \{t^3+t+1, t^4+2t^2+t,\ 1,\ t^2,\ t^3,\ t^4\}$ must be dependent. I found the following dependence by inspection (looking at it, an answer came to me). I got $v_2 - v_1 = -v_3 + 2v_5 - v_6 + v_7$. That is, subtracting v_1 from v_2 got rid of the coefficient of t and we had all the other powers of t to work with. Thus, we can get rid of any of v_3, v_5, v_6, v_7. If we get rid of v_6 just to be different, we would have a basis $\{v_1, v_2, v_3, v_5, v_7\} = \{t^3+t+1, t^4+2t^2+t,\ 1,\ t^2,\ t^4\}$.

What if we can't just spot dependence relations? For example, in a vector space without a standard basis there might not be any obvious ways to get dependence relations. So let's do this problem without using obvious dependencies. In this case, we set up a generic linear combination of the seven vectors, set that equal to 0, and use that to get a system of equations to help us out. I dislike subscripts so I will use the beginning of the alphabet for my scalars:

$$a(t^3+t+1) + b(t^4+2t^2+t) + c\cdot 1 + dt + et^2 + ft^3 + gt^4 = 0. \tag{1}$$

Rewrite this as a polynomial:

$$(b+g)t^4 + (a+f)t^3 + (2b+e)t^2 + (a+b+d)t + (a+c) = 0,$$

and to be 0, a polynomial must have all coefficients equal to 0. This gives us our system of equations:

$$a+c=0,\ a+b+d=0,\ 2b+e=0,\ a+f=0,\ b+g=0.$$

We get the coefficient matrix and reduce

$$\begin{pmatrix}1&0&1&0&0&0&0\\1&1&0&1&0&0&0\\0&2&0&0&1&0&0\\1&0&0&0&0&1&0\\0&1&0&0&0&0&1\end{pmatrix} \Rightarrow \begin{pmatrix}1&0&1&0&0&0&0\\0&1&-1&1&0&0&0\\0&2&0&0&1&0&0\\0&0&-1&0&0&1&0\\0&1&0&0&0&0&1\end{pmatrix} \Rightarrow \begin{pmatrix}1&0&1&0&0&0&0\\0&1&0&0&0&0&1\\0&1&-1&1&0&0&0\\0&2&0&0&1&0&0\\0&0&-1&0&0&1&0\end{pmatrix} \Rightarrow$$

$$\begin{pmatrix}1&0&1&0&0&0&0\\0&1&0&0&0&0&1\\0&0&-1&1&0&0&-1\\0&0&0&0&1&0&-2\\0&0&-1&0&0&1&0\end{pmatrix} \Rightarrow \begin{pmatrix}1&0&0&0&0&1&0\\0&1&0&0&0&0&1\\0&0&1&0&0&-1&0\\0&0&0&1&0&-1&-1\\0&0&0&0&1&0&-2\end{pmatrix}$$

This tells us that $a = -f,\ b = -g,\ c = f,\ d = f + g,\ e = 2g$. How does this give us dependence relations? Pick $f = 1,\ g = 0$ and we get $a = -1,\ b = 0,\ c = 1,\ d = 1,\ e = 0,\ f = 1,\ g = 0$. That is, our equation (1) above becomes

$$-1(t^3 + t + 1) + 1 + t + t^3 = 0,$$

or $v_1 = v_3 + v_4 + v_6$, as before. Setting $f = 0,\ g = 1$ gives the other dependence relation.

As a second example, let V be the space of all polynomials in P_3 that satisfy $p(2) = 0$ and $p(3) = 0$. The problem: Find a basis for V, and extend this to a basis for W, the set of polynomials in P_3 that only satisfy the condition $p(3) = 0$.

Finding a basis for V: if $p(x) = ax^3 + bx^2 + cx + d$ then we need $8a + 4b + 2c + d = 0$ and $27a + 9b + 3c + d = 0$. It is easier to use variables with the smallest coefficients as determined variables, so suppose we say a and b are free. Subtracting the first equation from the second, $19a + 5b + c = 0$, so $c = -19a - 5b$ and $d = -2c - 8a - 4b = 30a + 6b$. As an aside, this would happen if we used row reduction, but with columns d, c, b, a : $\begin{pmatrix}1&2&4&8\\1&3&9&27\end{pmatrix} \Rightarrow \begin{pmatrix}1&2&4&8\\0&1&5&19\end{pmatrix} \Rightarrow \begin{pmatrix}1&0&-6&-30\\0&1&5&19\end{pmatrix}$. We have $at^3 + bt^2 + ct + d = at^3 + bt^2 - (19a + 5b)t + (30a + 6b) = a(t^3 - 19t + 30) + b(t^2 - 5t + 6)$. This means that $\{t^3 - 19t + 30, t^2 - 5t + 6\}$ is a basis for V. Now we want to extend this to a basis for W. It turns out that W is 3-dimensional. If we knew this, then **any** three independent vectors in W would form a basis for W, and so the task is to find one extra vector in W, independent of the first two. In fact, $t - 3$ is in W and it is independent of the other two, so $\{t^3 - 19t + 30, t^2 - 5t + 6, t - 3\}$ is such a basis. I will let you check the independence of these vectors. If we don't know W is 3-dimensional, we can try to find a basis for it. Setting $27a + 9b + 4c + d = 0$ and writing $d = -27z - 9b - 3c$ we have that any vector in W can be written $p(t) = at^3 + bt^2 + ct - 27a - 9b - 3c = a(t^3 - 27) + b(t^2 - 9) + c(t - 3)$ so W is 3-dimensional, with basis $\{t^3 - 27, t^2 - 9, t - 3\}$. It turns out that any one of these vectors can be appended to the basis for V to get a new basis for W. If we did not want to use facts about dimension, we could still do this problem: Given our bases $\{t^3 - 19t + 30, t^2 - 5t + 6\}$ for V and $\{t^3 - 27, t^2 - 9, t - 3\}$ for W, the set $\{t^3 - 19t + 30, t^2 - 5t + 6, t^3 - 27, t^2 - 9, t - 3\}$

is a spanning set for W so we just remove dependent vectors, while making sure to keep the first two. If we call these vectors v_1 through v_5, I actually checked (via a system of equations) that $5v_1 - 19v_2 - 5v_3 + 19v_4 = 0$ and $v_1 - v_3 + 19v_5 = 0$ so v_4 and v_5 depend on v_1, v_2, v_3, so these three form a basis.

One final example. Let V be the the set of all 2×2 matrices who's entries add to 0. You should be able to check that this is a vector space. It has many nice bases, but suppose we are told that $\left\{ \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ 3 & -1 \end{pmatrix} \right\}$ is a basis. Our task: to find a basis that contains the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$. We could proceed as follows: form the set containing this and the other three matrices: $\left\{ \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ 3 & -1 \end{pmatrix} \right\}$, and look for a dependence relation (we need to remove one vector because $\dim(V) = 3$). Maybe a dependence relation can be found by inspection, but if not, we can use a system of equation. First, set a generic combination of the four matrices to 0:

$$a \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} + b \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + c \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 1 & -3 \\ 3 & -1 \end{pmatrix} = 0,$$

or $\begin{pmatrix} a+b+c+d & a-b-2c-3d \\ a+b+c+3d & -3a-b-d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Now we have four linear equations, we can form a matrix and reduce:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -2 & -3 \\ 1 & 1 & 1 & 3 \\ -3 & -1 & 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -3 & -4 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 3 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This says that we need $a = \frac{1}{2}c$, $b = -\frac{3}{2}c$ and $d = 0$. If we pick $c = 2$ then our dependence relation is

$$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} - 3 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + 2 \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix} = 0,$$

so we can eliminate either the second or third matrix, but not the fourth. One basis is

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ 3 & -1 \end{pmatrix} \right\}.$$