Math 5327 Homework 11
Fall 2016 **Solutions**

From the book:

Problem 6.1.4 Find the characteristic and minimal polynomials for each of the following matrices. Then use the Cayley-Hamilton theorem to invert each.

(a) $A = \begin{pmatrix} 4 & -5 & 3 \\ 2 & -3 & 2 \\ -1 & 1 & 0 \end{pmatrix}$

Solution:

$$\begin{aligned} c_A(x) &= \begin{vmatrix} x-4 & 5 & -3 \\ -2 & x+3 & -2 \\ 1 & -1 & x \end{vmatrix} = \begin{vmatrix} x+1 & 0 & 5x-3 \\ 0 & x+1 & 2x-2 \\ 1 & -1 & x \end{vmatrix} \\ &= x(x+1)^2 - (x+1)(5x-3) + 2(x+1)(x-1) \\ &= (x+1)(x^2 + x - 5x + 3 + 2x - 2) \\ &= (x+1)(x^2 - 2x + 1) = (x-1)^2(x+1). \end{aligned}$$

$$(A-I)(A+I) = \begin{pmatrix} 3 & -5 & 3 \\ 2 & -4 & 2 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & -5 & 3 \\ 2 & -2 & 2 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 2 \\ 0 & 0 & 0 \\ -2 & 2 & -2 \end{pmatrix} \neq 0$$

so the minimal polynomial must be $(x-1)^2(x+1)$. Note that you can check by multiplying the end matrix above by $A - I$ to see that you get 0.

To get A^{-1}, we multiply out the minimal polynomial to get $x^3 - x^2 - x + 1$. This means $A^3 - A^2 - A + I = 0$, or $A(-A^2 + A + I) = I$. This means $A^{-1} = A + I - A^2 =$
$$\begin{pmatrix} 4 & -5 & 3 \\ 2 & -3 & 2 \\ -1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & -2 & 2 \\ 0 & 1 & 0 \\ -2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ 2 & -3 & 2 \\ 1 & -1 & 2 \end{pmatrix}.$$

(b) $B = \begin{pmatrix} -2 & -6 & -9 \\ 3 & 7 & 9 \\ -1 & -2 & -2 \end{pmatrix}$

Solution:

$$\begin{aligned} c_B(x) &= \begin{vmatrix} x+2 & 6 & 9 \\ -3 & x-7 & -9 \\ 1 & 2 & x+2 \end{vmatrix} = \begin{vmatrix} x-1 & 0 & -3x+3 \\ 0 & x-1 & 3x-3 \\ 1 & 2 & x+2 \end{vmatrix} \\ &= (x-1)^2 \begin{vmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 1 & 2 & x+2 \end{vmatrix} = (x-1)^2(x-1) \\ &= (x-1)^3 \end{aligned}$$

$$(B-I)^2 = \begin{pmatrix} -3 & -6 & -9 \\ 3 & 6 & 9 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} -3 & -6 & -9 \\ 3 & 6 & 9 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so the minimal polynomial is $(x-1)^2 = x^2 - 2x + 1$.

Since $B^2 - 2B + I = 0,\ B^{-1} = 2I - B = \begin{pmatrix} 4 & 6 & 9 \\ -3 & -5 & -9 \\ 1 & 2 & 4 \end{pmatrix}$.

(c) $C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Solution: Trying to be more clever, $C^2 = I$ so the minimal polynomial is $x^2 - 1$. Both 1 and -1 have 2-dimensional eigenspaces so the characteristic polynomial is $(x-1)^2(x+1)^2$.

Since $C^2 - I = 0,\ C^{-1} = C$.

Problem 6.1.6 Suppose that $T : F^{n\times n} \to F^{n\times n}$ is the linear transformation defined by $T(A) = AB$ for some fixed $n \times n$ matrix B. Show that the minimal polynomial of T is the same as the minimal polynomial of B.

Solution: By induction, one can show that $T^k(A) = B^k A$. This means that for any polynomial $p(x),\ p(T)(A) = p(B)A$. Consequently, $p(x)$ annihilates T if and only if $p(x)$ annihilates B. Thus, the minimal polynomial for T is the same as the minimal polynomial for B.

Problem 6.2.2 Suppose that $T_A : F^n \to F^n$ and $T_B : F^n \to F^n$ both have a cyclic vector. Show that the $n \times n$ matrices A and B are similarly if and only if $c_A(x) = c_B(x)$.

Solution: First, similar matrices have the same characteristic polynomial, so if A is similar to B then $c_A(x) = c_B(x)$. Next, suppose that $c_A(x) = c_B(x)$. Since A has a cyclic vector, call it u, if we let $P_1 = (u \,|\, Au \,|\, A^2u \,|\, \cdots \,|\, A^{n-1}u)$ then $P_1^{-1}AP_1 = Q,\ A$'s companion matrix. Similarly, if v is a cyclic vector for B then $P_2 = (v \,|\, Bv \,|\, B^2v \,|\, \cdots \,|\, B^{n-1}v)$ has the property that $P_2^{-1}BP_2$ is B's companion matrix. But A and B have the same companion matrix, Q so $P_1^{-1}AP_1 = P_2^{-1}BP_2$, or $B = P_2P_1^{-1}AP_1P_2^{-1} = (P_1P_2^{-1})^{-1}AP_1P_2^{-1}$. That is, A and B are similar.

Problem 6.2.6 Find two sequences of generalized eigenvectors of maximal length with two distinct eigenvalues for $Q = \begin{pmatrix} 3 & 1 & 0 & 2 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Solution: The eigenvalues of Q are 2 and 3. We start by finding eigenvectors. In each case, the eigenspace is 1-dimensional. For 2, an eigenvector is $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = v$. Next, we want a vector w with $Qw = 2w + v$, which is to say, we need $(Q - 2I)w = v$. The general solution to this is $w = \begin{pmatrix} 6 \\ -7 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$. But this last piece is just cv, which means we can ignore it. That is, $(Q - 2I)v = 0$ so $(Q - 2I)(w + cv) = (Q - 2I)w$ for any c. Can we extend further? We need u with $(Q - 2I)u = w$. We can't extend because this is an inconsistent system (the bottom coordinate causes trouble.)

Next, for eigenvalue 3, by inspection we can see that $Qe_2 = 3e_2 + e_1$, $Qe_1 = 3e_1$. Again, we can't extend this further.

Problem 6.2.8 Determin which of the following matrices have cyclic vectors.

(a) $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

Solution: Given a vector v, we look at the vectors v, Av, A^2v, A^3v. We need to find a vector v for which these 4 are linearly independent in order for a cyclic vector to exist. However, if $v = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, then the sequence is $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, $\begin{pmatrix} b+d \\ a+c \\ b+c \\ a+c \end{pmatrix}$, $2\begin{pmatrix} a+c \\ b+d \\ a+c \\ b+d \end{pmatrix}$, and 4 times the second vector. That is, $A^3v = 4Av$ for all v so there are no cyclic vectors.

(b) $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

> **Solution:** e_1 is a cyclic vector since e_1, Be_1, B^2e_1 are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix}$, a linearly independent collection of vectors.

(c) $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

> **Solution:** If $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $Cv = \begin{pmatrix} a \\ b \\ 2c \end{pmatrix}$, $C^2v = \begin{pmatrix} a \\ b \\ 4c \end{pmatrix}$. Since the third is a combination of the first two, there are no cyclic vectors.

1. Given a vector space V with linear operator $T : V \to V$, the T-cyclic subspace of v is the Span$\{v, T(v), T^2(v), \ldots\}$.

   (a) If V is finite dimensional, prove that there is an n so that the cyclic subspace of v is Span$\{v, T(v), T^2(v), \ldots, T^{n-1}(v)\}$, and this set is linearly independent.

   > **Solution:** For $v = 0$, the subspace consists of just the 0-vector. For $v \neq 0$, let n be the smallest positive integer for which $T^n(v)$ is a combination of v, $T(v), \ldots, T^{n-1}(v)$. We know that $n \leq \dim(V)$. Let
   >
   > $$W = \text{Span}\{v, T(v), T^2(v), \ldots, T^{n-1}(v)\}.$$
   >
   > If we write $T^n(v) = c_{n-1}T^{n-1}(v) + \cdots + c_1T(v) + c_0v$, then this shows $T^n(v) \in W$. Now $T^{n+1}(v) = T(T^n(v)) = c_{n-1}T^n(v) + \cdots + c_1T^2(v) + c_0T(v)$, and $T(v), \ldots, T^n(v)$ are in W so $T^{n+1}(v)$ is also in W. An easy induction shows that $T^k(v)$ is in W for all positive integers k, showing that $W = \text{Span}\{v, T(v), T^2(v), \ldots\}$, as desired.

   (b) Let W be the cyclic subspace of v. If we define S by $S(w) = T(w)$ for all $w \in W$, show that S is a linear operator on W. (Technically, based on the definition $S : W \to V$ rather than $S : W \to W$.)

   > **Solution:** This is asking us to show that the range of S is a subspace of W. If W is n-dimensional then for any vector $w \in W$, $w = c_0v + c_1T(v) + \cdots + c_{n-1}T^{n-1}(v)$, for some scalars $c_0, \ldots, c_{n-1}$. Now $S(w) = T(w) = c_0T(v) + \cdots + c_{n-1}T^n(v) \in W$, by part (a). Since w was an arbitrary vector in W, the range of S is a subspace of W.

(c) Prove that $m_S(x)$ is a divisor of $m_T(x)$.

> **Solution:** We have $m_T(T) = 0$, the 0-operator on V. As such, $m_T(T)(v) = 0$ for every $v \in V$. In particular, $m_T(T)(w) = 0$ for every $w \in W$. Now $m_T(S)(w) = m_T(T)(w) = 0$, shows that $m_T(S)$ is the 0-operator on W. This means $m_T(x)$ annihilates S, so S's minimal polynomial divides this polynomial.

(d) Here is another proof of the Cayley-Hamilton Theorem. We show that $c_T(T) = 0$ as follows: We want the operator $c_T(T)$ to be the 0-operator, so we must show that $(c_T(T))(v) = 0$ for every vector $v \in V$. Given $v \in V$, let W be its cyclic subspace, and S, the operator on that subspace.

i. Why is $(c_S(S))(v) = 0$?

> **Solution:** v is a cyclic vector for W so by a theorem from class (Lemma 6.2.9 in the book), the minimal polynomial for S is the same as the characteristic polynomial for S. In particular, the characteristic polynomial for S annihilates S so $(c_S(S))(v) = 0$.

ii. Why does this mean that $(c_T(T))(v) = 0$? Conclude the proof of the Cayley-Hamilton theorem.

> **Solution:** As mentioned in class, we need one more fact, which you can just use: The characteristic polynomial of S divides the characteristic polynomial of T. Since every multiple of an annihilating polynomial is an annihilating polynomial, this means that $c_T(x)$ is an annihilating polynomial for S. In particular, $0 = c_T(S)(v) = c_T(T)(v)$. Since v was an arbitrary element of V, this means $c_T(T)(v) = 0$ for all $v \in V$. That is, $c_T(x)$ annihilates T.

2. Define T on P^2 by $T(ax^2 + bx + c) = \frac{2a+5c}{7}x^2 + bx + \frac{2a+5c}{7}$.

(a) Show that P^2 has T-cyclic subspaces of dimension 0, 1, 2 but not 3.

> **Solution:** 0 generates T-cyclic subspace of dimension 0, and any eigenvector, say x, generates a 1-dimensional space. Anything else will generate a 2-dimensional space. For example, if $v = 7$ then $T(v) = 5x^2 + 5 = T^2(v)$. In fact, T is a projection, so $T^2 = T$, so for any v, $T^2(v) = T(v)$, so $v, T(v), T^2(v)$ can't be an independent set.

(b) Let $v = x^2$. If W is the cyclic subspace of v, extend the basis for W to a basis for P^2. Then find the matrix of the transformation with respect to that basis.

Solution: By what we said in part (a), the subspace will be the span of v and $T(v)$, or $\text{Span}\{x^2, \frac{2}{7}(x^2+1)\}$. Appending x to this gives a basis for P^2, so let $B = \{x^2, \frac{2}{7}(x^2+1), x\}$. The matrix of T with respect to B is $[T]_B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

For extra credit:

3. Find, with proof, the characteristic and minimal polynomials for $T(p(x)) = \dfrac{p(x+1)+p(x-1)}{2}$ on P^n.

Solution: If we apply T to x^k we get $\frac{1}{2}((x+1)^k + (x-1)^k) = x^k + k(k-1)/2\; x^{k-2} +$ other terms. This means the matrix for T will be upper triangular with 1's on the diagonal, meaning $c_T(x) = (x-1)^{n+1}$. For the minimal polynomial, $(T-I)(p(x)) = \dfrac{p(x+1)+p(x-1)}{2} - p(x) = \dfrac{1}{2}[(p(x+1)-p(x)) - (p(x)-p(x-1))]$. You should check that the operator $D(p(x)) = p(x+1) - p(x)$ lowers the degree of $p(x)$ by 1, much like the derivative does. It is called the **Difference** operator. Our operator, $T - I$ is a scaled second difference operator, which drops the dimension of $p(x)$ by 2 each time it is applied. On P^4, degrees would go $4 \to 2 \to 0$, taking 3 steps to annihilate a degree 4 polynomial (we've gotten to a constant, one more application gives 0), and for 5, it also takes 3 steps. The minimal polynomial is $(x-1)^k$ where $k = \left\lfloor \dfrac{n+1}{2} \right\rfloor$.

4. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(a) Calculate the minimal polynomials of A and B by finding the gcd of the elments in $\text{Adj}(xI - A)$ and $\text{Adj}(xI - B)$.

Solution: Several people did these problems. In fact, several people used this technique to find minimal polynomials for the problems in exercise 6.1.4. I don't think this is the best way to calculate minimal polynomials–it is calculation intensive. But if you prefer, feel free.

The adjoint of $xI - A$ is

$$\operatorname{adj}\begin{pmatrix} x-1 & -1 & -1 & -1 \\ -1 & x-1 & -1 & -1 \\ -1 & -1 & x-1 & -1 \\ -1 & -1 & -1 & x-1 \end{pmatrix} = \begin{pmatrix} x^2(x-3) & x^2 & x^2 & x^2 \\ x^2 & x^2(x-3) & x^2 & x^2 \\ x^2 & x^2 & x^2(x-3) & x^2 \\ x^2 & x^2 & x^2 & x^2(x-3) \end{pmatrix}.$$

The characteristic polynomial is $x^3(x-4)$, and the greatest common divisor in the adjoint is x^2 so the minimal polynomial is the quotient, $x(x-4)$. The easiest way to verify this is to multiply A by $A - 4I$ and show the answer is 0. For B, the adjoint is

$$\begin{pmatrix} x^2(x-1) & x(x-1) & x(x-1) & x(x+2) \\ 0 & x(x-1)^2 & 0 & x(x-1) \\ 0 & 0 & x(x-1)^2 & x(x-1) \\ 0 & 0 & 0 & x^2(x-1) \end{pmatrix},$$

the characteristic polynomial is $x^2(x-1)^2$, and the greatest common divisor of the entries in $\operatorname{adj}(xI - B)$ is x so the minimal polynomial is $x(x-1)^2$, again checked by showing that $B(B-I)^2 = 0$ but $B(B-I) \neq 0$.

(b) Calculate the Smith-Normal forms for $xI - A$ and $xI - B$, and use these to calculate the minimal and characteristic polynomials for A and B.

Solution: The Smith-Normal forms for $xI - A$ are $xI - B$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x(x-4) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x(x-1)^2 \end{pmatrix},$$

respectively. The polynomial in the (4, 4) position is the minimal polynomial, the product of the diagonal entries is the characteristic polynomial, as we can see. To get these forms, we row/column reduce $xI - A$.

$$\begin{pmatrix} x-1 & -1 & -1 & -1 \\ -1 & x-1 & -1 & -1 \\ -1 & -1 & x-1 & -1 \\ -1 & -1 & -1 & x-1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & -1 & -1 & x-1 \\ -1 & x-1 & -1 & -1 \\ -1 & -1 & x-1 & -1 \\ x-1 & -1 & -1 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & -1 & -1 & x-1 \\ 0 & x & 0 & -x \\ 0 & 0 & x & -x \\ 0 & -x & -x & x^2-2x \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & -x \\ 0 & 0 & x & -x \\ 0 & 0 & -x & x^2-3x \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x^2-4x \end{pmatrix}.$$

The reduction is a little trickier for $xI - B$:

$$\begin{pmatrix} x-1 & -1 & -1 & -1 \\ 0 & x & 0 & -1 \\ 0 & 0 & x & -1 \\ 0 & 0 & 0 & x-1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & x-1 & -1 & -1 \\ x & 0 & 0 & -1 \\ 0 & 0 & x & -1 \\ 0 & 0 & 0 & x-1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x^2-x & 0 & -1 \\ 0 & 0 & x & -1 \\ 0 & 0 & 0 & x-1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & x^2-x \\ 0 & -1 & x & 0 \\ 0 & x-1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & x^2-x \\ 0 & 0 & x & -x^2+x \\ 0 & 0 & 0 & x(x-1)^2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x(x-1)^2 \end{pmatrix}.$$