Investigating Traces of Matrix Products

Submitted to the faculty of the graduate school
of the University of Minnesesota Duluth
by

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In partial fulfillment of the requirements
for the degree of
MS Applied and Computational Mathematics

Advisor: John Greene

May, 2015

Acknowledgements

I would first like to extend my deepest gratitude towards my advisor Dr. John Greene. Through great patience and wisdom, Dr. Greene has provided me with outstanding guidance during both the research and typing phases of this project. Thank you for reviewing, editing, correcting and reading multiple copies of this paper, I could not have asked for a more dedicated advisor.

I would also like to thank both Dr. Barry James and Dr. Marshall Hampton for serving on my committee. I thoroughly enjoyed your courses and I am grateful that you both have agreed to be part of this research by reading this paper, hearing my defense and providing feedback.

I also wish to thank the faculty and staff of the math department. I have learned so much from all of you and I am grateful for the many years and fond memories we have shared together. Finally I thank Angela Sharp, your friendship and constant support has had a profound impact on my success.

Dedication

To my Mother and Father, for providing nothing but support, praise and love throughout my entire life.

Abstract

The trace of a square matrix A is the sum of the diagonal entries in A, and is denoted $Tr(A)$. In this paper we investigate the relative size of the trace of a product of matrices. We consider how both the ordering of the product and the number of matrices in the product influences the size of the trace. Data was collected for products of real-valued matrices with independent random variable entries from a standard normal distribution. We first considered two $n \times n$ matrices A and B and compared $Tr(ABAB)$ vs $Tr(AABB)$ as n increased. We then considered products of $2A$'s and mB's for 2×2 matrices A and B. When $m = 4$, there are three possible traces to consider, $1 = Tr(AB^2AB^2), 2 = Tr(ABAB^3)$, and $3 = Tr(A^2B^4)$. Here, all possible orderings of the traces were investigated and it was found that the permutation 231 did not occur. This investigation was extended to larger numbers of B's, asking which permutations of the orders are possible and which are not.

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Chapter 1

Introduction

The trace of a square matrix A is the sum of the diagonal entries in A, and is denoted $Tr(A)$[7, p. 90]. As an example, if $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 5 & 4 & 2 & 5 \\ 8 & 6 & 6 & 2 \\ 3 & 6 & 9 & 0 \end{pmatrix}$ we find that $Tr(A) = 1 + 4 + 6 + 0 = 11$. When considering a product of matrices A and B, we have two possible products to examine, AB and BA. It turns out that if A and B are square matrices of the same size then $Tr(AB) = Tr(BA)$. For example, if $A = \begin{pmatrix} 2 & 1 & 8 \\ 3 & 4 & 3 \\ 1 & 6 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 5 & 5 \\ 8 & 1 & 2 \end{pmatrix}$, we have $AB = \begin{pmatrix} 69 & 13 & 21 \\ 34 & 23 & 26 \\ 64 & 37 & 44 \end{pmatrix}$, $BA = \begin{pmatrix} 4 & 2 & 16 \\ 22 & 51 & 58 \\ 21 & 24 & 81 \end{pmatrix}$ and $Tr(AB) = Tr(BA) = 136$.

Lemma 1.1. *If A and B are square matrices of the same size then*

$$Tr(AB) = Tr(BA)$$

Proof. Let $A = \begin{pmatrix} a_{ij} \end{pmatrix}$ and $B = \begin{pmatrix} b_{ij} \end{pmatrix}$ be $n \times n$ matrices. Then

$$Tr(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} = Tr(BA)$$

□

When considering the trace of a product of matrices, it is well known that the product of matrices is invariant under cyclic permutations[7, p. 110]. That is, for matrices A, B and C, $Tr(ABC) = Tr(CAB) = Tr(BCA)$, where the products CAB and BCA are cyclic permutations of ABC. Thus, these three permutations are equivalent when considering their traces.

Theorem 1.2. *Let $A_1, A_2, ..., A_n$ be $m \times m$ matrices. Then*

$$Tr(A_1...A_{n-1}A_n) = Tr(A_nA_1...A_{n-1})$$

Proof. By Lemma 1.1, for $m \times m$ matrices A_1 and A_2, $Tr(A_1A_2) = Tr(A_2A_1)$.

In general, for matrices

$$A_1, A_2, ..., A_n$$

Let

$$B = A_1A_2...A_{n-1}$$

Then we find that

$$Tr(BA_n) = Tr(A_nB)$$

or

$$Tr(A_1A_2...A_n) = Tr(A_nA_1...A_{n-1})$$

□

This does not hold for more general permutations however. In general $Tr(ABC) \neq Tr(CBA)$. For example, consider $A = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$. Then $ABC = \begin{pmatrix} 60 & 50 \\ 50 & 40 \end{pmatrix}$, $CBA = \begin{pmatrix} 47 & 58 \\ 39 & 46 \end{pmatrix}$ and so $Tr(ABC) = 100 \neq 93 = Tr(CBA)$.

Given a collection of matrices, we define a necklace as the set of all cyclically permuted products of the collection. For example, when considering a product of matrices containing two A's and two B's there are two necklaces to consider, $\{ABAB, BABA\}$ and

$\{AABB, BAAB, BBAA, ABBA\}$. For convenience we write the matrices of a necklace in lexicographic order and we let the first permutation of a necklace represent the entire set. Thus for two A's and two B's, we would say there are two necklaces represented by $AABB$ and $ABAB$. For three A's and three B's there are four necklaces represented by $AAABBB, AABABB, AABBAB$ and $ABABAB$. By Theorem 1.2 we can think of the traces as acting on necklaces of products, rather than on products.

In this paper we investigate the value of the trace of a product of matrices under certain conditions. We build on Huang's [5] work which served as the motivation behind this research. For example, for a product of mA's and nB's, as m and n increase, what happens to the trace of their product? For two $n \times n$ matrices A and B, how does increasing n affect the trace of their product? We considered which necklaces have the larger trace for different m and n values, however we almost exclusively restricted our simulations to the case of $m = 2$. We also investigated which traces can never be largest.

In Chapter 2, data for our initial simulations is presented. All of our simulations in this paper use real valued matrices with independent random variable entries from a standard normal distribution. We give the results for the products of two A's and two B's, where matrices A and B are $n \times n$, for n ranging from 2 to 1000. We also present simulation results for a product of two A's with two and three B's, where A and B are $2 \times 2, 3 \times 3$ and 4×4 matrices. In these simulations we considered the effects that complex eigenvalues have on the relative size of the trace. In the 2×2 case we explain most of the probabilities in our simulations for a product of two A's with three B's.

We then present more trace data in Chapter 3 and introduce the idea of a forbidden ordering. Chapter 4 is more theoretical. We introduce two variable Lucas Polynomials in order to derive a formula that relates a polynomial to the difference for the traces of a product of 2×2 matrices A and B. Using this formula we were able to verify that orders in our simulations are indeed forbidden. Suggestions for future work are contained within Chapter 5 and the appendix includes several samples of code used for collecting data in our simulations.

Chapter 2

Some Initial Data

We first considered a product of two A's and two B's, with independent random variable entries from a standard normal distribution. In general, $Tr(AABB) \neq Tr(ABAB)$ so we investigated how these traces compare for matrices of various sizes. Table 2.1 shows our results after 1,000,000 trials.

A and B are $n \times n$ matrices		
n	$Tr(ABAB) > Tr(AABB)$	$Tr(AABB) > Tr(ABAB)$
2	707359	292641
3	703919	296081
4	701421	298579
5	701513	298487
10	705472	294528
20	710109	289891
30	713266	286734
40	714237	285763
50	714643	285357
100	716399	283601
1000	718481	281519

Table 2.1: Simulations on $Tr(ABAB)$ vs $Tr(AABB)$ for $n \times n$ matrices as n varies.

Dr Greene [4] had done some research on products of matrices and found that for 2×2 matrices A and B, $Tr(ABAB) > Tr(AABB)$ with probability $\frac{1}{\sqrt{2}}$. This explains our results for $n = 2$ in Table 2.1 as $\frac{1}{\sqrt{2}} \approx 0.707$, so one expects roughly 707,000 vs 293,000. As we increased the size of matrices A and B, this probability decreased initially and then slowly began to increase. The probability that $Tr(ABAB) > Tr(AABB)$ appears to change slowly as the size of the matrices, n, increases. As Table 2.1 indicates, there are no sudden jumps from $n = 50$ to $n = 100$ or $n = 1000$, it appears to be fairly stable for larger n.

During our investigation we considered how complex eigenvalues might affect these traces. By Lemma 3.7 in [4], If matrix A has independent normally distributed elements of mean 0 and variance 1, then the probability that A has real eigenvalues is $\frac{1}{\sqrt{2}}$. Another result from Dr Greene [4] is that $Tr(ABAB) > Tr(A^2B^2)$ whenever either A or B has complex eigenvalues. We first considered when A and B are 2×2 matrices, as shown in Table 2.2, again with 1,000,000 trials. We use r to indicate the given matrix having real eigenvalues and c to indicate the given matrix having complex eigenvalues.

A eigenvalues	r	r	r	r	c	c	c	c
B eigenvalues	r	r	c	c	r	r	c	c
$Tr(ABAB) > Tr(A^2B^2)$	no	yes	no	yes	no	yes	no	yes
Count	292544	207597	0	207510	0	206602	0	85747

Table 2.2: Eigenvalues for 2×2 products of $2A$'s with $2B$'s.

These results can all be explained. We know that $Tr(ABAB)$ has the larger trace when either A or B has complex eigenvalues, thus explaining why the counts for columns three, five and seven are 0. Column one counts all the cases where $Tr(ABAB) < Tr(AABB)$, which is $(1 - \frac{1}{\sqrt{2}}) \approx 0.292$. Column eight counts the cases when A and B have complex eigenvalues, which happens with probability $(1 - \frac{1}{\sqrt{2}})(1 - \frac{1}{\sqrt{2}}) = \frac{3}{2} - \sqrt{2} \approx 0.085$. Now, columns four and six are the sum of total cases where one matrix has real eigenvalues and the other has complex. This happens with probability $\frac{1}{\sqrt{2}}(1-\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} - \frac{1}{2} \approx 0.207$. Finally, we know that $Tr(ABAB) > Tr(AABB)$ with probability $\frac{1}{\sqrt{2}}$, so column two occurs with probability $\frac{1}{\sqrt{2}} - (2(\frac{1}{\sqrt{2}} - \frac{1}{2}) + (\frac{3}{2} - \sqrt{2})) \approx 0.207$.

We also considered when A and B are $n \times n$ matrices, for $n = 3$ and $n = 4$, as shown in Table 2.3 and Table 2.4 respectively. We again ran our simulations for 1,000,000 trials. A 3×3 matrix will always have at least one real eigenvalue because any complex eigenvalues occur in conjugate pairs. As such, the columns in Table 2.3 labeled c correspond to that given matrix having one real and two complex eigenvalues.

A eigenvalues	r	r	r	r	c	c	c	c
B eigenvalues	r	r	c	c	r	r	c	c
$Tr(ABAB) > Tr(A^2B^2)$	no	yes	no	yes	no	yes	no	yes
Count	77386	47657	64586	164094	64326	164732	90363	326856

Table 2.3: Eigenvalues for 3×3 products of $2A$'s with $2B$'s.

We can explain some features of columns 3,4,5 and 6 using symmetry. Consider the entries in columns 3 and 5 for example, since we consider a product of two A's and two B's, the probability of A real B complex should equal A complex B real. Thus we can explain the cases when $Tr(A^2B^2) > Tr(ABAB)$. Likewise, the yes cases corresponding to columns 4 and 6 should have equal probabilities, which are observed in our simulations.

It was shown in [3] that if A is a 3×3 matrix, then three real eigenvalues should occur with probability $\frac{\sqrt{2}}{4}$ and one real eigenvalue should occur with probability $1 - \frac{\sqrt{2}}{4}$. In Table 2.3, if we consider the cases when matrix A has real eigenvalues, this occurred in $77386 + 47657 + 64586 + 164094 = 353723$ of 1,000,000 trials and fits nicely with $\frac{\sqrt{2}}{4} \approx 0.353553$. The remaining cases account for A having two complex and one real eigenvalues occurred $64326 + 164732 + 90363 + 326856 = 646277$ and again fits nicely with $1 - \frac{\sqrt{2}}{4} \approx 0.646447$.

With 4×4 matrices, the cases to consider are no real eigenvalues, two real eigenvalues or four real eigenvalues. The columns in Table 2.4 labeled 2c correspond to that matrix having two real and two complex eigenvalues. Similarly, the columns labeled 4r and 4c correspond to four real or four complex eigenvalues, respectively.

A eigenvalues	B eigenvalues	$Tr(ABAB) > Tr(A^2B^2)$	Count
4r	4r	no	10393
4r	4r	yes	5271
4r	2c	no	34885
4r	2c	yes	55621
4r	4c	no	4845
4r	4c	yes	14395
2c	4r	no	34591
2c	4r	yes	55706
2c	2c	no	151412
2c	2c	yes	369583
2c	4c	no	26145
2c	4c	yes	84145
4c	4r	no	4650
4c	4r	yes	14250
4c	2c	no	26065
4c	2c	yes	84794
4c	4c	no	5410
4c	4c	yes	17839

Table 2.4: Eigenvalues for 4×4 products of $2A$'s with $2B$'s.

It was also shown in [3] that a 4×4 matrix A has four real eigenvalues with probability $\frac{1}{8}$, two real eigenvalues with probability $-\frac{1}{4} + 11\frac{\sqrt{2}}{16}$ and no real eigenvalues with probability $\frac{9}{8} - 11\frac{\sqrt{2}}{16}$. From Table 2.4, consider those cases when matrix A has four real eigenvalues. These occurred $10393 + 5271 + 34885 + 55621 + 4845 + 14395 = 125410$ from 1,000,000 trials, in close agreement with $\frac{1}{8} = 0.125$. Next, the cases when A has two real and two complex eigenvalues occurred $34591 + 55706 + 151412 + 369583 + 26145 + 84145 = 721582$, closely matching $-\frac{1}{4} + 11\frac{\sqrt{2}}{16} \approx 0.722272$. The remaining cases account for A having no real eigenvalues and occurred $4650 + 14250 + 26065 + 84794 + 5410 + 17839 = 153008$ agreeing with $\frac{9}{8} - 11\frac{\sqrt{2}}{16} \approx 0.152728$.

We then considered the case of 2×2 matrices again but with the addition of another matrix B and compared $Tr(AABBB)$ vs $Tr(ABABB)$. With two A's and an odd number of B's we immediate noticed apparent symmetry. There is obvious symmetry in Table 2.5 among the yes and no values. The reason is that any involution on A and B which leaves them with standard normal variables should keep probabilities the same. An involution [1] is a map which, if done twice, gets you back where you started. As an example consider $f(x) = -x$, then $f(f(x)) = x$ as desired. The relevant involution for our work is $f(B) = -B$. This means that the probability that $Tr(ABAB^2) > Tr(A^2B^3)$ is the same as the probability that $Tr(A(-B)A(-B)^2) > Tr(A^2(-B)^3)$. But this simplifies to $-Tr(ABAB^2) > -Tr(A^2B^3)$, or $Tr(A^2B^3) > Tr(ABAB^2)$. Now, since B and $-B$ either both have real eigenvalues or both have complex eigenvalues, this involutions proves that the yes-no combinations occur with the same probability. In Table 2.5, $Tr(ABAB^2) > Tr(A^2B^3)$ in 500,549 of the 1,000,000 cases.

A eigenvalues	r	r	r	r	c	c	c	c
B eigenvalues	r	r	c	c	r	r	c	c
$Tr(ABAB^2) > Tr(A^2B^3)$	no	yes	no	yes	no	yes	no	yes
Count	250377	250144	103198	103557	103199	104059	42677	42789

Table 2.5: Eigenvalues for 2×2 products of $2A$'s with $3B$'s.

To explain these results in more detail we need to use the following.

Lemma 2.1. *For any* 2×2 *matrix* M*,*

$$M^2 = Tr(M)M - Det(M)I$$

where I *is the* 2×2 *identity matrix.*

Proof. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $M^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$

Now, $Tr(M)M = \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix}$ and $Det(M)I = \begin{pmatrix} -bc+ad & 0 \\ 0 & -bc+ad \end{pmatrix}$

So we have $Tr(M)M - Det(M)I = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$ as desired. □

When we consider $Tr(ABAB^2) > Tr(A^2B^3)$, this implies $Tr(ABAB^2) - Tr(A^2B^3) > 0$. By Lemma 2.1 $ABAB^2 = ABA[Tr(B)B - Det(B)I] = Tr(B)ABAB - Det(B)ABA$, and $A^2B^3 = AAB[Tr(B)B - Det(B)I] = Tr(B)A^2B^2 - Det(B)A^2B$. Thus

$$Tr(ABAB^2) - Tr(A^2B^3) = Tr(B)(Tr(ABAB) - Tr(AABB))$$

as $Tr(ABA) = Tr(A^2B)$. From our involution, $Tr(B) > 0$ has exactly the same probability that $Tr(-B) > 0$. As a result $Tr(B) > 0$ with probability exactly $\frac{1}{2}$. So if we add the yes-no entries we should get the probability of the combination of the eigenvalues.

We can use this information to explain the results in Table 2.3. Column one and two count the cases when A and B both have real eigenvalues, which happens with probability $\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2}$. Since the yes-no possibilities are evenly divided, the probability of each will be 0.25. Next, column three and four count the cases when A has real eigenvalues and B has complex eigenvalues, which happens with probability $\frac{1}{\sqrt{2}}(1 - \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} - \frac{1}{2}$. Similarly, column five and six count the cases when A has complex eigenvalues and B has real eigenvalues, which happens with probability $(1 - \frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{1}{2}$. As a result, each of these four probabilities should be $\frac{1}{2}(\frac{1}{\sqrt{2}} - \frac{1}{2}) \approx 0.103$. Finally, column seven and eight count the cases when A and B both have complex eigenvalues, which happens with probability $(1 - \frac{1}{\sqrt{2}})(1 - \frac{1}{\sqrt{2}}) = \frac{3}{2} - \sqrt{2}$. Since the yes-no possibilities are evenly divided, the probability of each will be $\frac{1}{2}(\frac{3}{2} - \sqrt{2}) \approx 0.042$.

This analysis becomes more and more complicated as the number of A's and B's increases. For example, consider the case of two A's and four B's. Now there are three necklaces denoted by $M_1 = AB^2AB^2, M_2 = ABAB^3$ and $M_3 = A^2B^4$. If we ask when $Tr(AB^2AB^2) > Tr(A^2B^4)$, we get Table 2.6,

A eigenvalues	r	r	r	r	c	c	c	c
B eigenvalues	r	r	c	c	r	r	c	c
$Tr(AB^2AB^2) > Tr(A^2B^4)$	no	yes	no	yes	no	yes	no	yes
Count	292393	207277	0	207536	0	207370	0	85424

Table 2.6: Eigenvalues $Tr(AB^2AB^2)$ vs $Tr(A^2B^4)$.

which is very much like Table 2.2. Dr Greene [4] found that if M is a product of A's and B's, then $Tr(M^2) > Tr(MM^R)$ if and only if $Tr(ABAB) > Tr(A^2B^2)$, where M^R is the reversal of M, the product in reverse order. For example, if $M = M_1M_2...M_n$ is a product of matrices, then the reversal of this product is defined as $M^R = M_nM_{n-1}...M_1$. Using this result for $M = AB^2$, this means that $Tr(AB^2AB^2) > Tr(AB^4A) = Tr(A^2B^4)$ with probability $\frac{1}{\sqrt{2}}$, explaining why Table 2.6 mimics Table 2.2.

However, if we ask when $Tr(ABAB^3) > Tr(A^2B^4)$ or when $Tr(AB^2AB^2) > Tr(ABAB^3)$ we obtain significantly different tables, Table 2.7 and Table 2.8

A eigenvalues	r	r	r	r	c	c	c	c
B eigenvalues	r	r	c	c	r	r	c	c
$Tr(ABAB^3) > Tr(A^2B^4)$	no	yes	no	yes	no	yes	no	yes
Count	292393	207277	87608	119928	0	207370	36218	49206

Table 2.7: Eigenvalues $Tr(A^2B^4)$ vs $Tr(ABAB^3)$.

A eigenvalues	r	r	r	r	c	c	c	c
B eigenvalues	r	r	c	c	r	r	c	c
$Tr(AB^2AB^2) > Tr(ABAB^3)$	no	yes	no	yes	no	yes	no	yes
Count	211671	287999	0	207536	146513	60857	0	85424

Table 2.8: Eigenvalues $Tr(ABAB^3)$ vs $Tr(AB^2AB^2)$

We first focus on Table 2.8. To explain these results in more detail we need to use the following.

By Lemma 2.1 $ABAB^3 = Tr(B)ABAB^2 - Det(B)ABAB$, and $AB^2AB^2 = Tr(B)ABAB^2 - Det(B)A^2B^2$. Thus

$$\begin{aligned} Tr(ABAB^2) - Tr(A^2B^3) &= -Det(B)(Tr(A^2B^2) - Tr(ABAB)) \\ &= Det(B)(Tr(ABAB) - Tr(AABB)) \end{aligned}$$

Therefore we need information about $Det(B)$. The involution $f(B) = -B$ does not affect the determinant, however, the involution where two rows of B are interchanged will change the sign of the determinant. This means $Det(B) > 0$ with probability $\frac{1}{2}$.

Lemma 2.2. *[6, p. 288] Let $\lambda_1, \lambda_2, ..., \lambda_n$ be all the eigenvalues of an $n \times n$ matrix M. Then*

$$\begin{aligned} Det(M) &= \lambda_1 \lambda_2 ... \lambda_n \\ Tr(M) &= \lambda_1 + \lambda_2 + ... + \lambda_n \end{aligned}$$

As a consequence, if B is a 2×2 matrix with complex eigenvalues $c + di$ and $c - di$ then by Lemma 2.2 $Det(B) = (c + di)(c - di) = c^2 + d^2 > 0$. So B will always have a positive determinant when it has complex eigenvalues. Since $Det(B) > 0$ half of the time, and all of the time when B has complex eigenvalues, and B has complex eigenvalues with probability $1 - \frac{1}{\sqrt{2}}$, it follows that the probability that B has real eigenvalues and $Det(B) > 0$ is $\frac{1}{2} - (1 - \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} - \frac{1}{2}$. Finally, since B has real eigenvalues with probability $\frac{1}{\sqrt{2}}$, we need $p(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} - \frac{1}{2}$, or $p = 1 - \frac{1}{\sqrt{2}}$.

What we have determined then is

$$Det(B) > 0 \quad \text{with probability} \begin{cases} 1, & \text{if } B \text{ has complex eigenvalues,} \\ 1 - \frac{1}{\sqrt{2}}, & \text{if } B \text{ has real eigenvalues.} \end{cases}$$

and

$$Det(B) < 0 \quad \text{with probability} \begin{cases} 0, & \text{if } B \text{ has complex eigenvalues,} \\ \frac{1}{\sqrt{2}}, & \text{if } B \text{ has real eigenvalues.} \end{cases}$$

It is useful to have a similar calculation for when $Tr(ABAB) > Tr(A^2B^2)$. We have to consider both A and B so there are four cases, depending on whether A and B have real or complex eigenvalues. We know that $Tr(ABAB)$ has the larger trace when either A or B has complex eigenvalues. For real eigenvalues, the probability that A has real eigenvalues and B has real eigenvalues and $Tr(ABAB) - Tr(A^2B^2) > 0$ is $\frac{1}{\sqrt{2}} - \frac{1}{2}$. So we need $p(\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} - \frac{1}{2}$, or $p = 2(\frac{1}{\sqrt{2}} - \frac{1}{2}) = \sqrt{2} - 1$.

Therefore we have determined that

$$Tr(ABAB) > Tr(A^2B^2) \quad \text{with probability} \begin{cases} 1, & \text{if complex, complex,} \\ 1, & \text{if complex, real,} \\ 1, & \text{if real, complex,} \\ \sqrt{2}-1, & \text{if real, real eigenvalues.} \end{cases}$$

With this information we can explain most of Table 2.8. The columns are linked in pairs, in other words, the sum of columns 1 and 2, 3 and 4, 5 and 6, 7 and 8 is the probability of a particular eigenvalue combination.

First, columns 1 and 2 count the cases when A and B both have real eigenvalues, this occurs with probability $(\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) = \frac{1}{2}$, which fits nicely with our data, $211671 + 287999 = 499670 \approx 500000$. Next, Columns 3 and 4 count the cases when A has real and B has complex eigenvalues, this occurs with probability $(\frac{1}{\sqrt{2}})(1 - \frac{1}{\sqrt{2}}) = \frac{1}{2}(\sqrt{2} - 1) \approx 0.207$, which matches our data, $0 + 207536 \approx 207000$. Columns 5 and 6 count the cases when A has complex and B has real eigenvalues, this occurs with probability $(1 - \frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) = \frac{1}{2}(\sqrt{2} - 1) \approx 0.207$, matching $146513 + 60857 = 207370 \approx 207000$. Finally, Columns 7 and 8 count the cases when A and B both have complex eigenvalues, this occurs with probability $(1 - \frac{1}{\sqrt{2}})(1 - \frac{1}{\sqrt{2}}) = \frac{1}{2}(3 - 2\sqrt{2}) \approx 0.085$, again matching our data $0 + 85424 = 85424 \approx 85000$.

Since the columns are linked if we explain one linked column we will explain the other by default. For example, column 8 counts the case when A has complex eigenvalues and B has complex eigenvalues and $Tr(ABAB) - Tr(A^2B^2) > 0$, this happens with probability $(1 - \frac{1}{\sqrt{2}})(1 - \frac{1}{\sqrt{2}})(1) = \frac{1}{2}(3 - 2\sqrt{2}) \approx 0.085$. Now, since we know columns 7 and 8 add to 85,000, column 7 must be 0. In fact, we already knew this because $Tr(ABAB) - Tr(A^2B^2) < 0$ with probability 0 when B has complex eigenvalues, and

$Det(B) > 0$ with probability 1 when B has complex eigenvalues.

Similarly column 3 must be 0 since B has complex eigenvalues and $Tr(ABAB) - Tr(A^2B^2) < 0$ with probability 0 when in this case. Thus, column 4 must be ≈ 0.207, matching our data nicely, 207536. Finally, column 6 counts the case when A has complex eigenvalues and B has real eigenvalues and $Tr(ABAB) - Tr(A^2B^2) > 0$, which means we need $Det(B) > 0$ as well. This occurs with probability $(1-\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}})(1-\frac{1}{\sqrt{2}}) = \frac{1}{4}(3\sqrt{2}-4) \approx 0.0607$, which is close to 60857. Now that we know column 6, column 5 must be $\frac{1}{2}(\sqrt{2}-1) - \frac{1}{4}(3\sqrt{2}-4) = \frac{1}{4}(2-\sqrt{2}) \approx 0.146$, matching 146536 nicely. We have explained columns 3,4,5,6,7 and 8. However, we were unable to explain columns 1 and 2.

The same analysis done in Table 2.8 will work in Table 2.7, except it is a bit more complicated. Since $Tr(ABAB^3) - Tr(A^2B^4) = (Tr(B)^2 - Det(B))(Tr(ABAB) - Tr(A^2B^2))$, things depend also on the sign of $Tr(B)^2 - Det(B)$. However, some features of Table 2.7 can be explained. If B is a 2×2 matrix with real eigenvalues λ_1 and λ_2 then by Lemma 2.2 $Tr(B)^2 - Det(B) = (\lambda_1 + \lambda_2)^2 - \lambda_1\lambda_2 = \lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2$. Since this is always nonnegative, when B has real eigenvalues, the sign of $Tr(ABAB^3) - Tr(A^2B^4)$ matches the sign of $Tr(ABAB) - Tr(A^2B^2)$. This means that columns where B has real eigenvalues should match the corresponding columns in Tables 2.6 and 2.7. This explains why columns 1,2,5 and 6 are equal in both Tables.

In general, if M_1, M_2 are each products of mA's and nB's, based on the involution $Tr(M_1) > Tr(M_2)$ with probability $\frac{1}{2}$ if either m or n is odd. Thus, the most interesting case is when m and n are both even.

Chapter 3

More Trace Data

In this chapter we investigate the frequencies of the possible orderings for products of matrices with mA's and nB's. For 2A's and 4B's there are three necklaces and we first investigated the frequencies of the 6 possible orderings of these necklaces.

Table 3.1 shows the count of each necklace when m = 2 and n = 4 and 5. We use numbers to represent the possible orderings. In general, with 2A's and nB's, necklace 1 represents the necklace with the A's separated (cyclically) as far as possible. If $n = 2m$, necklace 1 would be AB^mAB^m and the k^{th} necklace would be represented by $AB^{m-k+1}AB^{m+k-1}$. If $n = 2m + 1$, necklace 1 would be AB^mAB^{m+1} and the k^{th} necklace would be represented by $AB^{m-k+1}AB^{m+k}$.

As an example, with two A's and four B's let the ordering 123 correspond to $Tr(AB^2AB^2) > Tr(ABAB^3) > Tr(A^2B^4)$, and a count of 300975 means the the ordering $Tr(AB^2AB^2) > Tr(ABAB^3) > Tr(A^2B^4)$ occurs 300975 out of 1,000,000 trials.

2A4B		2A5B	
Ordering	Count	Ordering	Count
123	300975	123	159214
132	123560	132	28918
213	281835	213	311967
231	0	231	29048
312	217850	312	311800
321	75780	321	159053
5 permutations		6 permutations	
$1 = ABBABB$		$1 = ABBABBB$	
$2 = ABABBB$		$2 = ABABBBB$	
$3 = AABBBB$		$3 = AABBBBB$	

Table 3.1: Initial Data for 2×2 products containing $2A$'s and $4 - 5B$'s.

Notice that for $2A$'s and four B's the ordering 231 is, at best, very unlikely to occur, yet with $2A$'s and five B's the order 231 occurs with probability ≈ 0.029. We call an order that is not possible a forbidden order.

Table 3.2 lists our simulation results for m = 2, n = $6, 7, 8$ and 9. We only list those orderings that occurred a positive number of times in a sample size of $1,000,000$ trials. Notice that of the 4! possible orderings that could occur for two A's and six B's, only eight appear to be occurring. Similarly, only twelve of the twenty four possible orderings are occurring for two A's and seven B's. As we increase the number of B's, the proportion of the number of possible orderings appears to significantly decrease.

2A6B		2A7B		2A8B		2A9B	
Ordering	Count	Ordering	Count	Ordering	Count	Ordering	Count
1234	211402	1234	133813	12345	177271	12345	121673
1243	32432	1243	9990	12354	13281	12354	4732
1342	74880	1423	16273	12534	19865	12534	6768
1423	57865	2314	29034	13452	21283	15243	9905
1432	48690	2413	24458	13542	53833	23514	12164
3124	282379	3124	287396	14325	48936	24513	9117
4213	217214	3142	24531	14523	33747	25314	16696
4321	75138	3241	16047	15243	31961	25413	15087
		3421	9909	15423	24324	31452	15074
		4132	29248	42135	282247	31542	9182
		4213	286197	53124	217801	32415	15921
		4321	133104	54321	75451	34251	9841
						35124	10708
						41352	16738
						41532	12207
						42135	276783
						42153	10646
						43521	6488
						45321	4560
						51423	16154
						53124	276980
						54321	122576
8 permutations		12 permutations		12 permutations		22 permutations	
$1 = AB^3AB^3$		$1 = AB^3AB^4$		$1 = AB^4AB^4$		$1 = AB^4AB^5$	
$2 = AB^2AB^4$		$2 = AB^2AB^5$		$2 = AB^3AB^5$		$2 = AB^3AB^6$	
$3 = ABAB^5$		$3 = ABAB^6$		$3 = AB^2AB^6$		$3 = AB^2AB^7$	
$4 = A^2B^6$		$4 = A^2B^7$		$4 = ABAB^7$		$4 = ABAB^8$	
				$5 = A^2B^8$		$5 = A^2B^9$	

Table 3.2: 2×2 products products containing $2A$'s and $6 - 9B$'s.

2A10B		2A11B			
Ordering	Count	Ordering	Count	Ordering	Count
123456	161878	123456	116679	362451	12075
123465	6693	135642	271004	416253	9120
123645	9196	146325	15198	426153	10650
126354	13055	154263	12188	435261	9899
135642	41912	162534	9773	451362	8910
136542	12057	213456	2531	453162	6963
145236	33811	231456	3420	512463	4892
146325	30306	246513	6012	521463	11887
154263	24392	246531	270519	523641	15154
156243	18284	261354	6935	532614	9330
162534	19935	263154	9051	615243	6558
163452	21166	315642	5870	651423	4523
164325	18369	324156	4579	654132	3247
165243	14161	342516	6619	654312	2545
531246	282602	351624	10686	654321	116472
642135	217271	364125	11797		
654321	74912	364215	4914		
17 permutations		32 permutations			
$1 = AB^5AB^5$		$1 = AB^5AB^6$			
$2 = AB^4AB^6$		$2 = AB^4AB^7$			
$3 = AB^3AB^7$		$3 = AB^3AB^8$			
$4 = AB^2AB^8$		$4 = AB^2AB^9$			
$5 = ABAB^9$		$5 = ABAB^{10}$			
$6 = A^2B^{10}$		$6 = A^2B^{11}$			

Table 3.3: 2×2 products products containing $2A$'s and $10 - 11B$'s.

2A12B		2A13B			
Ordering	Count	Ordering	Count	Ordering	Count
1234567	152945	1234567	112376	5146237	9214
1234576	3949	1234576	1483	5164273	6945
1234756	5006	1234756	1931	5371264	6073
1237465	6546	1237465	2531	5463721	3267
1273645	9125	1273645	3274	5647321	2427
1356742	7685	1726354	4548	5731246	3847
1357642	34103	2563147	11822	6157243	5445
1365427	11779	2574136	10953	6175243	4374
1457236	10051	2653174	4943	6235714	4459
1467325	8421	2754136	4226	6237514	7655
1475236	23701	3425716	4523	6314572	4118
1476325	22039	3427516	5468	6314752	10813
1542673	15343	3516724	6325	6421357	267482
1542763	8964	3517624	4404	6421375	3760
1562473	18048	3615427	9049	6574321	1946
1634527	21247	3724615	7005	6754321	1555
1643725	18735	4157326	7634	7162534	6418
1652743	14186	4175326	4514	7245163	9241
1672534	10979	4267153	4339	7326415	9130
1726354	13018	4276153	6259	7413652	11914
1762534	8727	4352617	6560	7531246	267934
6421357	282070	4536271	4527	7654321	112507
7531246	217631	4621735	5958		
7654321	75702	4713562	4824		
24 permutations		46 permutations			
$1 = AB^6AB^6$		$1 = AB^6AB^7$			
$2 = AB^5AB^7$		$2 = AB^5AB^8$			
$3 = AB^4AB^8$		$3 = AB^4AB^9$			
$4 = AB^3AB^9$		$4 = AB^3AB^{10}$			
$5 = AB^2AB^{10}$		$5 = AB^2AB^{11}$			
$6 = ABAB^{11}$		$6 = ABAB^{12}$			
$7 = A^2B^{12}$		$7 = A^2B^{13}$			

Table 3.4: 2×2 products products containing $2A$'s and $12 - 13B$'s.

2A2B		2A3B		2A4B		2A5B	
Ordering	Count	Ordering	Count	Ordering	Count	Ordering	Count
12	296661	12	500326	123	74577	123	177022
21	703339	21	499674	132	161431	132	230613
				213	57709	213	92866
				231	190259	231	231017
				312	205206	312	91911
				321	310818	321	176571
2 permutations		2 permutations		6 permutations		6 permutations	
$1 = AABB$		$1 = AABBB$		$1 = AABBBB$		$1 = AABBBBB$	
$2 = ABAB$		$2 = ABABB$		$2 = ABABBB$		$2 = ABABBBB$	
				$3 = ABBABB$		$3 = ABBABBB$	

Table 3.5: 3×3 products products containing $2A$'s and $2-5B$'s.

2A6B		2A7B	
Ordering	Count	Ordering	Count
1234	26553	1234	81316
1243	28380	1243	45753
1324	10030	1324	16247
1342	79288	1342	144234
1423	35135	1423	40839
1432	52934	1432	45216
2134	8269	2134	23458
2143	10482	2143	22765
2314	5343	2314	17967
2341	43015	2341	45040
2413	35209	2413	43483
2431	117052	2431	144552
3124	11664	3124	7496
3142	34169	3142	43933
3214	3461	3214	10328
3241	25662	3241	41151
3412	21080	3412	22972
3421	49502	3421	46018
4123	42170	4123	10408
4132	56190	4132	17979
4213	56418	4213	7500
4231	55809	4231	16306
4312	52518	4312	23414
4321	139667	4321	81625
24 permutations		24 permutations	
$1 = AABBBBBB$		$1 = AABBBBBBB$	
$2 = ABABBBBB$		$2 = ABABBBBBB$	
$3 = ABBABBBB$		$3 = ABBABBBBB$	
$4 = ABBBABBB$		$4 = ABBBABBBB$	

Table 3.6: 3×3 products products containing $2A$'s and $6 - 7B$'s.

3A3B		3A4B		3A5B			
Ordering	Count	Ordering	Count	Ordering	Count	Ordering	Count
123	323986	1234	121637	12345	100129	35124	3229
132	114744	1243	13075	12354	5150	35421	9992
213	61405	1324	148925	12453	10151	41235	15723
231	114507	1342	27105	12543	21308	41325	11314
312	61084	1423	32342	13245	7507	41352	25033
321	324274	1432	28101	13254	3553	42153	3223
		2134	18789	13524	4284	42513	1856
		2143	5280	14235	9423	42531	4154
		2314	20331	14325	2532	43125	11860
		2341	28133	14352	15439	43152	11894
		2413	16791	15243	1967	43512	24263
		2431	27248	15342	9552	45213	1103
		3124	51770	21435	149840	45231	3482
		3142	16411	21534	24670	45321	5082
		3214	15942	24135	21772	51243	15002
		3241	32306	24315	3334	51342	3453
		3412	5006	24351	9507	52134	11951
		3421	12967	25134	11893	52314	11411
		4123	15814	25314	24882	52341	2377
		4132	20411	25341	15420	53142	21873
		4213	51907	31245	12806	53214	15463
		4231	149568	31254	1123	53241	9252
		4312	18799	31524	1845	53412	150054
		4321	121342	34215	14974	54213	12510
				34251	2001	54231	7460
				34521	21389	54321	100535
6 permutations		24 permutations		52 permutations			
$1 = ABABAB$		$1 = ABABABB$		$1 = ABABBABB$			
$2 = AABABB$		$2 = AABBABB$		$2 = ABABABBB$			
$3 = AAABBB$		$3 = AABABBB$		$3 = AABBABBB$			
		$4 = AAABBBB$		$4 = AABABBBB$			
				$5 = AAABBBBB$			

Table 3.7: 2×2 products products containing $3A$'s and $3 - 5B$'s.

Chapter 4

Forbidden Orders

In this chapter we demonstrate that certain orders are forbidden. For the remainder of this chapter, let $Tr(B) = x$, $Det(B) = y$ and $Tr(ABAB - A^2B^2) = z$.

From Lemma 2.1 we know that $B^2 = xB - yI$. Lets consider higher powers of such a matrix B.

$$\begin{aligned} B^3 = BB^2 &= B(xB - yI) \\ &= xB^2 - yBI \\ &= x(xB - yI) - yBI \\ &= (x^2 - y)B - xyI \end{aligned}$$

In a similar fashion we find:

$$\begin{aligned} B^4 = BB^3 &= (x^3 - 2xy)B - (x^2y - y^2)I \\ B^5 = BB^4 &= (x^4 - 3x^2y + y^2)B - (x^3y - 2xy^2)I \\ B^6 = BB^5 &= (x^5 - 4x^3y + 3xy^2)B - (x^4y - 3x^2y^2 + y^3)I. \end{aligned}$$

We now introduce two variable Lucas polynomials.

Definition 4.1. Define a sequence of polynomials $\{U_n(x, y)\}$ as

$$U_0 = 0, \quad U_1 = 1, \quad U_n(x, y) = xU_{n-1}(x, y) - yU_{n-2}(x, y)$$

for all $n \geq 2$

The first several terms of the sequence are given by the following:

n	0	1	2	3	4	5	6	...
$U_n(x,y)$	0	1	x	$x^2 - y$	$x^3 - 2xy$	$x^4 - 3x^2y + y^2$	$x^5 - 4x^3y + 3xy^2$	...

Table 4.1: Two variable Lucas Polynomials

Lucas polynomials, or Lucas sequences, have many uses in linear algebra and number theory. See [2, pp. 393-411], and [8, pp. 41-61], for general discussions of these polynomials.

Lemma 4.1. *If B is a 2×2 matrix with $Tr(B) = x$ and $Det(B) = y$, then*

$$B^n = U_n(x,y)B - yU_{n-1}(x,y)I$$

where I is the 2×2 identity matrix.

Proof. We prove this Lemma by induction on n.
Suppose $n = 1$. Then $B^1 = 1 \cdot B - 0 \cdot I = U_1B - yU_0I$ as desired.
Suppose $n = 2$. Then by Lemma 2.1, $B^2 = xB - yI = U_2B - yU_1I$ as desired.

Inductive Hypothesis: $B^n = U_n(x,y)B - yU_{n-1}(x,y)I$.

We will show that $B^{n+1} = U_{n+1}(x,y)B - yU_n(x,y)I$. We have

$$\begin{aligned}
B^{n+1} = B^nB &= [U_n(x,y)B - yU_{n-1}(x,y)I]B \qquad \text{by inductive hypothesis} \\
&= U_n(x,y)B^2 - yU_{n-1}(x,y)BI \\
&= U_n(x,y)[xB - yI] - yU_{n-1}(x,y)B \qquad \text{by Lemma 2.1} \\
&= xU_n(x,y)B - yU_n(x,y)I - yU_{n-1}(x,y)B \\
&= [xU_n(x,y) - yU_{n-1}(x,y)]B - yU_n(x,y)I \\
&= U_{n+1}(x,y)B - yU_n(x,y)I \qquad \text{by Definition 4.1.}
\end{aligned}$$

□

Lemma 4.2.

$$\begin{aligned} U_{m+n} &= xU_mU_n - yU_mU_{n-1} - yU_{m-1}U_n \\ -yU_{m+n-1} &= y^2U_{m-1}U_{n-1} - yU_mU_n. \end{aligned}$$

Proof. Using Lemma 4.1

$$\begin{aligned} B^mB^n &= U_{m+n}B - yU_{m+n-1}I \\ &= (U_mB - yU_{m-1}I)(U_nB - yU_{n-1})I \\ &= U_mU_nB^2 - yU_mU_{n-1}B - yU_{m-1}U_nB + y^2U_{m-1}U_{n-1}I \\ &= (xU_mU_n - yU_mU_{n-1} - yU_{m-1}U_n)B + (y^2U_{m-1}U_{n-1} - yU_mU_n)I. \end{aligned}$$

This tells us that

$$\begin{aligned} U_{m+n} &= xU_mU_n - yU_mU_{n-1} - yU_{m-1}U_n \\ -yU_{m+n-1} &= y^2U_{m-1}U_{n-1} - yU_mU_n. \end{aligned}$$

as desired. □

We now introduce a formula that relates a polynomial to the difference for the traces of a product of 2×2 matrices A and B.

Theorem 4.3. *If* $0 \le k \le m \le n$ *then*

$$Tr(AB^mAB^n - AB^{m-k}AB^{n+k}) = cTr(ABAB - A^2B^2)$$

where $c = y^{m-k}U_kU_{n-m+k}$.

Proof. Using Lemma 4.1,

$$\begin{aligned} B^nB^k &= B^{n+k} = U_nB^{k+1} - yU_{n-1}B^k, \\ &\Rightarrow B^n = U_{n-k}B^{k+1} - yU_{n-k-1}B^k, \\ &\Rightarrow B^n = U_kB^{n-k+1} - yU_{k-1}B^{n-k}. \end{aligned}$$

Now

$$\begin{aligned} AB^mAB^n &= U_mABAB^n - yU_{m-1}A^2B^n, \\ &= U_mU_{n-1}ABAB^2 - yU_mU_{n-2}ABAB \\ &\quad -yU_{m-1}U_{n-1}A^2B^2 + y^2U_{m-1}U_{n-2}A^2B. \end{aligned}$$

This means that

$$\begin{aligned} AB^mAB^n - AB^{m-k}AB^{n+k} &= ABAB^2(U_mU_{n-1} - U_{m-k}U_{n+k-1}) \\ &\quad -yABAB(U_mU_{n-2} - U_{m-k}U_{n+k-2}) \\ &\quad -yA^2B^2(U_{m-1}U_{n-1} - U_{m-k-1}U_{n+k-1}) \\ &\quad +y^2A^2B(U_{m-1}U_{n-2} - U_{m-k-1}U_{n+k-2}). \end{aligned}$$

If we write $ABAB^2 = xABAB - yABA$ then we have

$$\begin{aligned} AB^mAB^n - AB^{m-k}AB^{n+k} &= ABAB(xU_mU_{n-1} - xU_{m-k}U_{n+k-1}) \\ &\quad -yABAB(U_mU_{n-2} - U_{m-k}U_{n+k-2}) \\ &\quad -yA^2B^2(U_{m-1}U_{n-1} - U_{m-k-1}U_{n+k-1}) \\ &\quad -yABA(U_mU_{n-1} - U_{m-k}U_{n+k-1}) \\ &\quad +y^2A^2B(U_{m-1}U_{n-2} - U_{m-k-1}U_{n+k-2}). \end{aligned}$$

The expression multiplying ABAB

$$\begin{aligned} \text{is} \quad & xU_mU_{n-1} - xU_{m-k}U_{n+k-1} - yU_mU_{n-2} + yU_{m-k}U_{n+k-2} \\ = \quad & U_m(xU_{n-1} - yU_{n-2}) - U_{m-k}(xU_{n+k-1} - yU_{n+k-2}) \\ = \quad & U_mU_n - U_{m-k}U_{n+k}. \end{aligned}$$

This means that

$$\begin{aligned} AB^mAB^n - AB^{m-k}AB^{n+k} &= ABAB(U_mU_n - U_{m-k}U_{n+k}) \\ &\quad -yA^2B^2(U_{m-1}U_{n-1} - U_{m-k-1}U_{n+k-1}) \\ &\quad -yABA(U_mU_{n-1} - U_{m-k}U_{n+k-1}) \\ &\quad +y^2A^2B(U_{m-1}U_{n-2} - U_{m-k-1}U_{n+k-2}). \end{aligned}$$

Using Lemma 4.2 we find that $U_mU_n = U_{m+n-1} + yU_{m-1}U_{n-1}$. If we apply this to the coefficient of $ABAB$ we have

$$\begin{aligned} U_mU_n - U_{m-k}U_{n+k} &= U_{m+n-1} + yU_{m-1}U_{n-1} - (U_{m+n-1} + yU_{m-1-k}U_{n-1+k}) \\ &= y(U_{m-1}U_{n-1} - U_{m-1-k}U_{n-1+k}). \end{aligned}$$

That is, at the cost of a factor of y, all induces were dropped by 1. If we iterate this $m-k$ times we get

$$\begin{aligned} U_mU_n - U_{m-k}U_{n+k} &= y(U_{m-1}U_{n-1} - U_{m-1-k}U_{n-1+k}) \\ &= y^2(U_{m-2}U_{n-2} - U_{m-2-k}U_{n-2+k}) \\ &= ... \\ &= y^{m-k}(U_{m-(m-k)}U_{n-(m-k)} - U_{m-(m-k)-k}U_{n-(m-k)+k}) \\ &= y^{m-k}(U_kU_{n-m+k} - U_0U_{n-m+2k}) \\ &= y^{m-k}U_kU_{n-m+k}, \qquad \text{since} \quad U_0 = 0. \end{aligned}$$

This means the whole expression simplifies. We have

$$\begin{aligned} U_mU_n - U_{m-k}U_{n+k} &= y^{m-k}U_kU_{n-m+k}, \\ -y(U_{m-1}U_{n-1} - U_{m-k-1}U_{n+k-1}) &= -y \cdot y^{m-1-k}U_kU_{n-m+k}, \\ -y(U_mU_{n-1} - U_{m-k}U_{n+k-1}) &= -y \cdot y^{m-k}U_kU_{n-m-1+k}, \\ y^2(U_{m-1}U_{n-2} - U_{m-1-k}U_{n-2+k}) &= y^2 \cdot y^{m-1-k}U_kU_{n-m-1+k}. \end{aligned}$$

This means

$$\begin{aligned} AB^mAB^n - AB^{m-k}AB^{n+k} &= y^{m-k}U_kU_{n-m+k}(ABAB - A^2B^2) \\ &\quad -y^{m+1-k}U_kU_{n-m-1+k}(ABA - A^2B). \end{aligned}$$

Finally, taking the trace and noting that $Tr(ABA - A^2B) = 0$ we have

$$Tr(AB^mAB^n - AB^{m-k}AB^{n+k}) = y^{m-k}U_kU_{n-m+k}Tr(ABAB - A^2B^2)$$

as desired. □

As an example, if we let $z = Tr(ABAB - A^2B^2)$, when $m = n = k = 1$ we have $Tr(AB^1AB^1 - AB^{1-1}AB^{1+1}) = Tr(ABAB - A^2B^2) = z$.

For $n = 2$ we find

$$\begin{aligned} Tr(ABAB^2 - A^2B^3) &= xz \\ Tr(AB^2AB^2 - ABAB^3) &= yz \\ Tr(AB^2AB^2 - A^2B^4) &= x^2z \end{aligned}$$

We can use the formula in Theorem 4.3 to derive such tables, then use polynomial inequalities to show certain permutations cannot occur. For example, in Table 3.2, for two A's and six B's the ordering 4312 corresponding to $Tr(A^2B^6) > Tr(ABAB^5) > Tr(AB^3AB^3) > Tr(AB^2AB^4)$ occurred 0 times from 1,000,000 trials. With 1,000,000 trials we expect approximately three decimal places of accuracy, therefore we should be able to show this ordering cannot occur, or that this ordering is forbidden. Using Theorem 4.3

$$Tr(A^2B^6) > Tr(ABAB^5) > Tr(AB^3AB^3) > Tr(AB^2AB^4)$$

translates to the set of inequalities

$$-(x^4 - 3x^2y + y^2)z > 0 \tag{4.1}$$

$$-x^2yz > 0 \tag{4.2}$$

$$y^2z > 0. \tag{4.3}$$

Now (4.3) forces $z > 0$, so (4.2) implies that $y < 0$. Now (4.1) forces $(x^4-3x^2y+y^2)z < 0$, a contradiction.

Many such forbidden orders can be demonstrated in this fashion. Some are trickier. For example, in Table 3.1 the ordering 231 corresponding to $Tr(ABAB^3) > Tr(A^2B^4) > AB^2AB^2$ occurred 0 times from 1,000,000 trials. Using Theorem 4.3

$$Tr(ABAB^3) > Tr(A^2B^4) > AB^2AB^2$$

translates to

$$(x^2 - y)z > 0 \tag{4.4}$$

$$-x^2z > 0. \tag{4.5}$$

There is no obvious reason that we cannot have $z < 0$ and $x^2 - y < 0$. To explain these results we need the following:

Theorem 4.4. *If* $Tr(ABAB - A^2B^2) < 0$ *then* $x^2 - 4y \geq 0$*.*

Proof. If $Tr(ABAB - A^2B^2) = z < 0$ then B has real eigenvalues, call them λ_1 and λ_2.

$$\begin{aligned} \text{Then} \quad x^2 - 4y &= (\lambda_1 + \lambda_2)^2 - 4(\lambda_1\lambda_2) \quad \text{by Lemma 2.2} \\ &= \lambda_1^2 + 2\lambda_1\lambda_2 + \lambda_2^2 - 4\lambda_1\lambda_2 \\ &= \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 \\ &= (\lambda_1 - \lambda_2)^2 \geq 0, \quad \text{as desired.} \end{aligned}$$

□

We can now confirm that $Tr(ABAB^3) > Tr(A^2B^4) > AB^2AB^2$ is not possible because (4.4) forces $z < 0$, which implies $x^2 - 4y \geq 0$ by Theorem 4.4. Now this implies $x^2 - y \geq 0$, a contradiction.

Using Theorem 4.3 and Table 4.1 we are able to eliminate cases of trace inequalities to more efficiently verify which orders are forbidden. We use the same notation found in Table 3.2, for example $1 > 2$ if $Tr(AB^3AB^3) > Tr(AB^2AB^4)$. The associated polynomial with this order is obtained using Theorem 4.3. For $1 > 2$ we have $m = 3, n = 3$ and $k = 1$, so the resulting polynomial inequality will be $y^{3-1}U_1U_{3-3+1}Tr(ABAB - A^2B^2) > 0$ so $y^2U_1U_1z > 0$, or $y^2z > 0$.

For $m + n = 6$ we have:

Order	$1 > 2$	$1 > 3$	$1 > 4$	$2 > 3$	$2 > 4$	$3 > 4$
Polynomial	$y^2U_1^2z$	$y^1U_2^2z$	$y^0U_3^2z$	$y^1U_1U_3z$	$y^0U_2U_4z$	$y^0U_1U_5z$

Table 4.2: $2A6B$ Polynomials

or,

Order	$1 > 2$	$1 > 3$	$1 > 4$	$2 > 3$	$2 > 4$	$3 > 4$
Polynomial	y^2z	yx^2z	$(x^2 - y)^2z$	$y(x^2 - y)z$	$x^2(x^2 - 2y)z$	$(x^4 - 3x^2y + y^2)z$

Table 4.3: $2A6B$ Trace Inequalities

For two A's and six B's we can obtain all possible polynomials from Table 4.3, The order $2 > 1$ would correspond to $Tr(AB^3AB^3) > Tr(AB^2AB^4)$, or $-1(Tr(AB^3AB^3 - AB^2AB^4) > 0)$ or $-y^2z > 0$. That is, 1 is to the left of 2 in any allowable permutation when y^2z is positive and 1 is to the right of 2 when y^2z is negative. In general, if there are an even number of B's, say $2n$ total, then the product will have the form $y^jU_kU_{2n+k}z$ for some j and k.

Theorem 4.5. *If $n-m$ is even, $y > 0$ and $z < 0$ then $U_kU_{n-m+k} > 0$.*

Proof. Since $z < 0$ we know that $x^2 - 4y > 0$ by Theorem 4.4. By formula (IV.8) in [8] we have the formula

$$2^{n-1}U_n = \binom{n}{1}x^{n-1} + \binom{n}{3}x^{n-3}D + \binom{n}{5}x^{n-5}D^2 + ...$$

where $D = x^2 - 4y$. This means that

$$2^{k-1}U_k2^{n-m+k-1}U_{n-m+k}$$

$$= (\tbinom{k}{1}x^{k-1} + \tbinom{k}{3}x^{k-3}D + \tbinom{k}{5}x^{k-5}D^2 + ...)(\tbinom{j}{1}x^{j-1} + \tbinom{j}{3}x^{j-3}D + \tbinom{j}{5}x^{j-5}D^2 + ...)$$

$$= \tbinom{k}{1}\tbinom{j}{1}x^{j+k-2} + (\tbinom{k}{1}\tbinom{j}{3} + \tbinom{k}{3}\tbinom{j}{1})x^{j+k-4}(x^2-4y) + (\tbinom{k}{1}\tbinom{j}{5} + \tbinom{k}{3}\tbinom{j}{3} + \tbinom{k}{5}\tbinom{j}{1})x^{j+k-6}(x^2-4y)^2 + ...$$

where $j = n - m + k$. This means that $j + k = n - m + 2k$ is even, so all powers of x are even. Thus, we have collections of binomial coefficients, which are positive, times even powers of x, which are positive, times powers of $x^2 - 4y$, which are positive. That is, every term in the product of U_kU_{n-m+k} is positive, meaning that the product itself must be positive. □

Now, if $y < 0$, Table 4.3 becomes:

Order	$1 > 2$	$1 > 3$	$1 > 4$	$2 > 3$	$2 > 4$	$3 > 4$
Polynomial	z	$-z$	z	$-z$	z	z

Table 4.4: $2A6B$ Trace Inequalities for $y < 0$

This leads to two possible sign patterns, depending on when z is positive or z is negative. When z is positive, the only permutation possible is 3124, similarly when z is negative the only permutation possible is 4213.

Now if $y > 0$ and $z < 0$ then by Theorem 4.5, Table 4.3 has the form:

Order	$1 > 2$	$1 > 3$	$1 > 4$	$2 > 3$	$2 > 4$	$3 > 4$
Polynomial	$(-)$	$(-)$	$(-)$	$(-)$	$(-)$	$(-)$

Table 4.5: $2A6B$ Trace Inequalities for $y > 0$ and $z < 0$

This leads to only one possible permutation, namely 4321.

Finally, if $y > 0$ and $z > 0$, Table 4.3 becomes:

Order	$1 > 2$	$1 > 3$	$1 > 4$	$2 > 3$	$2 > 4$	$3 > 4$
Polynomial	$(+)$	$(+)$	$(+)$	$x^2 - y$	$x^2 - 2y$	$x^4 - 3x^2y + y^2$

Table 4.6: $2A6B$ Trace Inequalities for $y > 0$ and $z > 0$

We quickly see the only possible orderings remaining will begin with 1. Now everything depends on the possible signs of the polynomials involved. These signs depend on how x^2 compares to y. We can scale the y away by setting it equal to 1, and the problem reduces to how the zeros of $U_n(x, 1)$ are arranged. We also replace x^2 by x so the degrees drop by a factor of 2 while everything else remains unchanged. Thus we need to know about the zeros of $x-1, x-2$, and x^2-3x+1. Note that the zeros of x^2-3x+1 are approximately 0.382 and 2.618. Now, if we take all these zeros and put them in numerical order, then every time x passes from one range to another, some collection of signs will change. It is not hard to check that all sign patterns are different, as shown in Table 4.7.

	$x \in (0, .382)$	$x \in (0.382, 1)$	$x \in (1, 2)$	$x \in (2, 2.618)$	$x = (2.618, \infty)$
$x - 1$	(-)	(-)	(+)	(+)	(+)
$x - 2$	(-)	(-)	(-)	(+)	(+)
$x^2 - 3x + 1$	(+)	(-)	(-)	(-)	(+)

Table 4.7: Sign Patterns

Since we have verified that all sign patterns are different, we find the number of possible permutations is one more than the total number of distinct zeros among the U_n. Counting these zeros we get $1 + 1 + 2 = 4$ distinct zeros, and thus 5 total permutations.

Putting everything together, we have shown that of the 24 possible orderings when considering a product of two A's and six B's, only 8 permutations are actually possible; three which do not begin with 1 and five which do being with 1. Note, we are not saying these permutations must occur, but rather that these are the only permutations which could possible occur. Our data in Table 3.2 verifies these orders do indeed occur, so we have verified there are 16 forbidden orders.

Table 4.8 summarizes the number of occurring orders vs the number of possible orders for our simulations.

Number of B's	Necklaces	Possible Orders	Orders Occurring
2	2	2	2
3	2	2	2
4	3	6	5
5	3	6	6
6	4	24	8
7	4	24	12
8	5	120	12
9	5	120	22
10	6	720	17
11	6	720	32
12	7	5040	24
13	7	5040	46
14	8	40320	32
15	8	40320	64

Table 4.8: Confirmed Simulations

This same process can be extended to other combinations with more B's. The advantage of this process is the ability to eliminate large numbers of possibilities at once. For example, with two A's and ten B's there are 720 possible orderings, of which only 17 occurred. It would be a tedious exercise to verify each of the 703 forbidden orders one by one. We construct Table 4.10 to demonstrate. In this table, we write 12 rather than $1 > 2$. The meaning is the same as before, 1 is to the left of 2 if and only if y^4z is positive.

Order	Polynomial	Order	Polynomial
12	y^4z	26	$x^2(x^2-y)(x^2-2y)(x^2-3y)z$
13	x^2y^3z	34	$y^2(x^4-3x^2y+y^2)z$
14	$y^2(x^2-y)^2z$	35	$x^2y(x^2-y)(x^2-3y)z$
15	$x^2y(x^2-2y)^2z$	36	$(x^2-y)(x^6-5x^4y+6x^2y^2-y^3)z$
16	$(x^4-3x^2y+y^2)^2z$	45	$y(x^6-5x^4y+6x^2y^2-y^3)z$
23	$y^3(x^2-y)z$	46	$x^2(x^2-2y)(x^4-4x^2y+2y^2)z$
24	$x^2y^2(x^2-2y)z$	56	$(x^2-y)(x^6-6x^4y+9x^2y^2-y^3)z$
25	$y(x^2-y)(x^4-3x^2y+y^2)z$		

Table 4.9: $2A10B$ Trace Inequalities

If $y < 0$, Table 4.10 reduces to:

Order	Polynomial	Order	Polynomial
12	z	26	z
13	$-z$	34	z
14	z	35	$-z$
15	$-z$	36	z
16	z	45	$-z$
23	$-z$	46	z
24	z	56	z
25	$-z$		

Table 4.10: $2A10B$ Trace Inequalities for $y < 0$

This leads to two possible sign patterns, depending on when z is positive or z is negative. When z is positive, the only permutation possible is 531246, similarly when z is negative the only permutation possible is 642135. Next, much like the $2A6B$ case, when ($y > 0$ and $z < 0$), all polynomials are negative by Theorem 4.5, resulting in one possible permutation, 654321. In general, when considering both $y < 0$ or ($y > 0$ and $z < 0$), if the number of B's is even, there will be a total of 3 possible permutations, all of which do not begin with 1. All other allowable permutations are determined by combinations of the signs of the polynomials when ($y > 0$ and $z > 0$). This leads us to the following conjectures.

Conjecture 4.1. For a product of 2×2 matrices containing two A's with nB's, where n is even, if $y > 0$ and $z > 0$, there are $1 + k$ possible permutations, where k is the number of distinct positive zeros of $U_m(x, 1)$, for $3 \leq m \leq n - 1$. Consequently, the total number of possible permutations is $4 + k$.

For the case above, $n = 10$, the number of distinct zeros is 13, therefore we have $4 + 13 = 17$ possible permutations as observed in Table 4.8.

When there is an odd number n of B's, the permutations which arise from both $y < 0$ or ($y > 0$ and $z < 0$) are repeated in the situation with ($y > 0$ and $z > 0$). So it is enough to only consider this case.

Conjecture 4.2. For a product of 2×2 matrices containing two A's with nB's, where n is odd, there are $2(1 + k)$ possible permutations, where k is the number of distinct positive zeros of $U_m(x, 1)$, for $3 \leq m \leq n - 1$.

As an example, when considering the case of two A's seven B's, it can be shown using Theorem 4.3 that there are 5 distinct positive zeros. Therefore we expect $2(1 + 5) = 12$ possible permutations, as observed in Table 4.8.

Chapter 5

Future Work

There are several ways this work might be extended. To begin with, there is a simple extension to larger matrices. Data collected on products consisting of larger matrices suggests that forbidden orders are not limited to only the 2×2 case. We collected some data on products of two A's with nB's for 3×3 and 4×4 matrices A and B. For relatively small values of n, there appeared to be no forbidden orders. However, as n increases data suggests that orderings are a best, very unlikely, which leads us to believe that forbidden orders are prevalent for larger and larger n. In the 3×3 case, it was not until the number of B's reached seven until zeros start appearing in our simulations.

Another extension would include products of matrices with more A's and B's. For a product of mA's with nB's, we only looked at the case of $m = 3$ and $n = 3, 4, 5$ for 2×2 matrices A and B, forbidden orders started to appear when $n = 5$. Extending this to larger matrices and increasing both m and n could yield interesting results. One could also consider the effect that eigenvalues are having on products with more A's and B's.

Finally, it would be worth looking into proving the conjectures listed in Chapter 4. With these one could potentially derive a formula for the number of forbidden orders when taking a product of mA's and nB's. Or in another direction, deriving a formula similar to Theorem 4.3 but extending this to products with more than two A's would be extremely useful.

Chapter 6

Appendix. Source Code

In this section we provide several examples of the code used for our simulations. All code presented is written in `Mathematica`, and a brief description of what each piece of code does is presented.

Here is a sample of code used to collect data on nxn matrices, as n varies from 2 to 1000. This code takes advantage of parallel computing to increase the speed. The main idea behind this code is to have several different workstations collecting and sending data into one global list.

```
globalTable = Table[0, {i, 1, 2}];
set[] := Module[{},
   localTable = Table[0, {i, 1, 2}];
  ];
unset[] := Module[{},
   Clear[localTable];
  ];
do[] := Module[{a, b, c, d},
   a = RandomReal[NormalDistribution[0, 1], {1000, 1000}];
   b = RandomReal[NormalDistribution[0, 1], {1000, 1000}];

   c = SetPrecision[Tr[Dot[a, a, b, b]], 100];
   d = SetPrecision[Tr[Dot[a, b, a, b]], 100];

   If[c > d, localTable[[1]] ++, localTable[[2]] ++]
  ];
gether[] := Module[{},
   globalTable += localTable;
  ];
```

```
DistributeDefinitions[set, unset, do, gether];
SetSharedVariable[globalTable];
globalTable = Table[0, {i, 1, 2}];
ParallelEvaluate[set[]];
AbsoluteTiming[
 ParallelDo[
   Monitor[do[], {i, 1, 100}];, i]
  ParallelEvaluate[gether[]];
 ParallelEvaluate[unset[]];
]
```

```
{25 728.1080503, Null}
```

glovalTable is where all the individual simulations were collected, 281519 represents the number of occurances that Tr(AAB-B)>Tr(ABAB), and 718481 represents the number of occurances that Tr(ABAB)>Tr(AABB).

```
globalTable
```

```
{281 519, 718 481}
```

Here I am verifying that the globalTable sums to 1,000,000.

```
Total@globalTable
```

```
1 000 000
```

This is a sample of code used to collect eigenvalue data for 2x2 matrices A and B. g and h represent the discriminant of the characteristic polynomial of matrix A or B, respectively. The given matrix has complex eigenvalues if this discriminant is negative.

```
f = Table[0, {i, 1, 8}];

AbsoluteTiming[For[i = 10^6, i > 0, i--,
  a = RandomReal[NormalDistribution[0, 1], {2, 2}];
  b = RandomReal[NormalDistribution[0, 1], {2, 2}];

  c = SetPrecision[Tr[Dot[a, a, b, b, b]], 100];
  d = SetPrecision[Tr[Dot[a, b, a, b, b]], 100];

  g = (a[[1, 1]])^2 + 4 (a[[1, 2]]) (a[[2, 1]]) - 2 (a[[1, 1]]) (a[[2, 2]]) + (a[[2, 2]])^2;
  h = (b[[1, 1]])^2 + 4 (b[[1, 2]]) (b[[2, 1]]) - 2 (b[[1, 1]]) (b[[2, 2]]) + (b[[2, 2]])^2;

  (*real real*)
  If[g > 0 && h > 0,
   If[c > d, f[[1]] ++, (*12*)
    If[d > c, f[[2]] ++]]]; (*21*)
  (*real complex*)
  If[g > 0 && h < 0,
   If[c > d, f[[3]] ++, (*12*)
    If[d > c, f[[4]] ++]]]; (*21*)
  (*complex real*)
  If[g < 0 && h > 0,
   If[c > d, f[[5]] ++, (*12*)
    If[d > c, f[[6]] ++]]]; (*21*)
  (*complex complex*)
  If[g < 0 && h < 0,
   If[c > d, f[[7]] ++, (*12*)
    If[d > c, f[[8]] ++]]];(*21*)
 ]
]

{106.7074112, Null}

f

{250 007, 249 826, 103 847, 103 658, 103 445, 103 554, 42 791, 42 872}

Sum[f[[i]], {i, 1, 8}]

1 000 000
```

This is a sample of code used to collect eigenvalue data for 4x4 matrices A and B. Here we also take advantage of parallel computing to increase speed. To determine if a 4x4 matrix A has 0, 2, or 4 real eigenvalues we perform the following algorithm. If the Discriminant of the Characteristic Polynomial < 0 and the Minimum Value of this Discriminant > 0 then 4 real eigenvalues , if the Discriminant of the Characteristic Polynomial < 0 and the Minimum Value of this Discriminant < 0 then 2 real eigenvalues and if the Discriminant of the Characteristic Polynomial > 0 and the Minimum Value of this Discriminant > 0 then 0 real eigenvalues.

```
globalTable = Table[0, {i, 1, 18}];
set[] := Module[{},
   localTable = Table[0, {i, 1, 18}];
  ];
unset[] := Module[{},
   Clear[localTable];
  ];
do[] := Module[{a, b, c, d, e, f, g, h, i, j},
   a = RandomReal[NormalDistribution[0, 1], {4, 4}];
   b = RandomReal[NormalDistribution[0, 1], {4, 4}];

   c = CharacteristicPolynomial[a, x];
   d = CharacteristicPolynomial[b, x];

   e = Discriminant[c, x];
   f = Discriminant[d, x];

   g = NMinValue[c, x, MaxIterations → 10];
   (* Global Min for Matrix a Characteristic Polynomial *)
   h = NMinValue[d, x, MaxIterations → 10];
   (* Global Min for Matrix b Characteristic Polynomial *)

   i = SetPrecision[Tr[Dot[a, a, b, b]], 20];
   j = SetPrecision[Tr[Dot[a, b, a, b]], 20];

   If[e > 0 && g < 0,
    If[f > 0 && h < 0 && i > j, localTable[[1]] ++,
     If[f > 0 && h < 0  && i < j, localTable[[2]] ++,
      If[f < 0 && i > j, localTable[[3]] ++,
       If[f < 0 && i < j, localTable[[4]] ++,
        If[f > 0 && h > 0 && i > j, localTable[[5]] ++,
         If[f > 0 && h > 0 && i < j, localTable[[6]] ++
         ]]]]]]];

   If[e < 0 && f > 0,
    If[h < 0 && i > j, localTable[[7]] ++,
     If[h < 0 && i < j, localTable[[8]] ++,
      If[h > 0 && i > j, localTable[[11]] ++,
       If[h > 0 && i < j, localTable[[12]] ++
       ]]]]];
```

```
    If[e < 0 && f < 0,
     If[i > j, localTable[[9]] ++,
      If[i < j, localTable[[10]] ++]]];

    If[e > 0 && g > 0,
     If[f > 0 && h < 0 && i > j, localTable[[13]] ++,
      If[f > 0 && h < 0  && i < j, localTable[[14]] ++,
       If[f < 0 && i > j, localTable[[15]] ++,
        If[f < 0 && i < j, localTable[[16]] ++,
         If[f > 0 && h > 0 && i > j, localTable[[17]] ++,
          If[f > 0 && h > 0 && i < j, localTable[[18]] ++
          ]]]]]]]
   ];
gether[] := Module[{},
    globalTable += localTable;
   ];
DistributeDefinitions[set, unset, do, gether];
SetSharedVariable[globalTable];
globalTable = Table[0, {i, 1, 18}];
ParallelEvaluate[set[]];
AbsoluteTiming[
 ParallelDo[
  do[], {i, 1, 10^6}];
 ParallelEvaluate[gether[]];
 ParallelEvaluate[unset[]];
]
```

```
{2338.3597465, Null}
```

```
globalTable
```

```
{10 393, 5271, 34 885, 55 621, 4845, 14 395, 34 591, 55 706, 151 412,
 369 583, 26 145, 84 145, 4650, 14 250, 26 065, 84 794, 5410, 17 839}
```

```
Total@globalTable
```

```
1 000 000
```

This is a sample of code used to collect trace data for 2x2 matrices A and B. Here we create a list of all the possible permutations of the string abcd. We then compute the values for a,b,c,d and sort them. Once sorted, we increase the count in our table corresponding that that sorted permutation.

```
z = Table[0, {i, 1, 24}];

p = List["c", "d", "e", "f"];
perm = Permutations[p];

AbsoluteTiming[
 For[i = 10^6, i > 0, i--,
   a = SetPrecision[RandomReal[NormalDistribution[0, 1], {2, 2}], 100];
   b = SetPrecision[RandomReal[NormalDistribution[0, 1], {2, 2}], 100];

   c = SetPrecision[Tr[Dot[a, b, b, b, a, b, b, b]], 100];
   d = SetPrecision[Tr[Dot[a, b, b, a, b, b, b, b]], 100];
   e = SetPrecision[Tr[Dot[a, b, a, b, b, b, b, b]], 100];
   f = SetPrecision[Tr[Dot[a, a, b, b, b, b, b, b]], 100];

   temp = {{c, "c"}, {d, "d"}, {e, "e"}, {f, "f"}};
   temp = Sort[temp];
   z[[Flatten@Position[perm, temp[[All, 2]]][[1]]]] ++;
  ];
]

{445.0324544, Null}
```

Here we print the counts for each permutation of abcd. Note: The permutations are listed in the reverse order, that is, {c,d,e,f} is actually {f,e,d,c}, something we account for when inputing the data to a table.

```
For[i = 0, i < 24, i++
  If[z[[i]] > 0, Print[z[[i]], perm[[i]]]]]

75 316{c, d, e, f}

48 795{d, e, f, c}

75 127{d, f, e, c}

217 089{e, c, d, f}

57 683{e, d, f, c}

32 183{e, f, d, c}

282 590{f, d, c, e}

211 217{f, e, d, c}
```

This is what all the counts look like, including those which do not occur.

```
z

{75 316, 0, 0, 0, 0, 0, 0, 0, 0, 48 795, 0, 75 127,
 217 089, 0, 0, 57 683, 0, 32 183, 0, 0, 282 590, 0, 0, 211 217}
```

This is a sample of code used to collect trace data for 2x2 matrices A and B. Here we are verifying that the Lists code on the previous page is working how we intended. Each permutation of the string abcd is manually checked in nested If statements. There is no speed advantage to the List code, however there is a huge advantage in that we do not have to write out 8! = 40,320 nested If statements when in the 2A14B case for example.

```
In[1]:= z = Table[0, {i, 1, 24}];

In[2]:= AbsoluteTiming[For[i = 10^6, i > 0, i--,
          a = SetPrecision[RandomReal[NormalDistribution[0, 1], {2, 2}], 100];
          b = SetPrecision[RandomReal[NormalDistribution[0, 1], {2, 2}], 100];

          c = SetPrecision[Tr[Dot[a, b, b, b, a, b, b, b]], 100];
          d = SetPrecision[Tr[Dot[a, b, b, a, b, b, b, b]], 100];
          e = SetPrecision[Tr[Dot[a, b, a, b, b, b, b, b]], 100];
          f = SetPrecision[Tr[Dot[a, a, b, b, b, b, b, b]], 100];

          If[c > d > e > f, z[[1]] ++, (*1234*)
           If[c > d > f > e, z[[2]] ++, (*1243*)
            If[c > e > d > f, z[[3]] ++, (*1324*)
             If[c > e > f > d, z[[4]] ++, (*1342*)
              If[c > f > d > e, z[[5]] ++, (*1423*)
               If[c > f > e > d, z[[6]] ++, (*1432*)

                If[d > c > e > f, z[[7]] ++, (*2134*)
                 If[d > c > f > e, z[[8]] ++, (*2143*)
                  If[d > e > c > f, z[[9]] ++, (*2314*)
                   If[d > e > f > c, z[[10]] ++, (*2341*)
                    If[d > f > c > e, z[[11]] ++, (*2413*)
                     If[d > f > e > c, z[[12]] ++, (*2431*)

                      If[e > c > d > f, z[[13]] ++, (*3124*)
                       If[e > c > f > d, z[[14]] ++, (*3142*)
                        If[e > d > c > f, z[[15]] ++, (*3214*)
                         If[e > d > f > c, z[[16]] ++, (*3241*)
                          If[e > f > c > d, z[[17]] ++, (*3412*)
                           If[e > f > d > c, z[[18]] ++, (*3421*)

                            If[f > c > d > e, z[[19]] ++, (*4123*)
                             If[f > c > e > d, z[[20]] ++, (*4132*)
                              If[f > d > c > e, z[[21]] ++, (*4213*)
                               If[f > d > e > c, z[[22]] ++, (*4231*)
                                If[f > e > c > d, z[[23]] ++, (*4312*)
                                 If[f > e > d > c, z[[24]] ++  (*4321*)
                                 ]]]]]]]]]]]]]]]]]]]]]]]];
        ]

Out[2]= {406.9942788, Null}
```

In[3]:= **z**

Out[3]= {210 673, 31 829, 0, 75 256, 57 911, 48 940, 0, 0,
0, 0, 0, 0, 282 720, 0, 0, 0, 0, 0, 0, 217 099, 0, 0, 75 572}

In[5]:= **Sum[z[[i]], {i, 1, 24}]**

Out[5]= 1 000 000

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