VERIFYING AND DISCOVERING BBP-TYPE FORMULAS

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Chapter 1: Introduction

There are numerous well-known formulas for π, including

$$\pi = \frac{C}{2r} \text{ (where } C\text{=circumference and } r\text{=radius)},$$

$$\pi = \lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{6n^2}+4n\left(\frac{1}{n^2+1^2}+\frac{1}{n^2+2^2}+\cdots+\frac{1}{n^2+n^2}\right)\right] \text{ from [8, p. 152]},$$

$$\frac{4}{\pi}=1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\cfrac{11^2}{2+\cdots}}}}}} \text{ from [6, p. 127 formula 14]},$$

and

$$\frac{\pi^2}{6}=\sum_{k=1}^{\infty}\frac{1}{k^2} \text{ from [2, p. 225 formula 16.27]}.$$

Another notable formula is

(1.1) $$\pi=\sum_{k=0}^{\infty}\frac{1}{16^k}\left(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right),$$

credited to David Bailey, Peter Borwein, and Simon Plouffe in 1995 It is often called the BBP-formula. They wrote "*On the Rapid Computation of Various Polylogarithmic Constants*", in which the formula was first mentioned, [5, p. 1].

The formula is significant as it permits the computation of the n^{th} hexadecimal digit of π without calculation the preceding $n-1$ digits. The algorithm is explained in [5, section 3 p.6-8], [2, p.129], and [4, p.123]. Since its discovery, formulas of similar form have been discovered and have become known as BBP-type formulas. The general BBP-type formula has the form

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{b^k q(k)},$$

where α is a constant, p and q are polynomials with integer coefficients, $\deg(p) < \deg(q)$, $p(k)/q(k)$ is nonsingular for nonnegative k, and b is an integer, [3, p. 2].

Many nice BBP-type formulas can be written

$$\alpha = \sum_{k=0}^{\infty} x^{nk} \left(\frac{a_1}{nk+1} + \frac{a_2}{nk+2} + \cdots + \frac{a_n}{nk+n} \right),$$

where α is a well-known constant, $a_1, a_2, \ldots, a_n$ are constants (usually integers), and $x^n = \frac{1}{b}$. We call n the base of the BBP form. This is the only form I considered in this project.

A few examples of BBP-type formulas are the following:

$$\sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2k+1} \right) = \frac{\pi}{4},$$

which is known as the Leibniz formula,

$$\sum_{k=1}^{\infty} \left(\frac{1}{2} \right)^k \left(\frac{1}{k} \right) = \ln 2,$$

and

$$8\sqrt{2}\arctan\frac{1}{\sqrt{2}} = \sum_{k=0}^{\infty}\frac{1}{16^k}\left(\frac{8}{8k+1} - \frac{4}{8k+3} + \frac{2}{8k+5} - \frac{1}{8k+7}\right)$$ from [5, p. 6].

The following are not formulas of BBP-type:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(4)^{k+1}(2k+1)!} = \sin\left(\frac{1}{2}\right)$$ from [9, p. 617 formula 15]

and

$$\sum_{k=0}^{\infty}\frac{n!^2\,2^{n+1}}{(2n+1)!} = \pi$$ from [2, p. 229 (16.79)].

The objective of this project was to verify and discover BBP-type formulas for scaled values of π. In chapter 2, I begin by verifying previously discovered BBP-type formulas using their relationship to definite integrals. The chapter begins by explaining the method used, followed by two examples, and then a table containing the BBP-type formulas verified. The chapter ends with a verification of two formulas conjectured in [3, p. 18 formulas 72 & 73]. Next, chapter 3 shows how to search for BBP-type formulas. An algorithm is given and a detailed example of its use is provided. The next section, chapter 4, provides a search of the simplest cases where the base is 2, 3, or 4. The searches for base 6 and base 8 formulas are presented in chapters 5 and 6 respectively. The next chapter is the conclusion with a description of strengths, limitations, and future work related to this project. The paper ends with two appendixes. The first contains a table summarizing chapters 4, 5, and 6, while the second includes a copy from *Mathematica*'s output for integrating the non-alternating base 6 case.

Chapter 2: Unified Method to Verify

A technique was used to verify various BBP-type formulas. This chapter will provide a detail explanation of the technique's steps. Also provided is a table of formulas from the literature I have verified by this method.

Recall, in this project, I am interested in formulas of the type

$$\alpha = \sum_{k=0}^{\infty} x^{nk}\left(\frac{a_1}{nk+1}+\frac{a_2}{nk+2}+\cdots+\frac{a_n}{nk+n}\right),$$

where α is a well-known constant, $a_1, a_2, \ldots, a_n$ are constants (usually integers), and x is a constant. The key formulas are presented in Theorem 1 and Theorem 2.

Theorem 1: If $0<|x|<1$ or if $x=1$ and $\sum_{i=1}^{n} a_i = 0$, then

$$\sum_{k=0}^{\infty} x^{nk}\left(\frac{a_1}{nk+1}+\frac{a_2}{nk+2}+\cdots+\frac{a_n}{nk+n}\right) = \int_0^1 \frac{a_1 + a_2\cdot u + \ldots + a_n \cdot u^{n-1}}{1-x^n u^n} du .$$

Proof: The series is a power series with radius of convergence 1. If $|x|<1$, we can integrate and differentiate as stated in [1, p. 173 Theorem 6.5.7]. When $x=1$ and $\sum_{i=1}^{n} a_i = 0$, the resulting sum will converge by the limit comparison test with $\frac{1}{(k+1)^2}$.

We only verify the case when $|x|<1$ below.

Let

$$\alpha = \sum_{k=0}^{\infty} x^{nk}\left(\frac{a_1}{nk+1}+\frac{a_2}{nk+2}+\cdots+\frac{a_n}{nk+n}\right).$$

We break this into n summations,

$$\alpha = \sum_{k=0}^{\infty} \frac{a_1 \cdot x^{nk}}{nk+1} + \sum_{k=0}^{\infty} \frac{a_2 \cdot x^{nk}}{nk+2} + \cdots + \sum_{k=0}^{\infty} \frac{a_n \cdot x^{nk}}{nk+n}.$$

Taking each term individually, let $f_1(x) = \sum_{k=0}^{\infty} \frac{a_1 \cdot x^{nk}}{nk+1}$. Multiplying both sides by x gives $x \cdot f_1(x) = \sum_{k=0}^{\infty} \frac{a_1 \cdot x^{nk+1}}{nk+1}$. By taking the derivative we get,

$$\frac{d\left(x \cdot f_1(x)\right)}{dx} = \sum_{k=0}^{\infty} \frac{a_1 \cdot (nk+1) \cdot x^{nk+1-1}}{nk+1} = a_1 \sum_{k=0}^{\infty} x^{nk}.$$

This is a geometric series which can be summed, $\frac{d\left(x \cdot f_1(x)\right)}{dx} = \frac{a_1}{1-x^n}$. By the Fundamental Theorem of Calculus, $x \cdot f_1(x) = \int_0^x \frac{a_1}{1-t^n} dt$, which implies $f_1(x) = \frac{1}{x} \int_0^x \frac{a_1}{1-t^n} dt$ when $x \neq 0$.

Similarly, for the i^{th} summation, let $f_i(x) = \sum_{k=0}^{\infty} \frac{a_i \cdot x^{nk}}{nk+i}$. As before

$$\frac{d\left(x^i \cdot f_i(x)\right)}{dx} = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{a_i \cdot x^{nk+i}}{nk+i} \right) = \sum_{k=0}^{\infty} a_i \cdot x^{nk+i-1},$$

a geometric series that sums to $\frac{a_i \cdot x^{i-1}}{1-x^n}$. Integrating gives $x^i \cdot f_i(x) = \int_0^x \frac{a_i \cdot t^{i-1}}{1-t^n} dt$. Thus,

$$f_i(x) = \frac{1}{x} \int_0^x \frac{a_i \cdot \left(\frac{t}{x}\right)^{i-1}}{1-t^n} dt.$$

Adding the n terms,

$$\alpha=\sum_{k=0}^{\infty}\frac{a_1\cdot x^{nk}}{nk+1}+\sum_{k=0}^{\infty}\frac{a_2\cdot x^{nk}}{nk+2}+\cdots+\sum_{k=0}^{\infty}\frac{a_n\cdot x^{nk}}{nk+n}=\frac{1}{x}\int_0^x\frac{a_1+a_2\cdot\frac{t}{x}+\ldots+a_n\cdot\left(\frac{t}{x}\right)^{n-1}}{1-t^n}\,dt\ .$$

Finally, using the substitution $u=\frac{t}{x}$ we obtain

$$\alpha=\int_0^1\frac{a_1+a_2\cdot u+\ldots+a_n\cdot u^{n-1}}{1-x^nu^n}\,du\ .$$ ■

Therefore, verifying

$$\alpha=\sum_{k=0}^{\infty}x^{nk}\left(\frac{a_1}{nk+1}+\frac{a_2}{nk+2}+\cdots+\frac{a_n}{nk+n}\right)$$

is equivalent to verifying

$$\int_0^1\frac{a_1+a_2\cdot u+\ldots+a_n\cdot u^{n-1}}{1-x^nu^n}\,du=\alpha\ .$$

We were also interested in the alternating case of the form,

$$\alpha=\sum_{k=0}^{\infty}\left(-x^n\right)^k\left(\frac{a_1}{nk+1}+\frac{a_2}{nk+2}+\cdots+\frac{a_n}{nk+n}\right).$$

We have:

Theorem 2: If $0<|x|<1$ or $x=1$, then

$$\sum_{k=0}^{\infty}\left(-x^n\right)^k\left(\frac{a_1}{nk+1}+\frac{a_2}{nk+2}+\cdots+\frac{a_n}{nk+n}\right)=\int_0^1\frac{a_1+a_2\cdot u+\ldots+a_n\cdot u^{n-1}}{1+x^nu^n}du\ .$$

Proof: Again, we only give a proof in the case where $0<|x|<1$.

The proof is entirely analogous to that of Theorem 1 except that the geometric series is

$$\sum_{k=0}^{\infty} a_i \cdot x^{i-1}(-x^n)^k = \frac{a_i \cdot x^{i-1}}{1+x^n},$$

so adding gives (after the same change of variables in the integral)

$$\alpha = \int_0^1 \frac{a_1 + a_2 \cdot u + \ldots + a_n \cdot u^{n-1}}{1+x^n u^n}\, du. \qquad \blacksquare$$

From here, we will focus mostly on $\alpha = c\pi$, where c is an algebraic constant. It should be mentioned that this algorithm works for other constants that involve arctangents and logarithms.

As an example, we verify Leibniz's formula

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) = \sum_{k=0}^{\infty} \frac{4(-1)^k}{2k+1}.$$

In this example $x = 1$, $a_1 = 4$, $a_2 = 0$, $n = 2$, and the series is alternating. Using Theorem 2, we get

$$\sum_{k=0}^{\infty} \frac{(-1)^k \cdot 4}{2k+1} = \int_0^1 \frac{4}{1+u^2}\, du,$$

which can be easily integrated,

$$\int_0^1 \frac{4}{1+u^2} du = 4\arctan(u)\Big|_0^1 = 4[\arctan(1) - \arctan(0)] = 4\left(\frac{\pi}{4} - 0\right) = \pi.$$

This verifies $\sum_{k=0}^{\infty}\frac{4\left(-1\right)^k}{2k+1}=\pi$.

The original BBP formula is

$$\pi=\sum_{k=0}^{\infty}\frac{1}{16^k}\left(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right).$$

Here $x=\frac{1}{\sqrt{2}}$, $a_1=4,\ a_4=-2,\ a_5=-1,\ a_6=-1,\ a_2=a_3=a_7=a_8=0$, $n=8$, and it is non-alternating. To verify this formula is equivalent to verifying that

$$\int_0^1\frac{4-2u^3-u^4-u^5}{1-\frac{u^8}{16}}du=\pi,$$

by Theorem 1. Factoring the numerator and denominator, we cancel out common terms,

$$\int_0^1\frac{4-2u^3-u^4-u^5}{1-\frac{u^8}{16}}du=\int_0^1\frac{(16u-16)(u^4+2u^3+4u^2+4u+4)}{(u^4-2u^3+4u-4)(u^4+2u^3+4u^2+4u+4)}du$$
$$=\int_0^1\frac{16u-16}{u^4-2u^3+4u-4}du.$$

Using partial fractions, we get

$$\int_0^1\frac{16u-16}{u^4-2u^3+4u-4}du=\int_0^1\left(\frac{4u}{u^2-2}-\frac{4(u-2)}{u^2-2u+2}\right)du,$$

two quadratic terms which can be integrated by hand. We have

$$\int_0^1 \left(\frac{4u}{u^2-2} - \frac{4(u-2)}{u^2-2u+2} \right) du = \int_{-2}^{-1} \frac{2}{w}\, dw - \int_{-1}^{0} \frac{4v+4}{v^2+1}\, dv$$
$$= 2\ln\left|u^2-2\right| - 2\ln\left|(u-1)^2+1\right| - 4\arctan(u-1)\Big|_0^1$$
$$= -2\ln 2 + 2\ln 2 + \pi$$
$$= \pi.$$

As in the previous example, when verifying BBP-type formulas it is common that the rational polynomial's numerator and denominator will have common factors that cancel out. Also, the remaining rational polynomial usually has a nice partial fractions decomposition.

For convenience, we use the notation

$$BPP(x,n,(a_1,a_2,\ldots,a_n)) = \sum_{k=0}^{\infty} x^{nk} \left(\frac{a_1}{nk+1} + \frac{a_2}{nk+2} + \cdots + \frac{a_n}{nk+n} \right)$$
$$= \int_0^1 \frac{a_1 + a_2 \cdot u + \ldots + a_n \cdot u^{n-1}}{1-x^n u^n} du$$

and

$$ABPP(x,n,(a_1,a_2,\ldots,a_n)) = \sum_{k=0}^{\infty} \left(-x^n\right)^k \left(\frac{a_1}{nk+1} + \frac{a_2}{nk+2} + \cdots + \frac{a_n}{nk+n} \right)$$
$$= \int_0^1 \frac{a_1 + a_2 \cdot u + \ldots + a_n \cdot u^{n-1}}{1+x^n u^n} du.$$

The following tables contain BBP-type formulas from the literature which I have verified.

Table 1- BBP-type formulas verified for the bases 4, 6, and 8.

α	BBP -Type Formula	Partial Fractions Breakdown
π	$ABBP\left(\frac{1}{\sqrt{2}}, 4, (2, 2, 1, 0)\right)_{\text{a}}$	$\frac{4}{u^2-2u+2}$
$\frac{32\sqrt{3}\pi}{9}$	$BBP\left(\frac{1}{2}, 6, (16, 8, 0, -2, -1, 0)\right)_{\text{a}}$	$\frac{64}{u^2-2u+4}$
$\frac{32\sqrt{3}\pi}{9}$	$BBP\left(\frac{1}{2}, 6, (0, 32, 8, 4, -2, 0)\right)_{\text{c}}$	$\frac{-160}{3(u-2)}-\frac{32}{3(2+u)}+\frac{128(u-1)}{3(u^2-2u+4)}+\frac{64(u-2)}{3(u^2+2u+4)}$
$4\sqrt{3}\pi$	$BBP\left(\frac{1}{2}, 6, (20, 6, -1, -3, -1, 0)\right)_{\text{b}}$	$\frac{4}{2-u}+\frac{4}{u+2}-\frac{8(u-10)}{u^2-2u+4}$
π	$BBP\left(\frac{1}{\sqrt{2}}, 8, (4, 0, 0, -2, -1, -1, 0, 0)\right)_{\text{b}}$	$\frac{4u}{u^2-2}-\frac{4(u-2)}{u^2-2u+2}$
2π	$BBP\left(\frac{1}{\sqrt{2}}, 8, (0, 8, 4, 4, 0, 0, -1, 0)\right)_{\text{b}}$	$-\frac{8u}{u^2-2}+\frac{8u}{u^2-2u+2}$

a-These formulas came from [3, p. 5 formula 6 and p.7 formula 16].

b-These formulas came from [11, formulas 11, 2, and 3].

c-This formula comes from a combination of formulas from [3, p.7 formula 16 minus p.16 formula 59].

For example, for the fourth row,

$$\begin{aligned}
BBP\left(\frac{1}{2}, 6, (20, 6, -1, -3, -1, 0)\right) &= \sum_{k=0}^{\infty}\left(\frac{1}{64}\right)^k \left(\frac{20}{6k+1}+\frac{6}{6k+2}-\frac{1}{6k+3}-\frac{3}{6k+4}-\frac{1}{6k+5}\right) \\
&= \int_0^1 \frac{20+6u-u^2-3u^3-u^4}{1-\frac{u^6}{64}} \\
&= \int_0^1 \left(\frac{4}{u-2}+\frac{4}{u+2}-\frac{8(u-10)}{u^2-2u+4}\right)du \\
&= 4\ln|u-2|+4\ln|u+2|-4\ln|u^2-2u+4|+24\sqrt{3}\arctan\left(\frac{u-1}{\sqrt{3}}\right)\Bigg|_0^1 \\
&= 4(0-\ln 2)+4(\ln 3-\ln 2)-4(\ln 3-\ln 4)+24\sqrt{3}\left(0+\frac{\pi}{6}\right) \\
&= 4\sqrt{3}\pi
\end{aligned}$$

The second table includes the formulas where the base is 12.

Table 2- BBP-type formulas verified for base 12.

α	$BBP\left(x, n, (a_1, a_2, \ldots, a_n)\right)$	Partial Fractions Breakdown
$9\sqrt{3}\pi$	$BBP\left(\frac{1}{\sqrt{3}}, 12, (81, -54, 0, -9, 0, -12, -3, -2, 0, -1, 0, 0)\right)_{\text{a}}$	$\frac{162u}{u^2-3}-\frac{81(u-1)}{u^2+3}-\frac{81(u-2)}{u^2-3u+3}$
$9\sqrt{3}\pi$	$BBP\left(\frac{1}{\sqrt{3}}, 12, (81, 27, -162, -9, 27, 24, -3, 7, 6, 3, -1, 0)\right)_{\text{b}}$	$\frac{-324u}{u^2-3}+\frac{162(u+5)}{u^2+3}+\frac{81(2u-7)}{u^2-3u+3}$
$36\sqrt{3}\pi$	$BBP\left(\frac{1}{\sqrt{3}}, 12, (81, 189, 0, 45, 27, 24, -3, 1, 0, 1, -1, 0)\right)_{\text{b}}$	$\frac{-324u}{u^2-3}+\frac{162(u+2)}{u^2+3}+\frac{81(2u-1)}{u^2-3u+3}$
$8\sqrt{2}\pi$	$BBP\left(\frac{1}{\sqrt{2}}, 12, (32, 0, 8, 0, 8, 0, -4, 0, -1, 0, -1, 0)\right)_{\text{a}}$	$\frac{32}{2+u^2}+\frac{16}{u^2+\sqrt{6}u+2}+\frac{16}{u^2-\sqrt{6}u+2}$

a-These formulas came from [3, p. 14 formula 41 and p. 7 formula 15 respectively].

b-These formulas came from [11, formulas 13 and 14 respectively].

We also verified two more complicated cases conjectured in the paper *A Compendium of BBP-type Formulas for Mathematical Constants* by David H. Bailey. He conjectured that

1. $\sum_{k=0}^{\infty}\left(\frac{1}{729}\right)^{k}\left(\frac{81}{12k+2}-\frac{162}{12k+3}+\frac{27}{12+5}+\frac{36}{12+6}+\frac{9}{12k+8}+\frac{6}{12k+9}+\frac{4}{12+10}-\frac{1}{12k+11}\right)=0$

2. $\sum_{k=0}^{\infty}\left(\frac{1}{729}\right)^{k}\left(\frac{243}{12k+1}-\frac{324}{12k+2}-\frac{162}{12k+3}-\frac{81}{12k+4}-\frac{36}{12k+6}-\frac{9}{12k+7}+\frac{6}{12k+9}-\frac{1}{12k+10}\right)=0$

in [3, p. 18 formulas 72 & 73 respectively]

In our notation, we wish to show

$$BBP\left(\frac{1}{\sqrt{3}}, 12, (0, 81, -162, 0, 27, 36, 0, 9, 6, 4, -1, 0)\right)$$

and

$$BBP\left(\frac{1}{\sqrt{3}}, 12, (243, -324, -162, -81, 0, -36, -9, 0, 6, -1, 0, 0)\right)$$

are both zero.

Using Theorem 1,

$$
\begin{aligned}
&BBP\left(\frac{1}{\sqrt{3}}, 12, (0, 81, -162, 0, 27, 36, 0, 9, 6, 4, -1, 0)\right)\\
&\quad = \int_0^1 \frac{81u - 162u^2 + 27u^4 + 36u^5 + 9u^7 + 6u^8 + 4u^9 - u^{10}}{1 - \frac{u^{12}}{729}} du\\
&\quad = \int_0^1 \left(\frac{-486u}{u^2 - 3} + \frac{243(u+3)}{u^2+3} + \frac{243(u-3)}{u^2 - 3u + 3}\right) du\\
&\quad = -243\ln(u^2-3) + 243\left(\sqrt{3}\arctan\left(\frac{u}{\sqrt{3}}\right) + \frac{1}{2}\ln(u^2+3)\right) + 243\left(-\sqrt{3}\arctan\left(\frac{2u-3}{\sqrt{3}}\right) + \frac{1}{2}\ln(u^2 - 3u + 3)\right)\Bigg|_0^1\\
&\quad = 243(\ln 3 - \ln 2) + 243\left(\frac{\sqrt{3}\pi}{6} + \ln 2 - \frac{1}{2}\ln 3\right) - 243\left(\frac{\sqrt{3}\pi}{6} + \frac{1}{2}\ln 3\right)\\
&\quad = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&BBP\left(\frac{1}{\sqrt{3}}, 12, (243, -324, -162, -81, 0, -36, -9, 0, 6, -1, 0, 0)\right)\\
&\quad = \int_0^1 \frac{243 - 324u - 162u^2 - 81u^3 - 36u^5 - 9u^6 + 6u^8 - u^9}{1 - \frac{u^{12}}{729}} du\\
&\quad = \int_0^1 \left(\frac{486u}{u^2 - 3} - \frac{243(u-3)}{u^2+3} - \frac{243u}{u^2 - 3u + 3}\right) du\\
&\quad = 243\ln(u^2-3) + 243\left(\sqrt{3}\arctan\left(\frac{u}{\sqrt{3}}\right) - \frac{1}{2}\ln(u^2+3)\right) - 243\left(\sqrt{3}\arctan\left(\frac{2u-3}{\sqrt{3}}\right) + \frac{1}{2}\ln(u^2 - 3u + 3)\right)\Bigg|_0^1\\
&\quad = 243(\ln 2 - \ln 3) + 243\left(\frac{\sqrt{3}\pi}{6} - \ln 2 + \frac{1}{2}\ln 3\right) - 243\left(\frac{\sqrt{3}\pi}{6} - \frac{1}{2}\ln 3\right)\\
&\quad = 0.
\end{aligned}
$$

In fact, most of Bailey's conjectures for degree one BBP-type formulas from [3, p. 16 and 18] can be proven using Theorem 1 and formulas for combinations of arctangents.

Chapter 3: Searching and Discovering BBP-Type Formulas

In the previous chapter we described how to verify previously found BBP-type formulas. In this chapter we describe an algorithm for finding such formulas. This algorithm is also based on Theorem 1 and Theorem 2. Using Theorem 1, for example, we start with

(3.1) $$\sum_{k=0}^{\infty} x^{nk}\left(\frac{a_1}{nk+1}+\frac{a_2}{nk+2}+\cdots+\frac{a_n}{nk+n}\right)=\int_0^1 \frac{a_1+a_2\cdot u+\ldots+a_n\cdot u^{n-1}}{1-x^n u^n}du\ .$$

We integrate the right hand side of (3.1) symbolically as a function of $a_1, a_2, \ldots, a_n, x$ and look for values of the parameters that give a "nice" answer. For small cases, this can be done by-hand, but for most cases, a software program, i.e. *Mathematica*, was used. Once integrated, arctangents and logarithms of functions of x appear. We use one of the arctangents to give π (usually by setting its argument to be $\frac{1}{\sqrt{3}}, 1, \sqrt{3}$). This typically fixes our value of x.

Plugging in a "nice" x value, we get a function in terms of $a_1, a_2, \ldots, a_n$. We construct a system of linear equations involving $a_1, a_2, \ldots, a_n$, to eliminate non-π values. Row-reducing these equations in a matrix, we get a set of solutions that result in BBP-type formulas for a scaled values of π.

For example, suppose we wish to find a nice formula of type $ABPP(x,3,(a_1,a_2,a_3))$. We have

$$\begin{aligned} ABPP(x,3,(a_1,a_2,a_3)) &= \sum_{k=0}^{\infty}\left(-x^3\right)^k\left(\frac{a_1}{3k+1}+\frac{a_2}{3k+2}+\frac{a_3}{3k+3}\right) \\ &= \int_0^1 \frac{a_1+a_2\cdot u+a_3\cdot u^2}{1+x^3u^3}du. \end{aligned}$$

from Theorem 2. Using *Mathematica* to perform the integral,

$$ABPP(x,3,(a_1,a_2,a_3)) = \frac{\pi(a_1x+a_2)}{6\sqrt{3}x^2} + \frac{1}{6x^2}\left(-2\sqrt{3}x(a_1x+a_2)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)\right.$$
$$\left. + 2(a_1x^2 - a_2x + 2a_3)\ln(1+x) + (-a_1x^2 + a_2x + 2a_3)\ln(1-x+x^2)\right).$$

We can get an expression involving π using either $\frac{\pi(a_1x+a_2)}{6\sqrt{3}x^2}$ or $\arctan\left(\frac{1-2x}{\sqrt{3}}\right)$. When using $\frac{\pi(a_1x+a_2)}{6\sqrt{3}x^2}$, we need to eliminate other terms. The arctangent term can only be eliminated by setting $x=\frac{1}{2}$, (since we cannot use (a_1x+a_2) or our π value would be eliminated also). We have

$$ABPP\left(\frac{1}{2},3,(a_1,a_2,a_3)\right) = \frac{1}{9}\left(\left(\sqrt{3}a_1 + 2\sqrt{3}a_2\right)\pi + \left(3a_1 - 6a_2 + 48a_3\right)\ln 3 - 72a_3\ln 2\right).$$

Thus, we need $3a_1 - 6a_2 + 48a_3 = 0$ and $-72a_3 = 0$. Row reducing

$$\begin{bmatrix} 3 & -6 & 48 & 0 \\ 0 & 0 & -72 & 0 \end{bmatrix}$$

we get

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

implying $a_1 = 2a_2$ and $a_3 = 0$, resulting in the BBP-type formula

$$ABBP\left(\frac{1}{2},3,(2,1,0)\right) = \sum_{k=0}^{\infty}\left(-\frac{1}{8}\right)^k\left(\frac{2}{3k+1} + \frac{1}{3k+2}\right) = \frac{4\sqrt{3}\pi}{9}. \tag{3.2}$$

Alternatively, when $x=1$,

$$\arctan\left(\frac{1-2x}{\sqrt{3}}\right)=\arctan\left(\frac{-1}{\sqrt{3}}\right)=-\frac{\pi}{6}.$$

This results in

$$ABPP\left(1,3,(a_1,a_2,a_3)\right)=\frac{1}{9}\left((a_1+a_2)\sqrt{3}\pi+3(a_1-a_2+a_3)\ln 2\right).$$

Thus, we need $a_1-a_2+a_3=0$. There are two degrees of freedom, which implies two BBP-type formulas. Letting $a_3=0$ we get

(3.3) $$ABBP\left(1,3,(1,1,0)\right)=\sum_{k=0}^{\infty}\left(-1\right)^k\left(\frac{1}{3k+1}+\frac{1}{3k+2}\right)=\frac{2\sqrt{3}\pi}{9}.$$

If $a_2=0$, then we get

(3.4) $$ABBP\left(1,3,(1,0,-1)\right)=\sum_{k=0}^{\infty}\left(-1\right)^k\left(\frac{1}{3k+1}-\frac{1}{3k+3}\right)=\frac{\sqrt{3}\pi}{9}.$$

Hence, up to linear combinations of (3.2), (3.3) and (3.4), we have found three BBP-type formulas for the case $ABPP(x,3,(a_1,a_2,a_3))$. There are certainly more formulas if you allow more exotic x values.

In many cases, I had to simplify *Mathematica*'s output by hand. The output in this paper is not the exact output from *Mathematica*. See Appendix 2 for an example of the actual *Mathematica* printout for the non-alternating base 6 case. Simplification was needed for a general x value and when searching for a system of linear equations to eliminate the non-π values. The simplified version is shown throughout this paper.

Chapter 4: Examining Cases Where the Base is 2, 3, or 4

Using the algorithm described in the last section, we searched for BBP-type formulas for non-alternating and alternating bases 2, 3, 4, 6, and 8. In this chapter, we present the results for the simplest cases, bases 2, 3, and 4. The simplest case of all is the alternating base 2 case.

From Theorem 2 we have

$$\begin{aligned} ABBP\left(x, 2, (a_1, a_2)\right) &= \sum_{k=0}^{\infty}\left(-x^2\right)^k \left(\frac{a_1}{2k+1} + \frac{a_2}{2k+2}\right) \\ &= \int_0^1 \frac{a_1 + a_2 u}{1 + x^2 u^2}\, du. \end{aligned}$$

After integrating, we get

$$ABBP\left(x, 2, (a_1, a_2)\right) = \frac{1}{2x^2}\left(2a_1 x \arctan(x) + a_2 \ln(1 + x^2)\right).$$

From this equation we look for "nice" x-values. The criteria for x to be "nice" are that x^n needs to be rational and its use allows for the evaluate or eliminate an arctangent. In order to eliminate an arctangent, we need π to be a result of the integral which does not always occur.

In the above example, the two values $x = \frac{1}{\sqrt{3}}$ or $x = 1$ lead to interesting formulas since $\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ and $\arctan(1) = \frac{\pi}{4}$. When $x = \frac{1}{\sqrt{3}}$,

$$ABBP\left(\frac{1}{\sqrt{3}}, 2, (a_1, a_2)\right) = \frac{a_1 \pi}{2\sqrt{3}} + \frac{3a_2}{2}\ln\left(\frac{4}{3}\right).$$

Thus $a_2 = 0$, which results in the BBP-type formula

(4.1) $$ABBP\left(\frac{1}{\sqrt{3}}, 2, (1,0)\right) = \sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k \left(\frac{1}{2k+1}\right) = \frac{\pi\sqrt{3}}{6}.$$

When $x = 1$,

$$ABBP\left(1, 2, (a_1, a_2)\right) = a_1\frac{\pi}{4} + \frac{a_2}{2}\ln(2).$$

Thus $a_2 = 0$, which gives

(4.2) $$ABBP\left(1, 2, (1,0)\right) = \sum_{k=0}^{\infty}\left(-1\right)^k \left(\frac{1}{2k+1}\right) = \frac{\pi}{4},$$

the Leibniz formula again.

For non-alternating formulas with base $n = 3$, from Theorem 1, we have

$$\begin{aligned} BPP(x,3,(a_1,a_2,a_3)) &= \sum_{k=0}^{\infty}\left(x^{3k}\right)\left(\frac{a_1}{3k+1} + \frac{a_2}{3k+2} + \frac{a_3}{3k+3}\right) \\ &= \int_0^1 \frac{a_1 + a_2 \cdot u + a_3 \cdot u^2}{1 - x^3u^3} du. \end{aligned}$$

Using *Mathematica* to integrate,

$$\begin{aligned} BPP(x,3,(a_1,a_2,a_3)) = \frac{1}{18x^3}\Big(&(a_1x^2 + a_2x + a_3)6i\pi + (-a_1x^2 + a_2x)\sqrt{3}\pi \\ &+ 6\sqrt{3}(a_1x^2 - a_2x)\arctan\left(\frac{1+2x}{\sqrt{3}}\right) - 6(a_1x^2 + a_2x + a_3)\ln(x-1) + 3(a_1x^2 + a_2x - 2a_3)\ln(x^2+x+1)\Big). \end{aligned}$$

The only “nice” x value appears to be $x=1$ since we then have $\arctan\sqrt{3}=\frac{\pi}{3}$. With $x=1$

we have

$$BPP(1,3,(a_1,a_2,a_3))=\frac{1}{18}\Big((a_1+a_2+a_3)6i\pi+(a_1-a_2)\sqrt{3}\pi-3(a_1+a_2)\ln 3\Big).$$

This implies that $a_1+a_2+a_3=0$ and $a_1+a_2=0$. Row-reducing

$$\begin{bmatrix}1 & 1 & 1 & 0\\ 1 & 1 & 0 & 0\end{bmatrix}$$

yields

$$\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}.$$

Thus, $a_1=-a_2$ and $a_3=0$, resulting in the BBP-type formula

(4.3) $$BPP(1,3,(1,-1,0))=\sum_{k=0}^{\infty}\left(\frac{1}{3k+1}-\frac{1}{3k+2}\right)=\frac{\sqrt{3}\pi}{9}.$$

The alternating base 3 formulas were investigated in the previous chapter.

Next we will examine non-alternating base 4 formulas,

$$BBP\left(x,4,(a_1,a_2,a_3,a_4)\right)=\int_0^1\frac{a_1+a_2u+a_3u^2+a_4u^3}{1-x^4u^4}du.$$

Using *Mathematica*,

$$BBP\left(x,4,(a_1,a_2,a_3,a_4)\right)=\frac{1}{4x^2}\left((a_1x^3+a_2x^2+a_3x+a_4)i\pi+2(a_1x^3-a_3x)\arctan(x)\right.$$
$$\left.-(a_1x^3+a_2x^2+a_3x+a_4)\ln(x-1)+(a_1x^3-a_2x^2+a_3x-a_4)\ln(x+1)+(a_2x^2-a_4)\ln(x^2+1)\right).$$

The $\arctan(x)$ term can be evaluated if $x=\frac{1}{\sqrt{3}}$ or $x=1$. When $x=\frac{1}{\sqrt{3}}$,

$$BBP\left(\frac{1}{\sqrt{3}},4,(a_1,a_2,a_3,a_4)\right)=\frac{1}{4}\left((a_1-3a_3)\frac{\sqrt{3}\pi}{3}-(a_1+3a_3)\sqrt{3}\ln\left(3-\sqrt{3}\right)\right.$$
$$\left.+(a_1+3a_3)\sqrt{3}\ln\left(3+\sqrt{3}\right)+(a_2-9a_4)3\ln 2+18a_4\ln 3\right).$$

We row-reduce

$$\begin{bmatrix}1&0&3&0&0\\0&1&0&-9&0\\0&0&0&1&0\end{bmatrix},$$

which gives

$$\begin{bmatrix}1&0&3&0&0\\0&1&0&0&0\\0&0&0&1&0\end{bmatrix}.$$

Thus $a_1=-3a_3$ and $a_2=a_4=0$. Therefore, the BBP-type formula is

$$BBP\left(\frac{1}{\sqrt{3}},4,(3,0,-1,0)\right)=\sum_{k=0}^{\infty}\left(\frac{1}{9}\right)^k\left(\frac{3}{4k+1}-\frac{1}{4k+3}\right)=\frac{\sqrt{3}\pi}{2},$$

which is the same as 3 times the result in (4.1).

With $x=1$, $BBP\left(1,4,(a_1,a_2,a_3,a_4)\right)$ only converges if $a_1+a_2+a_3+a_4=0$. Using this,

$$BBP\left(1,4,(a_1,a_2,a_3,a_4)\right)=\frac{1}{4}\left((a_1-a_3)\frac{\pi}{2}+(a_1+a_3-2a_4)\ln 2\right).$$

Hence, we row-reduce

$$\begin{bmatrix}1 & 1 & 1 & 1 & 0\\ 1 & 0 & 1 & -2 & 0\end{bmatrix}$$

and get

$$\begin{bmatrix}1 & 0 & 1 & -2 & 0\\ 0 & 1 & 0 & 3 & 0\end{bmatrix}.$$

This produces the BBP-type formulas

$$BBP\left(1,4,(1,0,-1,0)\right)=\sum_{k=0}^{\infty}\left(\frac{1}{4k+1}-\frac{1}{4k+3}\right)=\frac{\pi}{4}$$

which is (4.2) again, and

$$BBP\left(1,4,(2,-3,0,1)\right)=\sum_{k=0}^{\infty}\left(\frac{2}{4k+1}-\frac{3}{4k+2}+\frac{1}{4k+4}\right)=\frac{\pi}{4}.$$

The alternating base 4 case gives

$$\begin{aligned}ABBP\left(x,4,(a_1,a_2,a_3,a_4)\right)=\frac{1}{8x^4}\Big(&a_2x^2\pi-(2\sqrt{2}a_1x^3+4a_2x^2+2\sqrt{2}a_3x)\arctan\left(1-\sqrt{2}x\right)\\ &+(2\sqrt{2}a_1x^3-4a_2x^2+2\sqrt{2}a_3x)\arctan\left(1+\sqrt{2}x\right)-(a_1x^3-a_3x-\sqrt{2}a_4)\sqrt{2}\ln(x^2-\sqrt{2}x+1)\\ &+(a_1x^3-a_3x+\sqrt{2}a_4)\sqrt{2}\ln(x^2+\sqrt{2}x+1)\Big).\end{aligned}$$

The only obvious nice x value is $x=\frac{1}{\sqrt{2}}$ which eliminates $\arctan(1-\sqrt{2}x)$. This implies that $a_2 \neq 0$ or we will eliminate our only π term. When $x=\frac{1}{\sqrt{2}}$

$$ABBP\left(\frac{1}{\sqrt{2}},4,(a_1,a_2,a_3,a_4)\right)=\frac{1}{4}\left(2a_2\pi+(a_1-2a_2+2a_3)2\arctan 2+(a_1-2a_3+4a_4)\ln 5-8a_4\ln 2\right).$$

By row-reducing

$$\begin{bmatrix} 1 & -2 & 2 & 0 & 0 \\ 1 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

we have

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus, we have the BBP-type formula

$$ABBP\left(\frac{1}{\sqrt{2}},4,(2,2,1,0)\right)=\sum_{k=0}^{\infty}\left(-\frac{1}{4}\right)^k\left(\frac{2}{4k+1}+\frac{2}{4k+2}+\frac{1}{4k+3}\right)=\pi .$$

It turns out that $x=1$ also produces BBP-type formulas. This is because $\arctan(1+\sqrt{2})=\frac{3\pi}{8}$ and $\arctan(1-\sqrt{2})=\frac{-\pi}{8}$. When $x=1$ we have

$$ABBP(1,4,(a_1,a_2,a_3,a_4))=\frac{1}{8}\Big((\sqrt{2}a_1+a_2+\sqrt{2}a_3)\pi+(-\sqrt{2}a_1+\sqrt{2}a_3+2a_4)\ln\left(2-\sqrt{2}\right)$$
$$+(\sqrt{2}a_1-\sqrt{2}a_3+2a_4)\ln\left(2+\sqrt{2}\right)\Big).$$

Row-reducing

$$\begin{bmatrix} -\sqrt{2} & 0 & \sqrt{2} & 2 & 0 \\ \sqrt{2} & 0 & -\sqrt{2} & 2 & 0 \end{bmatrix}$$

yields

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore, we have

$$ABBP\left(1,4,(0,1,0,0)\right) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{4k+2} = \frac{\pi}{8}$$

which is the Leibniz formula, (4.2), in disguise, and

(4.4) $$ABBP\left(1,4,(1,0,1,0)\right) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{4k+1} + \frac{1}{4k+3} \right) = \frac{\sqrt{2}\pi}{4}.$$

Chapter 5: Examining the Cases with a Base 6

We found no nice BBP-type formulas with base 5. In this chapter, we present results for base 6. For the non-alternating version,

$$BBP\left(x,6,(a_1,a_2,a_3,a_4,a_5,a_6)\right)=\frac{1}{18x^6}\left((a_2x^4-a_4x^2)\sqrt{3}\pi+(a_1x^5+a_2x^4+a_3x^3+a_4x^2+a_5x+a_6)3i\right)$$
$$+\frac{1}{12x^6}\Big((a_1x^5+a_2x^4-a_4x^2-a_5x)2\sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)$$
$$+(a_1x^5-a_2x^4+a_4x^2-a_5x)2\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+(a_1x^5+a_2x^4+a_3x^3+a_4x^2+a_5x+a_6)2\ln(x-1)$$
$$+(a_1x^5-a_2x^4+a_3x^3-a_4x^2+a_5x-a_6)2\ln(x+1)$$
$$-(a_1x^5-a_2x^4-2a_3x^3-a_4x^2+a_5x+2a_6)\ln(x^2-x+1)$$
$$+(a_1x^5+a_2x^4-2a_3x^3+a_4x^2+a_5x-2a_6)\ln(x^2+x+1)\Big).$$

The $\arctan\left(\frac{2x-1}{\sqrt{3}}\right)$ term suggests $x=\frac{1}{2}$ since $\arctan\left(\frac{2\left(\frac{1}{2}\right)-1}{\sqrt{3}}\right)=0$, or $x=1$ since $\arctan\left(\frac{2(1)-1}{\sqrt{3}}\right)=\arctan\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$. The $\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$ term also evaluates when $x=1$ since $\arctan\left(\frac{2(1)+1}{\sqrt{3}}\right)=\arctan\left(\sqrt{3}\right)=\frac{\pi}{3}$.

When $x=\frac{1}{2}$,

$$BBP\left(\frac{1}{2},6,(a_1,a_2,a_3,a_4,a_5,a_6)\right)=\frac{1}{18}\Big(4\sqrt{3}(a_2-4a_4)\pi+6\sqrt{3}\left(a_1-2a_2+8a_4-16a_5\right)\arctan\left(\frac{2}{\sqrt{3}}\right)$$
$$+192a_6\ln 2+3\left(a_1-2a_2+16a_3-8a_4+16a_5-128a_6\right)\ln 3$$
$$+3\left(a_1+2a_2-8a_3+8a_4+16a_5-64a_6\right)\ln 7\Big).$$

Row-reducing

$$\begin{bmatrix} 1 & -2 & 0 & 8 & -16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 192 & 0 \\ 1 & -2 & 16 & -8 & 16 & -128 & 0 \\ 1 & 2 & -8 & 8 & 16 & -64 & 0 \end{bmatrix}$$

we get

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 8 & 0 & 0 \\ 0 & 1 & 0 & -2 & 12 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus,

$$BBP\left(\frac{1}{2}, 6, (4, -2, -1, -1, 0, 0)\right) = \sum_{k=0}^{\infty}\left(\frac{1}{64}\right)^k \left(\frac{4}{6k+1} - \frac{2}{6k+2} - \frac{1}{6k+3} - \frac{1}{6k+4}\right) = \frac{4\sqrt{3}\pi}{9}$$

and

$$BBP\left(\frac{1}{2}, 6, (8, 12, 2, 0, -1, 0)\right) = \sum_{k=0}^{\infty}\left(\frac{1}{64}\right)^k \left(\frac{8}{6k+1} + \frac{12}{6k+2} + \frac{2}{6k+3} - \frac{1}{6k+5}\right) = \frac{8\sqrt{3}\pi}{3}.$$

When $x = 1$, we need $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0$ for convergence. Using this,

$$BBP\left(1, 6, (a_1, a_2, a_3, a_4, a_5, a_6)\right) = \frac{1}{36}\Big((3a_1 + a_2 - a_4 - 3a_5)\sqrt{3}\,\pi$$
$$+ (a_1 - a_2 + a_3 - a_4 + a_5 - a_6)6\ln 2 + (a_1 + a_2 - 2a_3 + a_4 + a_5 - 2a_6)3\ln 3\Big).$$

Row-reducing

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 0 \\ 1 & 1 & -2 & 1 & 1 & -2 & 0 \end{bmatrix}$$

yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus, we have

$$BBP\left(1,\ 6,\ (1,\ 0,\ 0,\ 0,\ -1,\ 0)\right) = \sum_{k=0}^{\infty}\left(\frac{1}{6k+1} - \frac{1}{6k+5}\right) = \frac{\sqrt{3}\pi}{6},$$

$$BBP\left(1,\ 6,\ (0, 1,\ 0,\ -1,\ 0,\ 0)\right) = \sum_{k=0}^{\infty}\left(\frac{1}{6k+2} - \frac{1}{6k+4}\right) = \frac{\sqrt{3}\pi}{18},$$

and

$$BBP\left(1,\ 6,\ (1,\ -1, -1,\ 0,\ 0,\ 1)\right) = \sum_{k=0}^{\infty}\left(\frac{1}{6k+1} - \frac{1}{6k+2} - \frac{1}{6k+3} + \frac{1}{6k+6}\right) = \frac{\sqrt{3}\pi}{18}.$$

The second of these is equivalent to the formula (4.3) for base 3.

In the alternating case, we have

$$\begin{aligned} ABBP\left((x,\ 6,\ (a_1, a_2, a_3, a_4, a_5, a_6)\right) = \frac{1}{12x^6}\Big(&(a_2x^2 + a_4)2\sqrt{3}\pi + (a_1x^5 + a_3x^3 + a_5x)4\arctan\left(x\right) \\ &-(a_1x^5 + \sqrt{3}a_2x^4 + 2a_3x^3 + \sqrt{3}a_4x^2 + a_5x)2\arctan\left(\sqrt{3} - 2x\right) \\ &+(a_1x^5 - \sqrt{3}a_2x^4 + 2a_3x^3 - \sqrt{3}a_4x^2 + a_5x)2\arctan\left(\sqrt{3} + 2x\right) + (a_2x^4 + a_4x^2 + a_6)2\ln\left(x^2 + 1\right) \\ &-(\sqrt{3}a_1x^5 + a_2x^4 - a_4x^2 - \sqrt{3}a_5x + 2a_6)\ln\left(x^2 - \sqrt{3}x + 1\right) \\ &+(\sqrt{3}a_1x^5 - a_2x^4 + a_4x^2 - \sqrt{3}a_5x + 2a_6)\ln\left(x^2 + \sqrt{3}x + 1\right)\Big). \end{aligned}$$

We can use $x = \frac{1}{\sqrt{3}}$ and $x = 1$ in the argument $\arctan(x)$. We can also eliminate $\arctan\left(\sqrt{3} - 2x\right)$ when $x = \frac{\sqrt{3}}{2}$. When $x = 1$, the terms $\arctan\left(\sqrt{3} + 2x\right)$ and $\arctan\left(\sqrt{3} - 2x\right)$ evaluate since $\arctan\left(\sqrt{3} + 2\right) = \frac{5\pi}{12}$ and $\arctan\left(\sqrt{3} - 2\right) = \frac{-\pi}{12}$.

When $x = \frac{1}{\sqrt{3}}$,

$$ABBP\left(\frac{1}{\sqrt{3}}, 6, (a_1, a_2, a_3, a_4, a_5, a_6)\right) = \frac{1}{12}\Big((a_1 + 9a_2 - 12a_3 + 27a_4 + 9a_5)\frac{\pi}{\sqrt{3}}$$
$$+\left(a_1 - 3a_2 + 6a_3 - 9a_4 + 9a_5\right)2\sqrt{3}\arctan\left(\frac{5}{\sqrt{3}}\right) + \left(a_2 - 3a_4 + 9a_6\right)12\ln 2$$
$$-162a_6\ln 3 + \left(a_1 - a_2 + 3a_3 - 9a_4 + 18a_5\right)3\ln 7\Big).$$

Row-reducing

$$\begin{bmatrix} 1 & -3 & 6 & -9 & 9 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 3 & -9 & 18 & 0 & 0 \end{bmatrix}$$

gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We have

$$ABBP\left(\frac{1}{\sqrt{3}}, 6, (9, 0, -3, 0, 1, 0)\right) = \sum_{k=0}^{\infty}\left(\frac{-1}{27}\right)^k\left(\frac{9}{6k+1} - \frac{3}{6k+3} + \frac{1}{6k+5}\right) = \frac{3\sqrt{3}\pi}{2}$$

and

$$ABBP\left(\frac{1}{\sqrt{3}}, 6, (0, 3, 3, 1, 0, 0)\right) = \sum_{k=0}^{\infty}\left(\frac{-1}{27}\right)^k\left(\frac{3}{6k+2} + \frac{3}{6k+3} + \frac{1}{6k+4}\right) = \frac{\sqrt{3}\pi}{2}.$$

When $x = \frac{\sqrt{3}}{2}$,

$$\begin{aligned} ABBP\left(\frac{\sqrt{3}}{2}, 6, (a_1, a_2, a_3, a_4, a_5, a_6)\right) &= \frac{1}{162}\Big((3a_2 + 4a_4)8\sqrt{3}\pi + (9a_1 - 12a_3 + 16a_5)4\sqrt{3}\arctan\left(\frac{\sqrt{3}}{2}\right) \\ &+ \left(9\sqrt{3}a_1 - 18\sqrt{3}a_2 + 24\sqrt{3}a_3 - 24\sqrt{3}a_4 + 16\sqrt{3}a_5\right)2\arctan\left(2\sqrt{3}\right) \\ &+ (27a_1 - 18a_2 + 24a_4 - 48a_5 + 64a_6)\ln 13 + (9a_2 - 12a_4 + 16a_6)4\ln 7 + 384a_6 \ln 2\Big). \end{aligned}$$

Row-reducing

$$\begin{bmatrix} 9 & 0 & -12 & 0 & 16 & 0 & 0 \\ 9 & -18 & 24 & -24 & 16 & 0 & 0 \\ 27 & -18 & 0 & 24 & -48 & 64 & 0 \\ 0 & 9 & 0 & -12 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

gives

$$\begin{bmatrix} 9 & 0 & 0 & 0 & -16 & 0 & 0 \\ 0 & 3 & 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 3 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

yielding

$$ABBP\left(\frac{\sqrt{3}}{2}, 6, (16, 24, 24, 18, 9, 0)\right) = \sum_{k=0}^{\infty}\left(\frac{-27}{64}\right)^k\left(\frac{16}{6k+1}+\frac{24}{6k+2}+\frac{24}{6k+3}+\frac{18}{6k+4}+\frac{9}{6k+5}\right) = \frac{64\sqrt{3}\pi}{9}.$$

.

Finally, when $x=1$,

$$ABBP\left(1, 6, (a_1, a_2, a_3, a_4, a_5, a_6)\right) = \frac{1}{36}\big((6a_1+2\sqrt{3}a_2+3a_3+2\sqrt{3}a_4+6a_5)\pi$$
$$+(a_2-a_4+a_6)6\ln 2+(a_1-a_5)6\sqrt{3}\ln\left(2+\sqrt{3}\right)\big).$$

Row-Reducing

$$\begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}.$$

Hence, this yields four BBP-type formulas,

$$ABBP\left(1,6,(0,0,1,0,0,0)\right) = \sum_{k=0}^{\infty}(-1)^k\frac{1}{6k+3} = \frac{\pi}{12},$$

which is Leibniz's formula in disguise again,

$$ABBP\left(1,6,(1,0,0,0,1,0)\right) = \sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{6k+1}+\frac{1}{6k+5}\right) = \frac{\pi}{3},$$

$$ABBP\left(1,6,(0,1,0,1,0,0)\right)=\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{6k+2}+\frac{1}{6k+4}\right)=\frac{\sqrt{3}\pi}{9},$$

and

$$ABBP\left(1,6,(0,1,0,1,0,0)\right)=\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{6k+2}-\frac{1}{6k+6}\right)=\frac{\sqrt{3}\pi}{18}.$$

The third and last formulas are equivalent to formula (3.3) and formula (3.4) respectively.

Chapter 6: Examining Cases with Base 8

This chapter deals with BBP-type formulas with base 8. There do not appear to be any "nice" formulas with base 7. In the non-alternating case,

$$
\begin{aligned}
BBP(x, 8, (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)) = \frac{1}{16x^2}\Big(&(a_1x^7 + a_2x^6 + a_3x^5 + a_4x^4 + a_5x^3 + a_6x^2 + a_7x + a_8)2i\pi \\
&(a_2x^6 - a_6x^2)2\pi - (a_1x^7 + a_2x^6 + a_3x^5 + a_4x^4 + a_5x^3 + a_6x^2 + a_7x + a_8)2\ln(x-1) \\
&+ (a_1x^7 - a_2x^6 + a_3x^5 - a_4x^4 + a_5x^3 - a_6x^2 + a_7x - a_8)2\ln(x+1) \\
&+ (a_1x^7 - a_3x^5 + a_5x^3 - a_7x)4\arctan(x) - (a_2x^6 - a_4x^4 + a_6x^2 - a_8)2\ln(x^2+1) \\
&- (\sqrt{2}a_1x^7 + a_2x^6 + \sqrt{2}a_3x^5 - \sqrt{2}a_5x^3 - a_6x^2 - \sqrt{2}a_7x)2\arctan\left(1-\sqrt{2}x\right) \\
&+ (\sqrt{2}a_1x^7 - a_2x^6 + \sqrt{2}a_3x^5 - \sqrt{2}a_5x^3 - a_6x^2 - \sqrt{2}a_7x)2\arctan\left(1+\sqrt{2}x\right) \\
&- (a_1x^7 - a_3x^5 - \sqrt{2}a_4x^4 - a_5x^3 + a_7x + \sqrt{2}a_8)\sqrt{2}\ln\left(x^2 - \sqrt{2}x + 1\right) \\
&+ (a_1x^7 - a_3x^5 + \sqrt{2}a_4x^4 - a_5x^3 + a_7x + \sqrt{2}a_8)\sqrt{2}\ln\left(x^2 + \sqrt{2}x + 1\right)\Big).
\end{aligned}
$$

Possible x-values are $x = \frac{1}{\sqrt{3}}$ and $x = 1$ from $\arctan(x)$. We can also eliminate $\arctan(1-\sqrt{2}x)$ if $x = \frac{1}{\sqrt{2}}$. The term $\arctan(1-\sqrt{2}x)$ also evaluates when x=1 since $\arctan(1+\sqrt{2}) = \frac{3\pi}{8}$ and $\arctan(1-\sqrt{2}) = \frac{-\pi}{8}$.

When $x = \frac{1}{\sqrt{3}}$,

$$
\begin{aligned}
BBP\left(\frac{1}{\sqrt{3}}, 8, (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)\right) = \frac{1}{8}\Bigg(&\left(\frac{1}{\sqrt{3}}a_1 + 3a_2 - \sqrt{3}a_3 + 3\sqrt{3}a_5 - 27a_6 - 9\sqrt{3}a_7\right)\pi \\
&+ \left(-\sqrt{\frac{3}{2}}a_1 - 3a_2 - 3\sqrt{\frac{3}{2}}a_3 + 9\sqrt{\frac{3}{2}}a_5 + 27a_6 + 27\sqrt{\frac{3}{2}}a_7\right)\arctan\left(1-\sqrt{\frac{2}{3}}\right) \\
&+ \left(\sqrt{\frac{3}{2}}a_1 - 3a_2 + 3\sqrt{\frac{3}{2}}a_3 - 9\sqrt{\frac{3}{2}}a_5 + 27a_6 - 27\sqrt{\frac{3}{2}}a_7\right)\arctan\left(1+\sqrt{\frac{2}{3}}\right) \\
&- 324a_8\ln 3 + (a_2 - 6a_4 + 9a_6 - 108a_8)3\ln 2 + (a_4 - 9a_8)\ln 5 \\
&+ (-a_1 + 3a_3 + 9a_5 - 27a_7)\sqrt{\frac{3}{2}}\ln\left(\frac{4-\sqrt{6}}{3}\right) + (a_1 - 3a_3 - 9a_5 + 27a_7)\sqrt{\frac{3}{2}}\ln\left(\frac{4+\sqrt{6}}{3}\right)\Bigg).
\end{aligned}
$$

Row-reducing

$$\begin{bmatrix} -\sqrt{\frac{3}{2}} & -3 & -3\sqrt{\frac{3}{2}} & 0 & 9\sqrt{\frac{3}{2}} & 27 & 27\sqrt{\frac{3}{2}} & 0 & 0 \\ \sqrt{\frac{3}{2}} & -3 & 3\sqrt{\frac{3}{2}} & 0 & -9\sqrt{\frac{3}{2}} & 27 & -27\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -6 & 0 & 9 & 0 & -108 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -9 & 0 \\ 1 & 0 & 3 & 0 & 9 & 0 & 27 & 0 & 0 \\ 1 & 0 & -3 & 0 & -9 & 0 & 27 & 0 & 0 \end{bmatrix}$$

gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 27 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -9 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus, we have the BBP-type formula

$$BBP\left(\frac{1}{\sqrt{3}}, 8, (27, 0, -9, 0, 3, 0, -1, 0)\right) = \sum_{k=0}^{\infty}\left(\frac{1}{81}\right)^k\left(\frac{27}{8k+1} - \frac{9}{8k+3} + \frac{3}{8k+5} - \frac{1}{8k+7}\right) = \frac{9\sqrt{3}\pi}{2}.$$

This formula is equivalent to the formula in (4.1).

When $x = \frac{1}{\sqrt{2}}$,

$$BBP\left(\frac{1}{\sqrt{2}},8,(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8)\right)=\frac{1}{8}\Big((a_2-4a_6)2\pi$$
$$+(a_1-2a_2+2a_3-4a_5+8a_6-8a_7)2\arctan(2)$$
$$+(a_1-2a_3+4a_5-8a_7)2\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\right)+(a_2-2a_4+4a_6-8a_8)2\ln 3 \quad .$$
$$+(a_1-2a_3+4a_4-4a_5+8a_7-16a_8)\ln 5+64a_8\ln 2+(a_1+2a_3+4a_5+8a_7)\ln\left(1+\frac{1}{\sqrt{2}}\right)$$
$$-(a_1+2a_3+4a_5+8a_7)\ln\left(1-\frac{1}{\sqrt{2}}\right)\Big).$$

Row-reducing

$$\begin{bmatrix} 1 & -2 & 2 & 0 & -4 & 8 & -8 & 0 & 0 \\ 1 & 0 & -2 & 0 & 4 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 4 & 0 & -8 & 0 \\ 1 & 0 & -2 & 4 & -4 & 0 & 8 & -16 & 0 \\ 1 & 0 & 2 & 0 & 4 & 0 & 8 & 0 & 0 \end{bmatrix}$$

we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives two BBP-type formulas,

$$BBP\left(\frac{1}{\sqrt{2}},8,(4,0,0,-2,-1,-1,0,0)\right)=\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\left(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right)=\pi$$

which is the original BBP formula (1.1), and

$$BBP\left(\frac{1}{\sqrt{2}},8,(0,8,4,4,0,0,-1,0)\right)=\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\left(\frac{8}{8k+2}+\frac{4}{8k+3}+\frac{4}{8k+4}-\frac{1}{8k+7}\right)=2\pi\,.$$

When $x=1$, $BBP(1,8,(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8))$ only converges if $a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8=0$. Using this,

$$\begin{aligned}BBP(1,8,(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8))&=\frac{1}{16}\big((a_1+a_2-a_3+a_5-a_6-a_7)\pi\\&+(a_1+a_3-a_5-a_7)\sqrt{2}\pi+(a_1+a_3-a_4+a_5+a_7-3a_8)2\ln 2\\&-(a_1-a_3-a_5+a_7)\sqrt{2}\ln\left(2-\sqrt{2}\right)+(a_1-a_3-a_5+a_7)\sqrt{2}\ln(2+\sqrt{2})\big).\end{aligned}$$

Row-reducing

$$\begin{bmatrix}1&0&1&-1&1&0&1&-3&0\\-1&0&1&0&1&0&-1&0&0\\1&0&-1&0&-1&0&1&0&0\\1&1&1&1&1&1&1&1&0\end{bmatrix}$$

yields

$$\begin{bmatrix}1&0&0&-\frac{1}{2}&0&0&1&-\frac{3}{2}&0\\0&1&0&2&0&1&0&4&0\\0&0&1&-\frac{1}{2}&1&0&0&-\frac{3}{2}&0\\0&0&0&0&0&0&0&0&0\end{bmatrix}.$$

Thus, we have the BBP-type formulas,

(6.1) $$BBP(1,\,8,\,(3,\,-8,\,3,\,0,\,0,\,0,0,\,2))=\sum_{k=0}^{\infty}\left(\frac{3}{8k+1}-\frac{8}{8k+2}+\frac{3}{8k+3}+\frac{2}{8k+8}\right)=\frac{(3\sqrt{2}-4)\pi}{8}\,,$$

(6.2) $$BBP(1,\, 8,\, (1,\, 0, 0,\, 0,\, 0,\, 0, -1, 0)) = \sum_{k=0}^{\infty}\left(\frac{1}{8k+1} - \frac{1}{8k+7}\right) = \frac{(\sqrt{2}+1)\pi}{8},$$

$$BBP(1,\, 8,\, (0,\, 1,\, 0,\, 0,\, 0,\, -1, 0, 0)) = \sum_{k=0}^{\infty}\left(\frac{1}{8k+2} - \frac{1}{8k+6}\right) = \frac{\pi}{8},$$

(6.3) $$BBP(1,\, 8,\, (0,\, 0, 1,\, 0,\, -1,\, 0, 0, 0)) = \sum_{k=0}^{\infty}\left(\frac{1}{8k+3} - \frac{1}{8k+5}\right) = \frac{(\sqrt{2}-1)\pi}{8},$$

and

(6.4) $$BBP(1, 8, (1, -4, 1, 2, 0, 0, 0, 0)) = \sum_{k=0}^{\infty}\left(\frac{1}{8k+1} - \frac{4}{8k+2} + \frac{1}{8k+3} + \frac{2}{8k+4}\right) = \frac{(\sqrt{2}-2)\pi}{8}.$$

For more elegant results, we can take three times (6.2) and subtract (6.1) to get

$$BBP(1, 8, (0, 8, -3, 0, 0, 0, -3, -2)) = \sum_{k=0}^{\infty}\left(\frac{8}{8k+2} - \frac{3}{8k+3} - \frac{3}{8k+7} - \frac{2}{8k+8}\right) = \frac{7\pi}{8}.$$

We can also take (6.1) and add four times (6.2) to get

$$BBP(1, 8, (7, -8, 3, 0, 0, 0, -4, 2)) = \sum_{k=0}^{\infty}\left(\frac{7}{8k+1} - \frac{8}{8k+2} + \frac{3}{8k+3} - \frac{4}{8k+7} + \frac{2}{8k+8}\right) = \frac{7\sqrt{2}\pi}{8}.$$

Also, (6.3) minus (6.4) is

$$BBP(1, 8, (-1, 4, 0, -2, -1, 0, 0, 0)) = \sum_{k=0}^{\infty}\left(\frac{-1}{8k+1} + \frac{4}{8k+2} - \frac{2}{8k+4} - \frac{1}{8k+5}\right) = \frac{\pi}{8}.$$

Lastly, we have two times (6.3) minus (6.4) which is

$$BBP\left(1,8,(-1,4,1,-2,-2,0,0,0)\right)=\sum_{k=0}^{\infty}\left(\frac{-1}{8k+1}+\frac{4}{8k+2}+\frac{1}{8k+3}-\frac{2}{8k+4}-\frac{2}{8k+5}\right)=\frac{\sqrt{2}\pi}{8}.$$

For the alternating case, we have eliminated the several pages of the integration. $ABBP\left(x,8,(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8)\right)$ involves the following arctangents

$$\arctan\left(\frac{-2+\sqrt{2+\sqrt{2}}x}{\sqrt{2-\sqrt{2}}}\right),$$

$$\arctan\left(\frac{2+\sqrt{2+\sqrt{2}}x}{\sqrt{2-\sqrt{2}}}\right),$$

$$\arctan\left(\frac{-2+\sqrt{2-\sqrt{2}}x}{\sqrt{2+\sqrt{2}}}\right),$$

and

$$\arctan\left(\frac{2+\sqrt{2-\sqrt{2}}x}{\sqrt{2+\sqrt{2}}}\right).$$

These arctangents evaluate if $x=1$ as follows

$$\arctan\left(\frac{-2+\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\right)=\frac{-\pi}{16},$$

$$\arctan\left(\frac{2+\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\right)=\frac{7\pi}{16},$$

$$\arctan\left(\frac{-2+\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right)=\frac{-3\pi}{16},$$

and

$$\arctan\left(\frac{2+\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right)=\frac{5\pi}{16}.$$

Using this information we were able to simplify $ABBP(1,8,(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8))$. To eliminate the logarithms we row-reduced

$$\begin{bmatrix} 0 & \frac{5}{4}\sqrt{2+\sqrt{2}} & 0 & 0 & 0 & -\frac{5}{4}\sqrt{2+\sqrt{2}} & 0 & (1-\sqrt{2})\sqrt{2+\sqrt{2}} & 0 \\ 1-\frac{1}{\sqrt{2}} & -\frac{1}{4}\sqrt{2-\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{4}\sqrt{2-\sqrt{2}} & -1+\frac{1}{\sqrt{2}} & -\sqrt{2-\sqrt{2}} & 0 \\ 1-\frac{1}{\sqrt{2}} & \frac{1}{4}\sqrt{2-\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{4}\sqrt{2-\sqrt{2}} & -1+\frac{1}{\sqrt{2}} & \sqrt{2-\sqrt{2}} & 0 \\ 2+\frac{3}{\sqrt{2}} & 0 & 1+\frac{1}{\sqrt{2}} & 0 & -1-\frac{1}{\sqrt{2}} & 0 & -2-\frac{3}{\sqrt{2}} & 0 & 0 \\ -2-\frac{3}{\sqrt{2}} & 0 & -1-\frac{1}{\sqrt{2}} & 0 & 1+\frac{1}{\sqrt{2}} & 0 & 2+\frac{3}{\sqrt{2}} & 0 & 0 \end{bmatrix},$$

which yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This leads to the following BBP-type formulas

$$ABBP\left(1,8,(0,0,0,1,0,0,0,0)\right)=\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{8k+4}\right)=\frac{\pi}{16},$$

$$ABBP\left(1,8,(1,0,0,0,0,0,1,0)\right)=\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{8k+1}+\frac{1}{8k+7}\right)=\frac{\sqrt{4+2\sqrt{2}}\pi}{8},$$

$$ABBP\left(1,8,(0,1,0,0,0,1,0,0)\right)=\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{8k+2}+\frac{1}{8k+6}\right)=\frac{\sqrt{2}\pi}{8},$$

and

$$ABBP\left(1,8,(0,0,1,0,1,0,0,0)\right)=\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{8k+3}+\frac{1}{8k+5}\right)=\frac{\sqrt{4-2\sqrt{2}}\pi}{8}.$$

The first formula is equivalent to Leibniz's formula, (4.2), and the third formula is equivalent to (4.4).

Chapter 7: Conclusions

In this project, I focused on BBP-type formulas for scaled values of π. It might be interesting to carry out this algorithm for other constants, such as arctangents, logarithms, and zero. The constant zero relates the null space of the system of linear equations. Except for an example in the alternating base 4 case and base 8 cases, our method only utilized a single arctangent term. Since there are many formulas for combinations of arctangents, like

$$2\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{3}{4}\right)=\frac{\pi}{2}.$$

A more sophisticated approach would take this into account.

BBP-type formulas can be extended to the formulas of the form

$$\alpha=\sum_{k=0}^{\infty}x^{nk}\left(\frac{a_1}{(nk+1)^s}+\frac{a_2}{(nk+2)^s}+\cdots+\frac{a_n}{(nk+n)^s}\right)$$

where s is an integer. This form leads to BBP-type formulas for constants such as π^2, as in the following example,

$$\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\left(\frac{16}{(8k+1)^2}-\frac{16}{(8k+2)^2}-\frac{8}{(8k+3)^2}-\frac{16}{(8k+4)^2}-\frac{4}{(8k+5)^2}-\frac{4}{(8k+6)^2}+\frac{2}{(8k+7)^2}\right)=\pi^2$$

,

from [3, p. 8 formula 19]. Our algorithm handles only cases where $s=1$. Future work could explore this area.

From [7, p. 737], we have the equation

$$4\sum_{k=0}^{\infty}\left(\frac{-1}{4}\right)^{k}\left(\frac{1}{2k+1}\right)-\frac{1}{64}\sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^{k}\left(\frac{32}{4k+1}+\frac{8}{4k+2}+\frac{1}{4k+3}\right)=\pi .$$

The above formula allows for the individual hex or binary digits of π to be calculated. It is over 40% faster than formula (1.1). It would be of interest to search for a faster equation or of BBP-type formulas that are combinations of different bases with different x-values.

Another limitation to our algorithm is that as we increase the base, we complicate the integration. Base 8 is the highest case that our algorithm can handle without too much complexity. Base 12 appears to have BBP-type formulas; it would have been nice to take a closer look at it.

Our algorithm is useful as it is a combination of elementary concepts. We simply used Theorem 1 and 2, along with row-reducing matrices of systems linear equations to eliminate non-π constants. Also, using integration provided a useful tool in discovering "nice" x-values. Although our algorithm for searching for BBP-type formulas does not work well for higher bases, it does work well for verifying most cases previously discovered of degree one.

References

[1] S. Abbott, *Understanding Analysis*, Springer, New York, 2001.

[2] J. Arndt and C. Haenel, *Pi – Unleashed*, Springer, Berlin, 2001.

[3] D. Bailey, "*A Compendium of BBP-Type Formulas for Mathematical Constants*," 2004, available at http://crd.lbl.gov/~dhbailey/dhbpapers/bbp-formulas.pdf.

[4] D. Bailey and J. Borwein, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, A K Peters, MA, 2004.

[5] D. Bailey, P. Borwein, and S. Plouffe, "*On the Rapid Computation of Various Polylogarithmic Constants*," Mathematics of Computation, vol. 66, no. 218, 1997, p. 903-913.

[6] P. Beckmann, *A History of Pi*, The Golem Press, Boulder, CO, 1971.

[7] L. Berggren, J. Borwein, and P. Borwein, *Pi: A Source Book*, 3rd ed., Springer, New York, 2004.

[8] I. Lehmann and A. S. Posamentier, *A Biography of the World's Most Mysterious Number*, Prometheus Books, Amherst, NY, 2004.

[9] J. Stewart, *Calculus: Concepts and Contexts*, 2nd ed., Brooks/Cole, Pacific Grove, CA, 2001.

[10] http://en.wikipedia.org/wiki/Bailey-Borwein-Plouffe_formula

[11] http://mathworld.wolfram.com/BBP-typeFormula.html

Appendix 1: Table Summarizing Bases 2, 3, 4, 5, 6, and 8

The following table summarizes the results of BBP-type formulas discovered in chapters 4, 5, and 6.

Table 3- BBP-type formulas discovered

#	α	$BBP\left(x, n, (a_1, a_2, \ldots, a_n)\right)$	Partial Fractions Breakdown
1	6	$ABBP\left(\frac{1}{\sqrt{3}}, 2, (1, 0)\right)$	$\frac{3}{u^2+3}$
2	$\frac{\pi}{4}$	$ABBP\left(1, 2, (1, 0)\right)$	$\frac{1}{u^2+1}$
3	$\frac{\sqrt{3}\pi}{9}$	$BBP\left(1, 3, (1, -1, 0)\right)_*$	$\frac{1}{u^2+u+1}$
4	$\frac{4\sqrt{3}\pi}{9}$	$ABBP\left(\frac{1}{2}, 3, (2, 1, 0)\right)_*$	$\frac{8}{u^2-2u+4}$
5	$\frac{2\sqrt{3}\pi}{9}$	$ABBP\left(1, 3, (1, 1, 0)\right)_*$	$\frac{1}{u^2-u+1}$
6	$\frac{\sqrt{3}\pi}{9}$	$ABBP\left(1, 3, (1, 0, -1)\right)_*$	$\frac{1-u}{u^2-u+1}$
7	$\frac{\sqrt{3}\pi}{2}$	$BBP\left(\frac{1}{\sqrt{3}}, 4, (3, 0, -1, 0)\right)$	$\frac{9}{u^2+3}$
8	$\frac{\pi}{4}$	$BBP\left(1, 4, (1, 0, -1, 0)\right)$	$\frac{1}{u^2+1}$
9	$\frac{\pi}{4}$	$BBP\left(1, 4, (2, -3, 0, 1)\right)_*$	$\frac{1}{u+1}-\frac{2u-1}{u^2+1}$

#	α	$BBP(x, n, (a_1, a_2, \ldots, a_n))$	Partial Fractions Breakdown
10	π	$ABBP\left(\frac{1}{\sqrt{2}}, 4, (2, 2, 1, 0)\right)$	$\frac{4}{u^2-2u+2}$
11	$\frac{\pi}{8}$	$ABBP(1,4,(0,1,0,0))$	$\frac{\sqrt{2}}{4(u^2-\sqrt{2}u+1)}-\frac{\sqrt{2}}{4(u^2+\sqrt{2}u+1)}$
12	$\frac{\sqrt{2}\pi}{4}$	$ABBP(1,4,(1,0,1,0))_*$	$\frac{1}{2(u^2-\sqrt{2}u+1)}+\frac{1}{2(u^2+\sqrt{2}u+1)}$
13	$\frac{4\sqrt{3}\pi}{9}$	$BBP\left(\frac{1}{2}, 6, (4, -2, -1, -1, 0, 0)\right)$	$\frac{4}{u-2}+\frac{4}{u+2}-\frac{8(u-2)}{u^2-2u+4}$
14	$\frac{8\sqrt{3}\pi}{3}$	$BBP\left(\frac{1}{2}, 6, (8, 12, 2, 0, -1, 0)\right)$	$\frac{-8}{u-2}-\frac{8}{u+2}+\frac{16(u+2)}{u^2-2u+4}$
15	$\frac{\sqrt{3}\pi}{6}$	$BBP(1, 6, (1, 0, 0, 0, -1, 0))_*$	$\frac{1}{2(u^2-u+1)}+\frac{1}{2(u^2+u+1)}$
16	$\frac{\sqrt{3}\pi}{18}$	$BBP(1, 6, (0, 1, 0, -1, 0, 0))$	$\frac{1}{2(u^2-u+1)}-\frac{1}{2(u^2+u+1)}$
17	$\frac{\sqrt{3}\pi}{18}$	$BBP(1, 6, (1, -1, -1, 0, 0, 1))_*$	$\frac{2u-1}{2(u^2-u+1)}+\frac{1}{2(u^2+u+1)}$
18	$\frac{3\sqrt{3}\pi}{2}$	$ABBP\left(\frac{1}{\sqrt{3}}, 6, (9, 0, -3, 0, 1, 0)\right)$	$\frac{27}{u^2+3}$
19	$\frac{\sqrt{3}\pi}{2}$	$ABBP\left(\frac{1}{\sqrt{3}}, 6, (0, 3, 3, 2, 0, 0)\right)_*$	$\frac{-9}{u^2+3}+\frac{9}{u^2-3u+3}$
20	$\frac{64\sqrt{3}\pi}{9}$	$ABBP\left(\frac{\sqrt{3}}{2}, 6, (16, 24, 24, 18, 9, 0)\right)_*$	$\frac{64}{3u^2-6u+4}$
21	$\frac{\pi}{12}$	$ABBP(1,6,(0,0,1,0,0,0))$	$\frac{-1}{3(u^2+1)}+\frac{1}{6(u^2-\sqrt{3}u+1)}+\frac{1}{6(u^2+\sqrt{3}u+1)}$

#	α	$BBP(x,n,(a_1,a_2,\ldots,a_n))$	Partial Fractions Breakdown
22	$\frac{\pi}{3}$	$ABBP(1,6,(1,0,0,0,1,0))$ *	$\frac{2}{3(u^2+1)}+\frac{1}{6(u^2-\sqrt{3}u+1)}+\frac{1}{6(u^2+\sqrt{3}u+1)}$
23	$\frac{\sqrt{3}\pi}{9}$	$ABBP(1,6,(0,1,0,1,0,0))$	$\frac{\sqrt{3}}{6(u^2-\sqrt{3}u+1)}-\frac{\sqrt{3}}{6(u^2+\sqrt{3}u+1)}$
24	$\frac{\sqrt{3}\pi}{18}$	$ABBP(1,6,(0,1,0,0,0,-1))$	$\frac{3u-2\sqrt{3}}{6(u^2-\sqrt{3}u+1)}-\frac{3u+2\sqrt{3}}{6(u^2+\sqrt{3}u+1)}$
25	$\frac{9\sqrt{3}\pi}{2}$	$BBP\left(\frac{1}{\sqrt{3}},8,(27,0,-9,0,3,0,-1,0)\right)$	$\frac{81}{u^2+3}$
26	π	$BBP\left(\frac{1}{\sqrt{2}},8,(4,0,0,-2,-1,-1,0,0)\right)$	$\frac{4u}{u^2-2}-\frac{4(u-2)}{u^2-2u+2}$
27	2π	$BBP\left(\frac{1}{\sqrt{2}},8,(0,8,4,4,0,0,-1,0)\right)$	$\frac{-8}{u^2-2}+\frac{8u}{u^2-2u+2}$
28	$\frac{7\pi}{8}$	$BBP(1,8,(0,8,-3,0,0,0,-3,-2))$ *	$\frac{-3}{2(u+1)}+\frac{5u+3}{2(u^2+1)}+\frac{2u+3\sqrt{2}}{4(u^2-\sqrt{2}u+1)}+\frac{2u-3\sqrt{2}}{4(u^2+\sqrt{2}u+1)}$
29	$\frac{7\sqrt{2}\pi}{8}$	$BBP(1,8,(7,-8,3,0,0,0,-4,2))$ *	$\frac{3}{2(u+1)}-\frac{5u-4}{2(u^2+1)}-\frac{2u-7+3\sqrt{2}}{4(u^2-\sqrt{2}u+1)}-\frac{2u-7-3\sqrt{2}}{4(u^2+\sqrt{2}u+1)}$
30	$\frac{\pi}{8}$	$BBP(1,8,(0,1,0,0,0,-1,0,0))$	$\frac{\sqrt{2}}{4(u^2-\sqrt{2}u+1)}-\frac{\sqrt{2}}{4(u^2+\sqrt{2}u+1)}$
31	$\frac{\pi}{8}$	$BBP(1,8,(-1,4,0,-2,-1,0,0,0))$ *	$\frac{-1}{2(u+1)}+\frac{3u-1}{2(u^2+1)}-\frac{2u-3\sqrt{2}}{4(u^2-\sqrt{2}u+1)}-\frac{2u+3\sqrt{2}}{4(u^2+\sqrt{2}u+1)}$
32	$\frac{\sqrt{2}\pi}{8}$	$BBP(1,8,(-1,4,1,-2,-2,0,0,0))$ *	$\frac{-1}{2(u+1)}-\frac{3u-2}{2(u^2+1)}-\frac{2u-1-3\sqrt{2}}{4(u^2-\sqrt{2}u+1)}-\frac{2u-1+3\sqrt{2}}{4(u^2+\sqrt{2}u+1)}$

#	α	$BBP\left(x,n,(a_1,a_2,\ldots,a_n)\right)$	Partial Fractions Breakdown
33	$\frac{\pi}{16}$	$ABBP\left(1, 8, (0, 0, 0, 1, 0, 0, 0, 0)\right)$	$\frac{1}{4\sqrt{4-2\sqrt{2}}}\left(\frac{-1}{u^2-\sqrt{2-\sqrt{2}}u+1}+\frac{1}{u^2+\sqrt{2-\sqrt{2}}u+1}+\frac{1}{u^2-\sqrt{2+\sqrt{2}}u+1}-\frac{1}{u^2+\sqrt{2+\sqrt{2}}u+1}\right)$
34	$\frac{\sqrt{4+2\sqrt{2}}\pi}{8}$	$ABBP\left(1, 8, (1, 0, 0, 0, 0, 0, 1, 0)\right)$ *	$\frac{1}{4\sqrt{2}}\left(\frac{2+\sqrt{2}}{\sqrt{2}(u^2-\sqrt{2-\sqrt{2}}u+1)}+\frac{1+\sqrt{2}}{u^2+\sqrt{2-\sqrt{2}}u+1}+\frac{\sqrt{2}-1}{u^2-\sqrt{2+\sqrt{2}}u+1}+\frac{\sqrt{2}-1}{u^2+\sqrt{2+\sqrt{2}}u+1}\right)$
35	$\frac{\sqrt{2}\pi}{8}$	$ABBP\left(1, 8, (0, 1, 0, 0, 0, 1, 0, 0)\right)$	$\frac{1}{4\sqrt{2-\sqrt{2}}}\left(\frac{1}{u^2-\sqrt{2-\sqrt{2}}u+1}+\frac{2-\sqrt{2}}{u^2+\sqrt{2-\sqrt{2}}u+1}+\frac{1}{u^2-\sqrt{2+\sqrt{2}}u+1}-\frac{1}{u^2+\sqrt{2+\sqrt{2}}u+1}\right)$
36	$\frac{\sqrt{4-2\sqrt{2}}\pi}{8}$	$ABBP\left(1, 8, (0, 0, 1, 0, 1, 0, 0, 0)\right)$ *	$\frac{1}{4\sqrt{2}}\left(\frac{-1}{u^2-\sqrt{2-\sqrt{2}}u+1}-\frac{1}{u^2+\sqrt{2-\sqrt{2}}u+1}+\frac{1}{u^2-\sqrt{2+\sqrt{2}}u+1}+\frac{1}{u^2+\sqrt{2+\sqrt{2}}u+1}\right)$

We will refer to each formula by its number corresponding to the first column.

Formulas 1, 7, 18, and 25 are equivalent to the power series of $\tan\left(\frac{1}{\sqrt{3}}\right)$.

Formulas 2, 8, 11, 21, 30, and 33 are equivalent to Leibniz's Formula, (4.2).

Formulas 3 and 16 are equivalent.

Formulas 5 and 23 are equivalent.

Formulas 6 and 24 are equivalent.

Formulas 10, 13, and 14 can be found in [3, p. 7 formula 16 and p. 16 formulas 58 & 59 respectively].

Formulas 12 and 35 are equivalent.

Formulas 26 and 27 can be found in [11, formulas 2 & 3 respectively].

Formulas 3, 4, 5, 6, 9, 12, 15, 17, 19, 20, 22, 28, 29, 31, 32, 34, and 36 are possibly new BBP-type formulas as we did not see them in the literature. They are also marked with an * in the above table. If two formulas are equivalent, only the formula from with a smaller base is considered new.

Appendix 2: *Mathematica*'s Output for the Non-Alternating Base 6

This is the actual *Mathematica* output for integrating the non-alternating base 6 case. The restrictions on x can be disregarded since we are concerned with $|x|<1$ and when $|x|=1$ we add the necessary constraints. Also, $a, b, c, d, e,$ and f are equivalent to $a_1, a_2, a_3, a_4, a_5,$ and a_6 respectively. For simplification, expressions were expanded and constants were factored. Logarithms were also expanded, i.e. $-2\,f\,\text{Log}\left[-1+x^6\right]$ is equivalent to $-2a_6\ln(-1+x^6)$, which was expanded to

$$-2a_6\left(\ln|x-1|+\ln(x+1)+\ln(x^2+x+1)+\ln(x^2-x+1)\right).$$

This often affected the system of linear equations and added a degree of freedom. Similar simplification was used once an x–value was plugged in.

In the base 8 case, partial fractions were used to convert the integral into four integrals. The four integrals were integrated separately and their results were summed together.

$$\int_0^1 \frac{a+b*t+c*t^2+d*t^3+e*t^4+f*t^5}{1-x^6*t^6}\,dt$$

$$\mathrm{If}\Big[(\mathrm{Re}[x]\geq -1\;||\;\mathrm{Im}[x]\neq 0)\;\&\&\;(\mathrm{Re}[x]\leq 1\;||\;\mathrm{Im}[x]\neq 0)\;\&\&$$

$$(\mathrm{Im}[x]\neq 0\;||\;(\mathrm{Im}[x]==0\;\&\&\;(((\mathrm{Re}[x]==-1\;||\;\mathrm{Re}[x]\geq -1)\;\&\&\;(\mathrm{Re}[x]==0\;||\;\mathrm{Re}[x]\leq 0))\;||$$

$$\mathrm{Re}[x]==0\;||\;\mathrm{Re}[x]\geq 0)))\;\&\&\;(\mathrm{Im}[x]\neq 0\;||$$

$$(\mathrm{Im}[x]==0\;\&\&\;(\mathrm{Re}[x]==0\;||\;\mathrm{Re}[x]\leq 0\;||\;((\mathrm{Re}[x]==1\;||\;\mathrm{Re}[x]\leq 1)\;\&\&\;(\mathrm{Re}[x]==0\;||\;\mathrm{Re}[x]\geq 0)))))\;\&\&$$

$$\left(-\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0\;||\;\left(-\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]==0\;\&\&\;\left(\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]==0\;||\right.\right.\right.$$

$$\left.\left.\left.-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 0\;||\;\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]==1\;||\;\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 1\right)\right)\right)\;\&\&$$

$$\left(\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0\;||\;\left(\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]==0\;\&\&\;\left(\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]==0\;||\right.\right.\right.$$

$$\left.\left.\left.-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 0\;||\;\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]==1\;||\;\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 1\right)\right)\right)\;\&\&$$

$$\left(-\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0\;||\;\left(-\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]==0\;\&\&\;\left(-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]==0\;||\right.\right.\right.$$

$$\left.\left.\left.\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 0\;||\;-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]==1\;||\;-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 1\right)\right)\right)\;\&\&$$

$$\left(\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0\;||\;\left(\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]==0\;\&\&\;\left(-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]==0\;||\right.\right.\right.$$

$$\left.\left.\left.\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 0\;||\;-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]==1\;||\;-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 1\right)\right)\right)\;\&\&$$

$$\left(\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]\geq\mathrm{Re}\left[\frac{1}{x}\right]\;||\;2+\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]\leq\mathrm{Re}\left[\frac{1}{x}\right]\;||\;\mathrm{Im}\left[\frac{1}{x}\right]+\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0\right)\;\&\&$$

$$\left(\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]\leq\mathrm{Re}\left[\frac{1}{x}\right]\;||\;\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]\geq 2+\mathrm{Re}\left[\frac{1}{x}\right]\;||\;\mathrm{Im}\left[\frac{1}{x}\right]+\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0\right)\;\&\&$$

$$\left(\mathrm{Im}[x]^2\leq 3\,\mathrm{Re}[x]^2\;||\;-\frac{1}{2}\leq\mathrm{Re}[x]<0\;||\;\mathrm{Re}[x]>0\right)\;\&\&$$

$$\left(\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\mathrm{Re}\left[\frac{1}{x}\right]\leq 0\;||\;\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\mathrm{Re}\left[\frac{1}{x}\right]\geq 2\;||\;\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq\mathrm{Im}\left[\frac{1}{x}\right]\right)\;\&\&$$

$$\left(\frac{\mathrm{Im}[x]^2}{\mathrm{Re}[x]^2}\leq 3\;||\;0<\mathrm{Re}[x]\leq\frac{1}{2}\;||\;\mathrm{Re}[x]<0\right)\;\&\&$$

$$\left(\mathrm{Im}[x]\neq 0\;||\;((\mathrm{Re}[x]==-1\;||\;\mathrm{Re}[x]\geq -1)\;\&\&\;(\mathrm{Re}[x]==0\;||\;\mathrm{Re}[x]\leq 0))\;||\right.$$

$$\left.\left((\mathrm{Im}[x]\neq 0\;||\;\mathrm{Re}[x]\geq 0)\;\&\&\;\frac{1}{x}\neq 0\right)\right)\;\&\&\;(\mathrm{Im}[x]\neq 0\;||\;\mathrm{Re}[x]==0\;||\;\mathrm{Re}[x]\leq 0\;||$$

$$\mathrm{Im}[x]\neq 0\;||\;((\mathrm{Re}[x]==1\;||\;\mathrm{Re}[x]\leq 1)\;\&\&\;(\mathrm{Re}[x]==0\;||\;\mathrm{Re}[x]\geq 0)))\;\&\&$$

$$\left(-\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0\;||\;\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]==0\;||\;-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 0\;||\right.$$

$$\left.\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0\;||\;-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]==-1\;||\;\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 1\right)\;\&\&$$

$$\left(\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0\;||\;\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]==0\;||\;-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 0\;||\right.$$

$$\left.-\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0\;||\;-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]==-1\;||\;\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 1\right)\;\&\&$$

$$\left(-\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0 \;\|\; -\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]=0 \;\|\; \frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 0 \;\|\;\right.$$

$$\left.\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0 \;\|\; \frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]=-1 \;\|\; -\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 1\right) \;\&\&$$

$$\left(\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0 \;\|\; -\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]=0 \;\|\; \frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 0 \;\|\;\right.$$

$$\left.-\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0 \;\|\; \frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]=-1 \;\|\; -\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 1\right),$$

$$-\frac{\pi\left(\sqrt{3}\,d x^2-\sqrt{3}\,b x^4-3 i\,(f+x\,(e+x\,(d+x\,(c+x\,(b+a x)))))\right)}{18 x^6}+$$

$$\frac{1}{12 x^6}\left(2\sqrt{3}\,x\left(-e-d x+x^3\,(b+a x)\right)\mathrm{ArcTan}\left[\frac{-1+2x}{\sqrt{3}}\right]+\right.$$

$$2\sqrt{3}\,x\left(-e+d x-b x^3+a x^4\right)\mathrm{ArcTan}\left[\frac{1+2x}{\sqrt{3}}\right]-2x\,(e+x\,(d+x\,(c+x\,(b+a x))))\,\mathrm{Log}[-1+x]+$$

$$2x\left(e+x\left(-d+x\left(c-b x+a x^2\right)\right)\right)\mathrm{Log}[1+x]-x\left(e+x\left(-d-2 c x-b x^2+a x^3\right)\right)\mathrm{Log}\left[1-x+x^2\right]+$$

$$\left.x\,(e+x\,(d+x\,(-2c+x\,(b+a x))))\,\mathrm{Log}\left[1+x+x^2\right]-2 f\,\mathrm{Log}\left[-1+x^6\right]\right),$$

$$\mathrm{Integrate}\left[\frac{a+t\,(b+t\,(c+t\,(d+t\,(e+f t))))}{1-t^6 x^6},\{t,0,1\},\right.$$

$$\mathrm{Assumptions}\to\;!\left((\mathrm{Re}[x]\geq -1 \;\|\; \mathrm{Im}[x]\neq 0) \;\&\&\; (\mathrm{Re}[x]\leq 1 \;\|\; \mathrm{Im}[x]\neq 0) \;\&\&\; (\mathrm{Im}[x]\neq 0 \;\|\; (\mathrm{Im}[x]=0 \;\&\&\right.$$

$$(((\mathrm{Re}[x]=-1 \;\|\; \mathrm{Re}[x]\geq -1) \;\&\&\; (\mathrm{Re}[x]=0 \;\|\; \mathrm{Re}[x]\leq 0)) \;\|\; \mathrm{Re}[x]=0 \;\|\; \mathrm{Re}[x]\geq 0))) \;\&\&$$

$$(\mathrm{Im}[x]\neq 0 \;\|\; (\mathrm{Im}[x]=0 \;\&\&\; (\mathrm{Re}[x]=0 \;\|\; \mathrm{Re}[x]\leq 0 \;\|\; ((\mathrm{Re}[x]=1 \;\|\; \mathrm{Re}[x]\leq 1) \;\&\&$$

$$(\mathrm{Re}[x]=0 \;\|\; \mathrm{Re}[x]\geq 0))))) \;\&\&\; \left(-\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0 \;\|\;\right.$$

$$\left(-\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]=0 \;\&\&\; \left(\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]=0 \;\|\; -\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 0 \;\|\;\right.\right.$$

$$\left.\left.\left.\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]=1 \;\|\; \frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 1\right)\right)\right) \;\&\&\; \left(\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq\right.$$

$$0 \;\|\; \left(\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]=0 \;\&\&\; \left(\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]=0 \;\|\; -\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq\right.\right.$$

$$\left.\left.\left.0 \;\|\; \frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]=1 \;\|\; \frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 1\right)\right)\right) \;\&\&$$

$$\left(-\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0 \;\|\; \left(-\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]=0 \;\&\&\right.\right.$$

$$\left(-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]=0 \;\|\; \frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 0 \;\|\; -\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]=1 \;\|\;\right.$$

$$\left.\left.\left.-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 1\right)\right)\right) \;\&\&\; \left(\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0 \;\|\;\right.$$

$$\left(\frac{1}{2}\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]=0 \;\&\&\; \left(-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]=0 \;\|\; \frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]-\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 0 \;\|\;\right.\right.$$

$$\left.\left.\left.-\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]=1 \;\|\; -\frac{1}{2}\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]+\frac{1}{2}\mathrm{Re}\left[\frac{1}{x}\right]\geq 1\right)\right)\right) \;\&\&$$

$$\left(\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]\geq \mathrm{Re}\left[\frac{1}{x}\right] \;\|\; 2+\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]\leq \mathrm{Re}\left[\frac{1}{x}\right] \;\|\; \mathrm{Im}\left[\frac{1}{x}\right]+\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0\right) \;\&\&$$

$$\left(\sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]\leq \mathrm{Re}\left[\frac{1}{x}\right] \;\|\; \sqrt{3}\,\mathrm{Im}\left[\frac{1}{x}\right]\geq 2+\mathrm{Re}\left[\frac{1}{x}\right] \;\|\; \mathrm{Im}\left[\frac{1}{x}\right]+\sqrt{3}\,\mathrm{Re}\left[\frac{1}{x}\right]\neq 0\right) \;\&\&$$

$$\left(\mathrm{Im}[x]^2 \le 3\,\mathrm{Re}[x]^2 \;||\; -\frac{1}{2} \le \mathrm{Re}[x] < 0 \;||\; \mathrm{Re}[x] > 0\right) \;\&\&$$

$$\left(\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] + \mathrm{Re}\left[\frac{1}{x}\right] \le 0 \;||\; \sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] + \mathrm{Re}\left[\frac{1}{x}\right] \ge 2 \;||\; \sqrt{3}\ \mathrm{Re}\left[\frac{1}{x}\right] \ne \mathrm{Im}\left[\frac{1}{x}\right]\right) \;\&\&$$

$$\left(\frac{\mathrm{Im}[x]^2}{\mathrm{Re}[x]^2} \le 3 \;||\; 0 < \mathrm{Re}[x] \le \frac{1}{2} \;||\; \mathrm{Re}[x] < 0\right) \;\&\&$$

$$\Big(\mathrm{Im}[x] \ne 0 \;||\; ((\mathrm{Re}[x] == -1 \;||\; \mathrm{Re}[x] \ge -1) \;\&\&\; (\mathrm{Re}[x] == 0 \;||\; \mathrm{Re}[x] \le 0)) \;||$$

$$\left((\mathrm{Im}[x] \ne 0 \;||\; \mathrm{Re}[x] \ge 0) \;\&\&\; \frac{1}{x} \ne 0\right)\Big) \;\&\&\; (\mathrm{Im}[x] \ne 0 \;||\; \mathrm{Re}[x] == 0 \;||\; \mathrm{Re}[x] \le 0 \;||$$

$$\mathrm{Im}[x] \ne 0 \;||\; ((\mathrm{Re}[x] == 1 \;||\; \mathrm{Re}[x] \le 1) \;\&\&\; (\mathrm{Re}[x] == 0 \;||\; \mathrm{Re}[x] \ge 0))) \;\&\&$$

$$\left(-\frac{1}{2}\,\mathrm{Im}\left[\frac{1}{x}\right] - \frac{1}{2}\sqrt{3}\ \mathrm{Re}\left[\frac{1}{x}\right] \ne 0 \;||\; \frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] - \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] == 0 \;||\; -\frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] + \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] \ge 0 \;||\right.$$

$$\left.\frac{1}{2}\,\mathrm{Im}\left[\frac{1}{x}\right] + \frac{1}{2}\sqrt{3}\ \mathrm{Re}\left[\frac{1}{x}\right] \ne 0 \;||\; -\frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] + \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] == -1 \;||\; \frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] - \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] \ge 1\right) \;\&\&$$

$$\left(\frac{1}{2}\,\mathrm{Im}\left[\frac{1}{x}\right] - \frac{1}{2}\sqrt{3}\ \mathrm{Re}\left[\frac{1}{x}\right] \ne 0 \;||\; \frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] + \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] == 0 \;||\; -\frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] - \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] \ge 0 \;||\right.$$

$$\left.-\frac{1}{2}\,\mathrm{Im}\left[\frac{1}{x}\right] + \frac{1}{2}\sqrt{3}\ \mathrm{Re}\left[\frac{1}{x}\right] \ne 0 \;||\; -\frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] - \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] == -1 \;||\; \frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] + \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] \ge 1\right) \;\&\&$$

$$\left(-\frac{1}{2}\,\mathrm{Im}\left[\frac{1}{x}\right] + \frac{1}{2}\sqrt{3}\ \mathrm{Re}\left[\frac{1}{x}\right] \ne 0 \;||\; -\frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] - \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] == 0 \;||\; \frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] + \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] \ge 0 \;||\right.$$

$$\left.\frac{1}{2}\,\mathrm{Im}\left[\frac{1}{x}\right] - \frac{1}{2}\sqrt{3}\ \mathrm{Re}\left[\frac{1}{x}\right] \ne 0 \;||\; \frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] + \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] == -1 \;||\; -\frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] - \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] \ge 1\right) \;\&\&$$

$$\left(\frac{1}{2}\,\mathrm{Im}\left[\frac{1}{x}\right] + \frac{1}{2}\sqrt{3}\ \mathrm{Re}\left[\frac{1}{x}\right] \ne 0 \;||\; -\frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] + \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] == 0 \;||\; \frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] - \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] \ge 0 \;||\right.$$

$$\left.-\frac{1}{2}\,\mathrm{Im}\left[\frac{1}{x}\right] - \frac{1}{2}\sqrt{3}\ \mathrm{Re}\left[\frac{1}{x}\right] \ne 0 \;||\; \frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] - \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] == -1 \;||\; -\frac{1}{2}\sqrt{3}\ \mathrm{Im}\left[\frac{1}{x}\right] + \frac{1}{2}\,\mathrm{Re}\left[\frac{1}{x}\right] \ge 1\right)\Big]\Big]$$