Patterns of Non-Simple Continued Fractions

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Contents

1 Introduction

1.1 Real Numbers and Simple Continued Fractions

Take a moment to reflect on the real numbers and the fact that they fall into one of two categories: rational or irrational. It is known that, when written in decimal form, all rational numbers are either terminating, such as $\frac{11}{4} = 2.75$, or are non-terminating but eventually periodic, such as $\frac{10}{3} = 3.333\ldots$. On the other hand, irrational numbers possess neither of these properties, being non-terminating and non-periodic, such as $\pi = 3.14159\ldots$. These expansions rely on base-10 notation, which is the most common. However, one could just as easily write these numbers using other bases, such as base-3, which would produce $\frac{11}{4} = 2.2020\ldots$, $\frac{10}{3} = 10.2$, and $\pi = 11.001001\ldots$. These examples show that a change of base may produce expansions that have different properties (such as terminating). Notice that there are many different ways to write the same number, including the relatively less common method of continued fraction expansion. We will first introduce this technique and then show why it may be useful.

A *continued fraction* is an expression of the form

$$a_0 + \cfrac{r_0}{a_1 + \cfrac{r_1}{a_2 + \cfrac{r_2}{\ddots}}},$$

but we shall restrict ourselves to considering a constant r value, that is,

$$a_0 + \cfrac{r}{a_1 + \cfrac{r}{a_2 + \cfrac{r}{\ddots}}}.$$

For a continued fraction of this form, we will occasionally use the notation

$$a_0 + \frac{r}{a_1 +} \frac{r}{a_2 +} \cdots,$$

but we will more often use the compact notation $[a_0, a_1, a_2, \ldots]_r$. The r in this context will usually be referred to as the "numerator" of the continued fraction. We may say that the expansion is eventually periodic (generally referred to just as periodic) with period length n if for some m, $a_{m+i} = a_{m+n+i}$ for $i \geqslant 0$, and use the notation $[a_0, \ldots, a_{m-1}, \overline{a_m, \ldots, a_{m+n-1}}]_r$. Moreover, an expansion is purely periodic if $m = 0$ in the context we just introduced.

The most common restriction imposed on continued fractions is to have $r = 1$ and then call the expression a *simple continued fraction*.

Example. To demonstrate the recursive approach to expanding a number into a simple continued fraction, we will calculate the expansion for $\frac{20}{7}$. Follow each step as

$$\frac{20}{7} = 2 + \frac{6}{7} = 2 + \frac{1}{\frac{7}{6}} = 2 + \frac{1}{1 + \frac{1}{6}} = 2 + \frac{1}{1 + \frac{1}{5 + \frac{1}{1}}}.$$

This calculation demonstrates an elementary fact of simple continued fractions — each rational number has exactly two expansions, differing only at the end of the expression. In general, $[a_0, a_1, \ldots, a_n]_1 = [a_0, a_1, \ldots, a_n - 1, 1]_1$ if $a_n > 1$. The method for expanding negative numbers differs only in the first step, as seen with

$$\frac{-19}{7} = -3 + \frac{2}{7} = -3 + \frac{1}{\frac{7}{2}} = -3 + \frac{1}{3 + \frac{1}{2}} = -3 + \frac{1}{3 + \frac{1}{1 + \frac{1}{1}}}.$$

Recall the earlier observation that some representations of a number have different properties (such as terminating or periodic) depending on the base being used. More precisely, if $x = a_0.a_1a_2\ldots$, then $0 \leqslant a_i \leqslant b - 1$ where b is the base, so the values of each a_i depend on b. This is not a problem with continued fraction expansions since the entries are integers and can be altered independently. Take as example that $\frac{10}{3} = [3, 1]_1$ with base-10 while $\frac{10}{3} = [10, 1]_1$ with base-3. This means that one could use a different base for a continued fraction which only changes how the integers are expressed, but not what they are.

One of the frequently used tools for working with continued fractions is *convergents*. Convergents are the values formed by partial expansions of a continued fraction. These have many uses for simple continued fractions, such as finding rational approximations of an irrational number. For $[a_0, a_1, a_2, \ldots]_r$, the convergents are denoted as c_n where $c_n = [a_0, a_1, \ldots, a_n]_r$.

Example. To demonstrate the convergents of a simple continued fraction, consider the expansion of $\frac{20}{7} = [2, 1, 6]_1$. Then,

$$c_0 = 2, \qquad c_1 = 2 + \frac{1}{1} = 3, \qquad c_2 = 2 + \frac{1}{1 + \frac{1}{6}} = \frac{20}{7}.$$

As mentioned, we can calculate rational approximations of irrational numbers such as π. We know that the expansion of π begins as $[3, 7, 15, 1, 292, 1, \ldots]_1$[2]. By calculating the first few convergents, we find

$$c_0 = 3, \qquad c_1 = 3 + \frac{1}{7} = \frac{22}{7}, \qquad c_2 = 3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106},$$

and see that $\frac{333}{106} = 3.141509\ldots$, which is a decent approximation.

There are a number of interesting properties to find from convergents, in particular by looking at the numerator and denominator parts. For this reason, we often write $c_n = \frac{p_n}{q_n}$.

1.2 Generalized Continued Fractions

Simple continued fractions have many applications, particularly for number theory, and have been studied thoroughly in the past as demonstrated in [2]. However, the same is not true for more general forms of continued fractions. While [1] looks into the option of integer numerator values, we wish to go further. The primary extension in this work is to allow for positive rational numerators. That is, we will work with continued fractions of the form

$$a_0 + \frac{u/v}{a_1 + \frac{u/v}{a_2 + \frac{u/v}{\ddots}}},$$

where $u, v \in \mathbb{Z}^+$, $a_0 \in \mathbb{Z}$, and $a_i \in \mathbb{Z}^+$ for $i > 0$. Additionally, we will suppose that $u \geqslant v$ and $gcd(u, v) = 1$. For a continued fraction of this form, we will similarly use the notation

$$a_0 + \frac{u/v}{a_1} \underset{+}{} \frac{u/v}{a_2} \underset{+}{} \cdots$$

and the compact notation $[a_0, a_1, a_2, \ldots]_{u/v}$. An algorithm for computing continued fractions of this form will be shown and examples given.

Section 2 of this paper works ground up, starting with an algorithm that we can use for continued fractions with rational numerators. We develop relations that hold for convergents, which end up having many similarities to those that hold for simple continued fractions. We go on to find other theorems that have analogues to theorems for simple continued fractions. Importantly, these findings are applicable to any expansion.

Section 3 of this paper has extra focus on continued fraction expansions of rational numbers that are periodic. We develop relations that hold when an expansion is periodic, which gives us a way to find and verify periodicity. Furthermore, there is a special case for period length 1, to which we give attention.

Section 4 contains some potentially useful findings that do not necessarily apply to other places or have yet to be proven. Section 5 gives future work possibilities, mostly relating to unproven ideas.

1.3 Motivation

The motivation for this work comes from a common observation. When presenting continued fractions, many authors start by giving the general form and then restricting to the simple case. By allowing more generality and working out examples, it does not take long to see potential patterns and wonder what can proven. In addition, one may wonder what happens to theorems that have been found for simple continued fractions and if they carry over. This work aims to fulfill some of these wonders.

2 General Results

2.1 Continued Fraction Algorithm and Convergents

We start by defining a recursive algorithm for computing continued fractions with rational numerators and verifying that the algorithm gives successful results.

Theorem 2.1. *Given $x \in \mathbb{R}$ and $r \in \mathbb{Q}^+$ with $r \geqslant 1$, choose sequences (x_n) and (a_n) by*

$$x_0 = x, \qquad \lfloor x_n \rfloor - \lfloor r \rfloor \leqslant a_n \leqslant \lfloor x_n \rfloor, \qquad x_n - a_n \leqslant r, \qquad x_n = \frac{r}{x_{n-1} - a_{n-1}},$$

with $a_i > 0$ for $i > 0$, and terminating when x_n is an integer. This creates the expansion

$$[a_0, a_1, a_2, \ldots]_r \, .$$

Proof. There is only one potential issue with this algorithm and it is analogous to the problem discussed in [1, Theorem 1.5 on p.2445]. That is, an issue arises if $x_i < 1$ for some $i > 0$, for then no choice is available for a_i. We can inductively show that this issue cannot arise. As a base case, a valid choice exists for a_0 since it is allowed to be 0. Then assume that a valid $a_k > 0$ is chosen for some $k > 0$. Subsequently, $0 \leqslant x_k - \lfloor x_k \rfloor \leqslant x_k - a_k \leqslant r$. As stated, if x_k is an integer, then the algorithm terminates, otherwise, $0 < x_k - \lfloor x_k \rfloor \leqslant x_k - a_k \leqslant r$ and $x_{k+1} = \frac{r}{x_k - a_k} \geqslant \frac{r}{r} = 1$. Thus, $x_{k+1} \geqslant 1$, so we can make a valid choice for a_{k+1}. By the principle of mathematical induction, a valid choice is available for any a_n. □

Remark. A question that could be raised at this point is whether or not an infinite expansion converges to the desired value. That is, for an infinite expansion, can we truly write $x = [a_0, a_1, \ldots]_{u/v}$. We can refer this question to the proof of [1, Theorem 1.4 on p.2444]. This proof can be applied here after demonstrating that $q_{2n} \geqslant (u+v)^n$ and $q_{2n+1} \geqslant v(u+v)^n$, both of which can be shown inductively. The rest of the proof follows, and we see that convergence holds.

With the described algorithm, one can create many different expressions for a particular value. Each expansion depends on the choices of a_n, where there are as many as $\lceil r \rceil$ possibilities. Sometimes we will desire a certain method for the selections.

Definition 2.2. For a choice of a_n in the continued fraction algorithm, if a_n is the smallest integer such that $a_n \geqslant \lfloor x_n \rfloor - \lfloor r \rfloor$ and $x_n - a_n \leqslant r$, then it is the *min* choice for a_n. If $a_n = \lfloor x_n \rfloor$, then it is the *max* choice. If the min choice is used for every a_n, then this is referred to as the *min algorithm*. Similarly, we define the *max algorithm*. Note that these two may produce identical expansions in some cases.

With these algorithms, it is useful to demonstrate the numerous ways to apply them.

Example. We found earlier that $\frac{20}{7} = [2, 1, 6]_1$. Now instead we can make an expression that uses numerator $\frac{3}{2}$. Starting with the max algorithm,

$$\frac{20}{7} = 2 + \frac{6}{7} = 2 + \frac{3/2}{\frac{7}{4}} = 2 + \frac{3/2}{1 + \frac{3/2}{2}},$$

so $\frac{20}{7} = [2, 1, 2]_{3/2}$. With the min algorithm, $\lfloor x_0 \rfloor - \lfloor r \rfloor = \lfloor \frac{20}{7} \rfloor - \lfloor \frac{3}{2} \rfloor = 1$, so $1 \leqslant a_0 \leqslant 2$. However, $x_0 - 1 = \frac{13}{7} > \frac{3}{2}$, so the only valid (and thus smallest) choice is $a_0 = 2$. Continuing in this fashion, the min algorithm actually produces the same expansion as the max algorithm.

Suppose instead we wish to expand $\frac{20}{7}$ with $\frac{5}{2}$ as a numerator. While omitting the calculations, we find $\frac{20}{7} = [2, 2, 2, 3, 5, 3, 5]_{5/2}$ with the max algorithm. Then, using the min algorithm,

$$\frac{20}{7} = 1 + \frac{5/2}{\frac{35}{26}} = 1 + \frac{5/2}{1 + \frac{5/2}{\frac{65}{9}}} = 1 + \frac{5/2}{1 + \frac{5/2}{5 + \frac{5/2}{\frac{9}{8}}}} = 1 + \frac{5/2}{1 + \frac{5/2}{5 + \frac{5/2}{1 + \frac{5/2}{20}}}},$$

so $\frac{20}{7} = [1, 1, 5, 1, 20]_{5/2}$, a different expression!

These examples have shown how rational numbers may have expansions that terminate. In other cases however, we may find this does not happen. For example, when we attempt to expand $\frac{10}{3}$ with numerator $\frac{16}{9}$ and the max algorithm, then

$$\frac{10}{3} = 3 + \frac{16/9}{\frac{16}{3}} = 3 + \frac{16/9}{5 + \frac{16}{3}} = 3 + \frac{16/9}{5 + \frac{16/9}{\frac{16}{3}}} = 3 + \frac{16/9}{5 + \frac{16/9}{5 + \frac{16/9}{\ddots}}}.$$

It appears that this pattern continues indefinitely, suggesting that $\frac{10}{3} = \left[3, \overline{5}\right]_{16/9}$. Later, we will show how this periodicity can be proved.

It also appears that some expansions of rational numbers can continue indefinitely without becoming periodic. For example, when we expand $\frac{23}{13}$ with numerator $\frac{29}{14}$ and the max algorithm, the expansion begins as $\frac{23}{13} = [1, 2, 2, 2, 22, \ldots]_{29/14}$. This appears to never terminate nor become periodic, even when checking up to one million iterations! Unfortunately, we are unable to prove this continues indefinitely; it could, for example, terminate later in the expansion.

Remark. When we calculate the expansion of $\frac{29}{14}$ with numerator $\frac{23}{13}$ and the max algorithm, one million steps can be used without terminating or becoming periodic. However, some expansions are quicker to expose behavior. Using the max algorithm up to 100 steps, neither expansion of $\frac{10}{3}$ with numerator $\frac{17}{7}$ or $\frac{23}{7}$ terminate or become periodic. However, using a sufficiently large number of steps, both become periodic! The moral is that using a greater number of iterations can potentially expose the true properties.

The convergents of continued fractions play a number of roles in these results, so they are developed first in the generalized setting. We start by showing how we can determine the convergents, $c_n = \frac{p_n}{q_n}$, by "collapsing" an expansion.

Example. We wish to find the convergents of $\frac{40}{11}$ when expanded with numerator $\frac{3}{2}$ and the max algorithm. A series of calculations gives $\frac{40}{11} = [3, 2, 4, 7, 3]_{3/2}$. Now, for example, the calculation for c_2 is

$$c_2 = [3, 2]_{3/2} = 3 + \frac{3/2}{2} = \frac{3 * 2 + 3/2}{2} = \frac{15/2}{2} = \frac{15}{4}.$$

We also calculate the rest of the convergents, but omit the calculations, to find

$$(c_n) = \left(\frac{3}{1}, \frac{15}{4}, \frac{69}{19}, \frac{1011}{278}, \frac{40}{11}\right).$$

The convergents of a continued fraction have many interesting uses, requiring a few relations. The relations found here have known analogues for simple continued fractions.

Theorem 2.3. *Suppose that $x = [a_0, a_1, \ldots, a_{n-1}, a_n, \ldots]_{u/v}$. Let $x_n = [a_n, a_{n+1}, \ldots]_{u/v}$, so that $x = [a_0, a_1, \ldots, a_{n-1}, x_n]_{u/v}$. For the convergents of x, $\frac{p_n}{q_n}$, the following relations are satisfied:*

$$p_{-1} = 1 \qquad p_0 = a_0 \qquad q_{-1} = 0 \qquad q_0 = 1$$

$$p_{2n} = a_{2n}p_{2n-1} + up_{2n-2} \quad p_{2n+1} = va_{2n+1}p_{2n} + up_{2n-1}$$
$$q_{2n} = a_{2n}q_{2n-1} + uq_{2n-2} \quad q_{2n+1} = va_{2n+1}q_{2n} + uq_{2n-1} \tag{2.1}$$

$$p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1}u^n \tag{2.2}$$

$$x = x_0 \qquad x = \frac{p_{2n-1}x_{2n} + up_{2n-2}}{q_{2n-1}x_{2n} + uq_{2n-2}} \qquad x = \frac{vp_{2n}x_{2n+1} + up_{2n-1}}{vq_{2n}x_{2n+1} + uq_{2n-1}} \tag{2.3}$$

Proof. The proof of (2.1) uses a similar strategy to that of [2, Theorem 1.3 on p.21]. As a base case,

$$\frac{p_1}{q_1} = a_0 + \frac{u/v}{a_1} = \frac{va_0a_1 + u}{va_1} = \frac{va_1p_0 + up_{-1}}{va_1q_0 + uq_{-1}}$$

and

$$\frac{p_2}{q_2} = a_0 + \frac{u/v}{a_1 +} \frac{u/v}{a_2} = \frac{va_0a_1a_2 + ua_0 + ua_2}{va_1a_2 + u} = \frac{a_2p_1 + up_0}{a_2q_1 + uq_0}.$$

Then, assume that (2.1) holds for integers $2k-1$ and $2k$ for some $k \in \mathbb{N}$. With this assumption, we shall show that the relations also hold for $2k+1$ and $2k+2$. First, observe that for any n,

$$\frac{p_{n+1}}{q_{n+1}} = a_0 + \frac{u/v}{a_1 +} \cdots + \frac{u/v}{a_n +} \frac{u/v}{a_{n+1}} = a_0 + \frac{u/v}{a_1 +} \cdots + \frac{u/v}{a_{n-1} +} \frac{u/v}{a_n + \frac{u/v}{a_{n+1}}},$$

which allows the calculation of $\frac{p_{2k+1}}{q_{2k+1}}$ by replacing a_{2k}. There must be some care with this observation, again following the strategy used in [2, Theorem 1.3 on p.21]. Note that if we replace a_{2k}, the values of p_{2k+1} and q_{2k+1} are unaltered. This is due to the manner of calculation, where the values of p_{2k+1} and q_{2k+1} depend on the preceding equations and not the further a values. Additionally, the same technique is used for calculating p_{2k+2} and q_{2k+2}. Now, starting with the assumptions for $2k-1$ and $2k$, we calculate

$$\begin{aligned}\frac{p_{2k+1}}{q_{2k+1}} &= \frac{(a_{2k} + \frac{u/v}{a_{2k+1}})p_{2k-1} + up_{2k-2}}{(a_{2k} + \frac{u/v}{a_{2k+1}})q_{2k-1} + uq_{2k-2}} = \frac{(va_{2k}a_{2k+1} + u)p_{2k-1} + up_{2k-2}}{(va_{2k}a_{2k+1} + u)q_{2k-1} + uq_{2k-2}} \\ &= \frac{va_{2k+1}(a_{2k}p_{2k-1} + up_{2k-2}) + up_{2k-1}}{va_{2k+1}(a_{2k}q_{2k-1} + uq_{2k-2}) + uq_{2k-1}} = \frac{va_{2k+1}p_{2k} + up_{2k-1}}{va_{2k+1}q_{2k} + uq_{2k-1}}\end{aligned}$$

and

$$\begin{aligned}\frac{p_{2k+2}}{q_{2k+2}} &= \frac{v(a_{2k+1} + \frac{u/v}{a_{2k+2}})p_{2k} + up_{2k-1}}{v(a_{2k+1} + \frac{u/v}{a_{2k+2}})q_{2k} + uq_{2k-1}} = \frac{(va_{2k+1}a_{2k+2} + u)p_{2k} + ua_{2k+2}p_{2k-1}}{(va_{2k+1}a_{2k+2} + u)q_{2k} + ua_{2k+2}q_{2k-1}} \\ &= \frac{a_{2k+2}(va_{2k+1}p_{2k} + up_{2k-1}) + up_{2k}}{a_{2k+2}(va_{2k+1}q_{2k} + uq_{2k-1}) + uq_{2k}} = \frac{a_{2k+2}p_{2k+1} + up_{2k}}{a_{2k+2}q_{2k+1} + uq_{2k}}.\end{aligned}$$

By the principle of mathematical induction, (2.1) holds for any $n \in \mathbb{N}$.

Proof of (2.2) also follows from induction. For the base case, calculations show that

$$p_0 q_{-1} - p_{-1} q_0 = a_0 * 0 - 1 * 1 = -1 = (-1)^{0-1} u^0$$

and

$$p_1 q_0 - p_0 q_1 = v a_0 a_1 + u - v a_0 a_1 = u = (-1)^{1-1} u^1.$$

Now, assume that (2.2) holds for some $2k$, $k \in \mathbb{N}$. We shall use this assumption to show that the relation also holds for both $2k+1$ and $2k+2$. Assume that

$$p_{2k-1} q_{2k} = p_{2k} q_{2k-1} - (-1)^{2k+1} u^{2k}.$$

Then,

$$\begin{aligned} p_{2k+1} q_{2k} &= v a_{2k+1} p_{2k} (a_{2k} q_{2k-1} + u q_{2k-2}) + u p_{2k-1} (a_{2k} q_{2k-1} + u q_{2k-2}) \\ &= v a_{2k+1} p_{2k} q_{2k} + u p_{2k-1} q_{2k} = v a_{2k+1} p_{2k} q_{2k} + u (p_{2k} q_{2k-1} - (-1)^{2k-1} u^{2k}) \\ &= v a_{2k+1} p_{2k} q_{2k} + u p_{2k} q_{2k-1} + (-1)^{2k} u^{2k+1} = p_{2k} q_{2k+1} + (-1)^{2k} u^{2k+1}. \end{aligned}$$

With the calculations for $2k+2$, it is nearly identical to show that $p_{2k+2} q_{2k+1} = p_{2k+1} q_{2k+2} + (-1)^{2k+1} u^{2k+2}$. Thus, by the principle of mathematical induction, (2.2) holds for $n \in \mathbb{N}$.

Proof of (2.3) will also follow from induction. For the base case,

$$x = [a_0, x_1] = a_0 + \frac{u/v}{x_1} = \frac{v a_0 x_1 + u}{v x_1} = \frac{v p_0 x_1 + u p_{-1}}{v q_0 x_1 + u q_{-1}}$$

and

$$x = [a_0, a_1, x_2] = a_0 + \frac{u/v}{a_1} + \frac{u/v}{x_2} = \frac{v a_0 a_1 x_2 + u x_2 + a_0 u}{v a_1 x_2 + u} = \frac{p_1 x_2 + u p_0}{q_1 x_2 + u q_0}.$$

Now assume that (2.3) holds for some $2k$, $k \in \mathbb{N}$. With this we can show that the relation holds for both $2k+1$ and $2k+2$. In a similar manner to the proof of (2.1), and for the same reasoning that allows it, we can replace an x_{2k+1} term with a term using both a_{2k} and x_{2k+1}. That is,

$$\begin{aligned} x = [a_0, a_1, \ldots, x_{2k+1}] &= \left[a_0, a_1, \ldots, a_{2k} + \frac{u/v}{x_{2k+1}}\right] = a_0 + \frac{u/v}{a_1} + \cdots + \frac{u/v}{a_{2k} + \frac{u/v}{x_{2k+1}}} \\ &= \frac{p_{2k}(a_{2k} + \frac{u/v}{x_{2k+1}}) + u p_{2k-2}}{q_{2k}(a_{2k} + \frac{u/v}{x_{2k+1}}) + u q_{2k-2}} = \frac{v p_{2k} x_{2k+1} (a_{2k} + u) + u v p_{2k-2} x_{2k+1}}{v q_{2k} x_{2k+1} (a_{2k} + u) + u v q_{2k-2} x_{2k+1}} \\ &= \frac{v x_{2k+1}(a_{2k} p_{2k-1} + u p_{2k-2}) + u p_{2k-1}}{v x_{2k+1}(a_{2k} q_{2k-1} + u q_{2k-2}) + u q_{2k-1}} = \frac{v p_{2k} x_{2k+1} + u p_{2k-1}}{v q_{2k} x_{2k+1} + u q_{2k-1}}. \end{aligned}$$

Therefore, the relation holds for $2k+1$, and a similar result follows for $2k+2$. Thus, by the principle of mathematical induction, (2.3) holds for $n \in \mathbb{N}$. □

It is convenient to have a table of some convergent values on hand to avoid the computations each time they are needed.

n	p_n	q_n
-1	1	0
0	a_0	1
1	$va_0a_1 + u$	va_1
2	$va_0a_1a_2 + ua_0 + ua_2$	$va_1a_2 + u$
3	$va_0a_1a_2a_3 + uva_0a_1 + uva_0a_3 + uva_2a_3 + u^2$	$v^2a_1a_2a_3 + uva_1 + uva_3$

Table 1: Values of p_n and q_n for $[a_0, a_1, \ldots]_{u/v}$ with small n.

Corollary 2.4. *For $n \in \mathbb{N}$ and the convergents, $\frac{p_n}{q_n}$, of a continued fraction using numerator $\frac{u}{v}$,*

(i) $v \mid q_{2n-1}$.

(ii) $gcd(p_n, q_n) \mid u^n$.

Proof. For (i), note $q_1 = va_1q_0 + uq_{-1} = va_1$, and $v \mid va_1$. Assume that for some k, $v \mid q_{2k-1}$. Then $q_{2k+1} = va_{2k+1}q_{2k} + uq_{2k-1}$. Since $v \mid va_{2k+1}q_{2k}$ and $v \mid q_{2k-1}$, then $v \mid q_{2k+1}$.
For (ii), let $d = gcd(p_n, q_n)$. Then $d \mid p_n$ and $d \mid q_n$, so $d \mid p_nq_{n-1} - p_{n-1}q_n$. By (2.2), $d \mid (-1)^nu^n$, so $d \mid u^n$. □

The relations shown above can be demonstrated by extending a previous example.

Example. We calculated $\frac{40}{11} = [3, 2, 4, 7, 3]_{3/2}$. By (2.1),

$$(p_n) = (3, 15, 69, 1011, 3240)$$

and

$$(q_n) = (1, 4, 19, 278, 891).$$

Subsequently,

$$\left(\frac{p_n}{q_n}\right) = \left(\frac{3}{1}, \frac{15}{4}, \frac{69}{19}, \frac{1011}{278}, \frac{3240}{891}\right) = \left(\frac{3}{1}, \frac{15}{4}, \frac{69}{19}, \frac{1011}{278}, \frac{40}{11}\right).$$

2.2 Continued Fraction Expansions

This section focuses on methods for using expansions that have already been calculated. We first turn our sights to finding continued fraction expansions that are very similar to one another. As observed, a given expansion may have terminating or periodic behavior, or even appear to go on forever without becoming periodic.

Definition 2.5. Two continued fraction expansions are *equivalent* if they have the same form (i.e., periodic, terminating, etc.) with the same lengths (e.g., periodic with period length 2), and eventually become the same expansion from some point on.

Example. Let $r = \frac{7}{4}$. We calculate the following.

$$\frac{5}{21} = [0,7,5]_r \qquad \frac{5}{41} = [0,14,5]_r \qquad \frac{87}{41} = [2,14,5]_r$$
$$\frac{5}{22} = \left[0,7,2,\overline{3}\right]_r \qquad \frac{5}{42} = \left[0,14,2,\overline{3}\right]_r \qquad \frac{89}{42} = \left[2,14,2,\overline{3}\right]_r$$

Note how these share certain characteristics. The observations to make are that $41 = 21+5*4$, $87 = 5 + 41 * 2$, $42 = 22 + 5 * 4$, and $89 = 5 + 42 * 2$.

Following from the previous example, we shall now show two results that will assist us in finding equivalent continued fractions.

Lemma 2.6. *For integers u, v, x, and y, if*

$$\frac{x}{y} = [a_0, a_1, a_2, \ldots]_{u/v},$$

then

$$\frac{x+yi}{y} = [a_0 + i, a_1, a_2, \ldots]_{u/v}$$

for any positive integer i.

Proof. Notice that $\frac{(x+yi)}{y} = \frac{x}{y} + i$. The result follows immediately by construction. □

Lemma 2.7. *For integers u, v, x, and $xj + k$ with $j \geqslant 1$ and $0 \leqslant k < x$, if*

$$\frac{x}{xj+k} = [0, a_1, a_2, \ldots]_{u/v},$$

then

$$\frac{x}{x(j+vi)+k} = [0, a_1 + ui, a_2, \ldots]_{u/v}$$

for any positive integer i.

Proof. Let $x_0 = \frac{x}{xj+k}$ with $j \geqslant 1$ and $0 < k < x$, then $gcd(x, xj + k) = 1$. Suppose that

$$x_0 = [0, a_1, a_2, \ldots]_{u/v} = [0, a_1, x_2]_{u/v}.$$

By the algorithm that we introduced earlier,

$$a_0 = 0, \qquad x_1 = \frac{u(mj+k)}{vm}, \qquad a_1 = \lfloor x_1 \rfloor, \qquad x_2 = \frac{u/v}{x_1 - \lfloor x_1 \rfloor},$$

are the beginning pieces for constructing the expansion. Then we shall do a similar construction for $\frac{x}{x(j+vi)+k}$ where i is a positive integer. With the same algorithm,

$$\begin{aligned} a'_0 &= 0, \\ x'_1 &= \frac{u(m(j+vi))+k)}{vm} = x_1 + \frac{uvim}{vm} = x_1 + ui, \\ a'_1 &= \lfloor x_1 + ui \rfloor = a_1 + ui, \end{aligned}$$

and thus,

$$x_2' = \frac{u/v}{x_1' - a_1'} = \frac{u/v}{x_1 + ui - a_1 - ui} = \frac{u/v}{x_1 - \lfloor x_1 \rfloor}.$$

Therefore, $\frac{x}{x(j+vi)+k} = [0, a_1 + ui, a_2, \ldots]_{u/v}$. □

With the results of the previous lemmas, the following can be shown.

Theorem 2.8. *With numerator $\frac{u}{v}$ and integers x and y, when considering continued fraction expansions under the same algorithm,*

$$\frac{x}{y} \sim \frac{x \ (mod\ y)}{y} \sim \frac{x \ (mod\ y)}{y \ (mod\ v * (x \ (mod\ y)))}.$$

Proof. By definition, $x \pmod{y} \equiv x + yi$ for some i. The first equivalence follows from Lemma 2.6. Now let $l = x \pmod{y}$. Then $l < y$, and we can rewrite y as $y = lj + k$ for some $j \geqslant 1$ and $0 < k < l$. Next, $y \pmod{vl} \equiv y + vli \equiv l(j + vi) + k$ for some i. The second equivalence follows from Lemma 2.7. □

One application of Theorem 2.8 is the ability to classify continued fractions more efficiently, especially if we are only looking for general properties.

Example. We wish to know whether there are any periodic continued fraction expansions for numbers of the form $\frac{1}{y}$, where y is an integer, with numerator $\frac{3}{2}$. We calculate

$$\frac{1}{2} = [0, 3]_{3/2}$$

and

$$\frac{1}{3} = [0, 4, 3]_{3/2}\,.$$

By Theorem 2.8, every number $\frac{1}{y}$ will have one of the above forms, so no periodic continued fraction expansions arise from the form $x = \frac{1}{y}$.

Example. We would like to determine all unique period lengths that continued fraction expansions have for numerator $\frac{17}{9}$ when we expand all $\frac{x}{y}$ where $x, y < 1000$. Calculations show that $\frac{9}{11}$ becomes periodic with period length 44 under the max algorithm. We now can save significant time by no longer needing to compute the expansions for $\frac{9}{92}$, $\frac{9}{173}$, $\frac{355}{173}$, $\frac{9}{254}$, $\frac{263}{254}$, etc., since they are all equivalent (each will be periodic with period length 44).

As demonstrated, time can be saved by calculating some continued fraction expansions and then no longer needing to do the same for a large class of others, possibly infinite.

The next results are extensions of classic work on simple continued fractions.

Theorem 2.9. *For integers u, v, x, and y, if $0 < x < y$ and*

$$\frac{x}{y} = [0, a_1, a_2, a_3, \ldots]_{u/v}\,,$$

then

$$\frac{uy}{vx} = [a_1, a_2, a_3, \ldots]_{u/v}\,.$$

Proof. With $0 < x < y$ and $x_0 = \frac{x}{y}$, $\lfloor x_0 \rfloor = 0$, so $x_1 = \frac{u/v}{x/y - 0} = \frac{uy}{vx}$. $\square$

Theorem 2.9 gives the ability to "extract" a piece of a continued fraction, as demonstrated in the next example.

Example. After calculating $\frac{5}{3} = [1,2,2,2,9,5,7]_{7/4}$, we wish to "pull off" the front of the expansion. Using Theorem 2.8 and Theorem 2.9, we know $\frac{2}{3} = [0,2,2,2,9,5,7]_{7/4}$ and $\frac{7*3}{4*2} = \frac{21}{8} = [2,2,2,9,5,7]_{7/4}$. We can continue in this fashion, repeatedly applying the two theorems, finding expansions such as $\frac{35}{16} = [2,9,5,7]_{7/4}$.

Theorem 2.10. *If* $\frac{p_n}{q_n} = [a_0, a_1, a_2, \ldots, a_n]_{u/v}$ *and* $k \in \mathbb{N}$, *then*

(i) *for* $2 \leqslant 2k \leqslant n$, $\frac{q_{2k}}{q_{2k-1}} = [a_{2k}, ..., a_2, a_1]_{u/v}$.

(ii) *if* $a_0 \neq 0$, *for* $0 \leqslant 2k \leqslant n$, $\frac{p_{2k}}{p_{2k-1}} = [a_{2k}, ..., a_2, a_1, a_0]_{u/v}$.

(iii) *if* $a_0 = 0$, *for* $2 \leqslant 2k \leqslant n$, $\frac{p_{2k}}{p_{2k-1}} = [a_{2k}, ..., a_4, a_3, a_2]_{u/v}$.

Additionally, if $v \mid a_{2i}$ *for all* $0 \leqslant i \leqslant n$, *then*

(iv) *for* $1 \leqslant 2k+1 \leqslant n$, $\frac{q_{2k+1}}{q_{2k}} = \left[va_{2k+1}, ..., \frac{a_2}{v}, va_1\right]_{u/v}$.

(v) *if* $a_0 \neq 0$, *for* $1 \leqslant 2k+1 \leqslant n$, $\frac{p_{2k+1}}{p_{2k}} = \left[va_{2k+1}, ..., \frac{a_2}{v}, va_1, \frac{a_0}{v}\right]_{u/v}$.

(vi) *if* $a_0 = 0$, *for* $3 \leqslant 2k+1 \leqslant n$, $\frac{p_{2k+1}}{p_{2k}} = \left[va_{2k+1}, ..., \frac{a_4}{v}, va_3, \frac{a_2}{v}\right]_{u/v}$.

Proof. The proof of (i) follows by induction. We are interested in an index on $2k$ and for $2k = 2$,

$$\frac{q_2}{q_1} = \frac{a_2 q_1 + u q_0}{q_1} = a_2 + \frac{u q_0}{q_1} = a_2 + \frac{u}{v a_1} = a_2 + \frac{u/v}{a_1} = [a_2, a_1]_{u/v},$$

which completes the base case. Now assume that the result holds for some positive integer $2l$. That is,

$$\frac{q_{2l}}{q_{2l-1}} = [a_{2l}, ..., a_2, a_1]_{u/v} = a_{2l} + \frac{u/v}{a_{2l-1}} + \cdots + \frac{u/v}{a_2} + \frac{u/v}{a_1}.$$

Then,

$$\begin{aligned}\frac{q_{2l+2}}{q_{2l+1}} &= \frac{a_{2l+2} q_{2l+1} + u q_{2l}}{q_{2l+1}} = a_{2l+2} + \frac{u q_{2l}}{q_{2l+1}} = a_{2l+2} + \frac{u/v}{(q_{2l+1}/q_{2l})/v} \\ &= a_{2l+2} + \frac{u/v}{((v a_{2l+1} q_{2l} + u q_{2l-1})/q_{2l})/v} = a_{2l+2} + \frac{u/v}{a_{2l+1} + (u q_{2l-1})/(v q_{2l})} \\ &= a_{2l+2} + \frac{u/v}{a_{2l+1} + \frac{u/v}{q_{2l}/q_{2l-1}}} = a_{2l+2} + \frac{u/v}{a_{2l+1}} + \cdots + \frac{u/v}{a_1} = [a_{2l+2}, ..., a_2, a_1]_{u/v},\end{aligned}$$

as desired for $2l + 2$, which proves the result. The proofs for (ii), (iii), (iv), (v), and (vi) are very similar. $\square$

Example. We calculate $\frac{41}{17} = [2, 3, 2, 4, 3]_{3/2}$. Furthermore,

$$(q_n) = (1, 6, 15, 138, 459).$$

Then, by Theorem 2.10.i, $\frac{15}{6} = [2, 3]_{3/2}$ and $\frac{459}{138} = [3, 4, 2, 3]_{3/2}$.

Example. We calculate $\frac{32}{5} = [6, 3, 2]_{3/2}$. Furthermore,

$$(p_n) = (6, 39, 96).$$

Then, by Theorem 2.10.v, $\frac{39}{6} = \left[2 * 3, \frac{6}{2}\right]_{3/2} = [6, 3]_{3/2}$.

3 Periodic Results

3.1 General Periodicity

As mentioned earlier, some continued fraction expansions appear to eventually become periodic. Here, we go into further detail about what this entails and find special results. Periodic continued fraction expansions consist of a leading tail and an orbit, both of which can vary in length. These have the form

$$[a_0, a_1, \ldots, \overline{a_k, a_{k+1}, \ldots, a_{k+n-1}, a_{k+n-1}}]_{u/v}$$

with the part under the line repeating indefinitely. Periodicity can be proven, as demonstrated next.

Example. We wish to compute the continued fraction expansion of $\frac{15}{34}$ with numerator $\frac{7}{4}$ and the max algorithm. When calculated, there are sequences $(a_n) = (0, 3, 1, 2, 10, \ldots)$ and $(x_n) = (\frac{119}{30}, \frac{105}{58}, \frac{203}{94}, \frac{329}{30}, \frac{105}{58}, \ldots)$. At that point, we are able to prove periodicity. The term $\frac{105}{58}$ reappears in the (x_n) sequence, so the steps will repeat at that point forward, causing period length 3. Indeed, $\frac{15}{34} = \left[0, 3, \overline{1, 2, 10}\right]_{7/4}$.

Remark. A remark must be made about the previous example – the algorithm for choosing terms should be consistent. For instance, if the max choice is made at one point and the min choice at another, the expansion will not necessarily have periodic behavior as the steps for choosing the terms may not follow the same pattern.

There is slight difficulty in classifying periodic expansions when we have to work with the tails. It would be helpful to "extract" the periodic part in the sense of taking the value of the orbit into a new continued fraction. By doing this, we will have one that is purely periodic with period length n of the form $[\overline{a_0, a_1, \ldots, a_{n-1}}]_{u/v}$. Previously, we showed how Theorem 2.9 could be used to "extract" the end of an expansion.

Example. Calculations show that $\frac{49}{30} = \left[1, 2, \overline{2, 5, 1, 2, 9}\right]_{7/4}$, but we would like to work with an expansion that is purely periodic with period length 5. Applying ideas from Theorem 2.8 and Theorem 2.9 twice, we arrive at $\frac{(19*4)*7}{((30*7)-2*(19*4))*4} = \frac{532}{232} = \frac{133}{58} = \left[\overline{2, 5, 1, 2, 9}\right]_{7/4}$.

Going further, a purely periodic continued fraction can be used in a recursive manner. Recall that periodic behavior occurs when a value in the sequence (x_n) reappears. Thus, for a purely periodic expansion of x with a period defined by $a_0, a_1, \ldots, a_{n-1}$, we can write

$$x = [a_0, a_1, \ldots, a_{n-1}, x]_{u/v}\,.$$

With this form of a periodic continued fraction at hand, previous results can be used to derive several properties. First, a known fact of simple continued fractions is that if x is a real number and the expansion of x is eventually periodic, then x is a quadratic surd [2, p.89]. This is not, in general, true for continued fractions with rational numerators, as we have demonstrated. However, if

$$x = [a_0, a_1, \ldots, \overline{a_k, a_{k+1}, \ldots, a_{k+n-1}, a_{k+n-1}}]_{u/v}\,,$$

and we let $b = [\overline{a_k, a_{k+1}, \ldots, a_{k+n-1}, a_{k+n-1}}]_{u/v}$, then (2.3) tells us

$$b = \frac{p_{n-1}b + up_{n-2}}{q_{n-1}b + uq_{n-2}}$$

or

$$b = \frac{vp_{n-1}b + up_{n-2}}{vq_{n-1}b + uq_{n-2}}.$$

That is, b satisfies a quadratic equation, so b is either a rational number or a quadratic surd. Since $x = [a_0, a_1, \ldots, b]_{u/v}$, x can be simplified and will be either a rational number or quadratic surd as well.

For the next property, we can go on to say more about expansions of rational numbers using a rational numerator.

Theorem 3.1. *If a continued fraction expansion of x, a rational number, with numerator $\frac{u}{v}$ is purely periodic with period length n, then there exists an integer solution to*

$$(uq_{n-2} + p_{n-1})^2 - 4u^n = s^2, \qquad \text{if } n \text{ is even} \qquad (3.1)$$
$$(uq_{n-2} + vp_{n-1})^2 + 4vu^n = s^2, \qquad \text{if } n \text{ is odd} \qquad (3.2)$$

where $\frac{p_{n-2}}{q_{n-2}}$ and $\frac{p_{n-1}}{q_{n-1}}$ are the $n-2$ and $n-1$ convergents, respectively, of x and s is an integer.

Proof. Assume that we have a purely periodic continued fraction expansion

$$x = [a_0, a_1, \ldots, a_{n-1}, x]_{u/v}$$

where n is even. Recall the x_{2n} version of (2.3) for x, so

$$x = \frac{p_{n-1}x_n + up_{n-2}}{q_{n-1}x_n + uq_{n-2}} = \frac{p_{n-1}x + up_{n-2}}{q_{n-1}x + uq_{n-2}},$$

which can be rearranged; then apply the quadratic formula to

$$q_{n-1}x^2 + (uq_{n-2} - p_{n-1})x - up_{n-2} = 0.$$

Since x is rational, the discriminant of the quadratic must be a perfect square. Here, the discriminant is

$$u^2q_{n-2}^2 + 4uq_{n-1}p_{n-2} - 2uq_{n-2}p_{n-1} + p_{n-1}^2,$$

and we can simplify by utilizing (2.2), then

$$u^2q_{n-2}^2 + 2uq_{n-2}p_{n-1} + 4(-1)^nu^n + p_{n-1}^2 = (uq_{n-2} + p_{n-1})^2 - 4u^n.$$

With even n, this results gives (3.1). By a very similar process for the case when n is odd, we arrive at (3.2). □

With these equations for periodicity, we take note of an observation about odd period lengths, which can now be proved.

Theorem 3.2. *If there exists integers x and y such that the expansion of $\frac{x}{y}$ with numerator r has odd period length, then for some $u, w \in \mathbb{N}$, $r = \frac{u}{w^2}$.*

Proof. Suppose that we have a continued fraction with odd period length n and numerator $\frac{u}{v}$. Then there must also exist a purely periodic continued fraction with the same period length and numerator. Since v is an integer, and we know that integers can be decomposed into a "square" part and a "square-free" part, we can write $v = w^2v'$ where w is an integer and v' is a square-free integer. Substitute this into (3.2) and we know there exists an integer solution to

$$(uq_{n-2} + w^2v'p_{n-1})^2 + 4w^2v'u^n = s^2. \tag{3.3}$$

Now suppose that there is a prime number p such that $p > 2$ and $p \mid v'$. Since v' is square-free, we know $p \parallel v'$. If $p \mid w^2$, since w^2 is a perfect square, then $p^{2m} \parallel w^2$ for some m. Next, following from Corollary 2.4.i, since $w^2v' \mid q_{n-2}$, then $p * p^{2m} = p^{2m+1} \mid q_{n-2}$. With these facts, we know $p^{2m+1} \mid (uq_{n-2} + w^2v'p_{n-1})^2$, and since $gcd(w^2v', u) = 1$, we know $p^{2m+1} \parallel 4w^2v'u^n$, which ultimately gives the result that $p^{2m+1} \parallel (uq_{n-2}+w^2v'p_{n-1})^2+4w^2v'u^n$. However, s^2 is a perfect square so the prime decomposition under the fundamental theorem of arithmetic must have all primes to an even power. We have shown $p^{2m+1} \parallel s^2$, a contradiction.

Now instead suppose that $v' = 2$, and consider two cases. First, if $2^{2m} \parallel w^2$ for some $m > 0$, then $2^{2(2m+1)} = 2^{4m+2} \mid (uq_{n-2} + w^2v'p_{n-1})^2$ and $2^22^{2m}2 = 2^{2m+3} \parallel 4w^2v'u^n$. This implies that $2^{2m+3} \parallel s^2$. Since s^2 is a perfect square, we find a contradiction.

On the other hand, if $gcd(w^2, 2) = 1$, then $4 \mid (uq_{n-2} + w^2v'p_{n-1})^2$ and $8 \parallel 4w^2v'u^n$. In that case, (3.3) can be divided by 4 as

$$\left(\frac{uq_{n-2} + w^2v'p_{n-1}}{2}\right)^2 + 2w^2u^n = \left(\frac{s}{2}\right)^2.$$

Well-known properties of squares tell us that both sides of the above must be 0 (mod 4) or 1 (mod 4). This is not the case, however, as $2w^2u^n \equiv 2 \pmod 4$.

Thus, v' is neither 2 nor divisible by an odd prime, so $v' = 1$ and $v = w^2 * 1 = w^2$. □

As previously shown, Theorem 2.9 can be used to slightly alter periodic behavior. To further the possibilities of altering expansions, we set out to determine reversals of purely periodic continued fractions. First, we need to prove a useful fact about convergents that we will make use of.

Lemma 3.3. *If* $x = [a_0, a_1, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots]_{u/v}$ *and* $x' = [a_n, a_{n-1}, \ldots, a_1, a_0, b_1, b_2, \ldots]_{u/v}$*, then for* $\frac{p_k}{q_k}$*, the* k *convergent of* x*, and* $\frac{p'_k}{q'_k}$*, the* k *convergent of* x'*,*

$$p_n = p'_n, \tag{3.4}$$

$$q_{n-1} = q'_{n-1}, \tag{3.5}$$

$$q_n = \begin{cases} p'_{n-1}, & \text{if } n \text{ is even,} \\ vp'_{n-1} & \text{if } n \text{ is odd,} \end{cases} \tag{3.6}$$

$$q'_n = \begin{cases} p_{n-1}, & \text{if } n \text{ is even,} \\ vp_{n-1} & \text{if } n \text{ is odd.} \end{cases} \tag{3.7}$$

Proof. For convenience and conciseness, it is possible to think of p_n and q_n as polynomials in the variables $u, v, a_0, a_1, \ldots$. For example, let $q_2(a_0, a_1, a_2) = va_1a_2 + u$, then $q'_2 = q_2(a_2, a_1, a_0)$. This way, we avoid the prime notation and write

$$p_n(a_0, a_1, \ldots, a_n) = p_n(a_n, a_{n-1}, \ldots, a_0),$$
$$q_{n-1}(a_0, a_1, \ldots, a_n) = q_{n-1}(a_n, a_{n-1}, \ldots, a_0),$$
$$q_n(a_0, a_1, \ldots, a_n) = \begin{cases} p_{n-1}(a_n, a_{n-1}, \ldots, a_0), & \text{if } n \text{ is even,} \\ vp_{n-1}(a_n, a_{n-1}, \ldots, a_0) & \text{if } n \text{ is odd,} \end{cases}$$
$$q_n(a_n, a_{n-1}, \ldots, a_0) = \begin{cases} p_{n-1}(a_0, a_1, \ldots, a_n), & \text{if } n \text{ is even,} \\ vp_{n-1}(a_0, a_1, \ldots, a_n) & \text{if } n \text{ is odd.} \end{cases}$$

There are several important details to notice about p_n and q_n. First, one can see from the formulas that p_n depends only on the first $n+1$ arguments, that is, $p_n(a_0, a_1, \ldots, a_n, a_{n+1}, \ldots) = p_n(a_0, a_1, \ldots, a_n)$. Similarly, q_n does not depend on the first argument nor arguments past $n + 1$, that is, $q_{n-1}(a_0, a_1, \ldots, a_n, a_{n+1}, \ldots) = q_{n-1}(a_1, \ldots, a_{n-1})$ (one must take care with the fact that this cannot be applied repeatedly, that is, if $q_{n-1}(a_0, a_1, \ldots, a_n, a_{n+1}, \ldots) = q_{n-1}(a_1, \ldots, a_{n-1})$, it is not necessarily true that $q_{n-1}(a_1, \ldots, a_{n-1}) = q_{n-1}(a_2, \ldots, a_{n-1})$). By using these facts, we can reconsider the above formulas and aim to prove

$$p_n(a_0, a_1, \ldots, a_n) = p_n(a_n, a_{n-1}, \ldots, a_0), \tag{3.8}$$

$$q_{n-1}(a_0, a_1, \ldots, a_{n-1}) = q_{n-1}(a_n, a_{n-1}, \ldots, a_1), \tag{3.9}$$

$$q_n(a_0, a_1, \ldots, a_n) = \begin{cases} p_{n-1}(a_n, a_{n-1}, \ldots, a_1), & \text{if } n \text{ is even,} \\ vp_{n-1}(a_n, a_{n-1}, \ldots, a_1) & \text{if } n \text{ is odd,} \end{cases} \tag{3.10}$$

$$q_n(a_n, a_{n-1}, \ldots, a_1) = \begin{cases} p_{n-1}(a_0, a_1, \ldots, a_{n-1}), & \text{if } n \text{ is even,} \\ vp_{n-1}(a_0, a_1, \ldots, a_{n-1}) & \text{if } n \text{ is odd.} \end{cases} \tag{3.11}$$

Again using the same facts introduced above, we find several additional formulas that will be of use,

$$q_n(a_0, a_1, \ldots, a_n) = \begin{cases} p_{n-1}(a_1, \ldots, a_n), & \text{if } n \text{ is even,} \\ vp_{n-1}(a_1, \ldots, a_n) & \text{if } n \text{ is odd,} \end{cases} \tag{3.12}$$

$$p_n(a_0, a_1, \ldots, a_n) = a_0 q_n(a_0, a_1, \ldots, a_n) + \begin{cases} \frac{u}{v} q_{n-1}(a_1, \ldots, a_n), & \text{if } n \text{ is even,} \\ uq_{n-1}(a_1, \ldots, a_n) & \text{if } n \text{ is odd.} \end{cases} \tag{3.13}$$

When viewed as four formulas, (3.12) and (3.13) can be proved by induction. To start, see that $q_0(a_0) = 1 = p_{-1}(a_0)$ and $q_1(a_0, a_1) = va_1 = vp_0(a_1)$ as base cases. For even n, assume that $q_{n-1}(a_0, a_1, \ldots, a_{n-1}) = vp_{n-2}(a_1, \ldots, a_{n-1})$ and $q_n(a_0, a_1, \ldots, a_n) = p_{n-1}(a_1, \ldots, a_n)$, then

$$\begin{aligned} q_{n+1} &= q_{n+1}(a_0, a_1, \ldots, a_n, a_{n+1}) \\ &= va_{n+1} q_n(a_0, a_1, \ldots, a_n, a_{n+1}) + uq_{n-1}(a_0, a_1, \ldots, a_n, a_{n+1}) \\ &= va_{n+1} q_n(a_0, a_1, \ldots, a_n) + uq_{n-1}(a_0, a_1, \ldots, a_{n-1}) \\ &= va_{n+1} p_{n-1}(a_1, \ldots, a_n) + uvp_{n-2}(a_1, \ldots, a_{n-1}) \\ &= vp_n(a_1, \ldots, a_n, a_{n+1}), \end{aligned}$$

as desired for (3.12). The case for odd n follows almost the same steps. After showing base cases and assuming induction hypotheses, (3.13) can be proved in a similar manner.

Now we can prove (3.8) through (3.11), also using induction. The bases are trivial, as we have shown in similar situations. Assume that each equation holds up to n. Starting with the even n case of (3.8), we have

$$\begin{aligned} p_{n+1}(a_0, a_1, \ldots, a_n, a_{n+1}) &= va_{n+1} p_n(a_0, \ldots, a_n) + up_{n-1}(a_0, \ldots, a_{n-1}) \\ &= va_{n+1} a_0 q_n(a_0, \ldots, a_n) + ua_{n+1} q_{n-1}(a_1, \ldots, a_{n-1}) \\ &\quad + ua_0 q_{n-1}(a_0, \ldots, a_{n-1}) + u^2 q_{n-2}(a_1, \ldots, a_{n-1}) \\ &= va_{n+1} a_0 q_n(a_{n+1}, \ldots, a_1) + ua_{n+1} q_{n-1}(a_n, \ldots, a_2) \\ &\quad + ua_0 q_{n-1}(a_n, \ldots, a_1) + u^2 q_{n-2}(a_{n-1}, \ldots, a_2) \\ &= a_{n+1} q_{n+1}(a_{n+1}, \ldots, a_1, a_0) + uq_n(a_n, \ldots, a_0) \\ &= p_{n+1}(a_{n+1}, a_n, \ldots, a_2, a_1, a_0), \end{aligned}$$

as desired. The case for odd n of (3.8) follows in a similar manner. The proofs for (3.9), (3.10), and (3.11) are omitted for brevity, but use similar strategies. By proving those equations, the relations that we originally introduced hold as well. □

Theorem 3.4. *If* $\frac{x}{y} = [\overline{a_0, a_1, \ldots, a_{k-1}}]_{u/v}$ *and* $\frac{x'}{y'} = [\overline{a_{k-1}, \ldots, a_1, a_0}]_{u/v}$*, then*

$$\frac{x'}{y'} = \frac{xq_{k-1}}{yvp_{k-2}}, \qquad \textit{if } k \textit{ is even} \tag{3.14}$$

$$\frac{x'}{y'} = \frac{xq_{k-1}}{yp_{k-2}}, \qquad \textit{if } k \textit{ is odd.} \tag{3.15}$$

where $\frac{p_{k-2}}{q_{k-2}}$ *and* $\frac{p_{k-1}}{q_{k-1}}$ *are the* $k-2$ *and* $k-1$ *convergents of* $\frac{x}{y}$*, respectively.*

Proof. Assume that we have

$$\frac{x}{y} = [\overline{a_0, a_1, \ldots, a_{k-1}}]_{u/v} = \left[a_0, a_1, \ldots, a_{k-1}, \frac{x}{y}\right]_{u/v}$$

where k is even. We can apply the x_{2n} version of (2.3) to get

$$\frac{x}{y} = \frac{p_{k-1}\frac{x}{y} + up_{k-2}}{q_{k-1}\frac{x}{y} + uq_{k-2}} = \frac{p_{k-1}\frac{x}{y} + up_{k-2}}{q_{k-1}\frac{x}{y} + uq_{k-2}} * \frac{\frac{q_{k-1}}{vp_{k-2}}}{\frac{q_{k-1}}{vp_{k-2}}} = \frac{p_{k-1}\frac{xq_{k-1}}{yvp_{k-2}} + \frac{uq_{k-1}}{v}}{q_{k-1}\frac{xq_{k-1}}{yvp_{k-2}} + \frac{uq_{k-1}q_{k-2}}{vp_{k-2}}}.$$

Through multiplication,

$$\frac{xq_{k-1}}{yvp_{k-2}} = \frac{x}{y} * \frac{1}{\frac{1}{q_{k-1}}} * \frac{1}{vp_{k-2}} = \frac{p_{k-1}\frac{xq_{k-1}}{yvp_{k-2}} + \frac{uq_{k-1}}{v}}{q_{k-1}\frac{xq_{k-1}}{yvp_{k-2}} + \frac{uq_{k-1}q_{k-2}}{vp_{k-2}}} * \frac{1}{\frac{1}{q_{k-1}}} * \frac{1}{vp_{k-2}} = \frac{p_{k-1}\frac{xq_{k-1}}{yvp_{k-2}} + \frac{uq_{k-1}}{v}}{vp_{k-2}\frac{xq_{k-1}}{yvp_{k-2}} + uq_{k-2}}.$$

Suppose that $\frac{x'}{y'} = [\overline{a_{k-1}, \ldots, a_1, a_0}]_{u/v}$. By the results of Lemma 3.3,

$$\frac{xq_{k-1}}{yvp_{k-2}} = \frac{p'_{k-1}\frac{xq_{k-1}}{yvp_{k-2}} + up'_{k-1}}{q'_{k-2}\frac{xq_{k-1}}{yvp_{k-2}} + uq'_{k-2}}.$$

Thus, $\frac{x'}{y'} = \frac{xq_{k-1}}{yvp_{k-2}} = [\overline{a_{k-1}, \ldots, a_1, a_0}]_{u/v}$.

The proof for odd k is nearly identical. □

The utility of the previous theorem can be demonstrated through example.

Example. We previously discovered

$$\frac{133}{58} = \left[\overline{2, 5, 1, 2, 9}\right]_{7/4}.$$

By Theorem 3.4,

$$\frac{133q_4}{58p_3} = \frac{819}{86} = \left[\overline{9, 2, 1, 5, 2}\right]_{7/4}.$$

Note that this does not, in some cases, give results through the same algorithm. Here, for example, $\frac{133}{58} = \left[\overline{2, 5, 1, 2, 9}\right]_{7/4}$ with the max algorithm while $\frac{819}{86} = \left[9, 3, 5, 21, 2, \overline{3}\right]_{7/4}$ with the same algorithm.

With these results, we have tools to find and confirm periodic continued fraction expansions. However, a question still remains about how many different purely periodic expansions can be made for a particular numerator.

Theorem 3.5. *If there exists*

$$\frac{x_0}{y_0} = [\overline{a_0, a_1, \ldots, a_{k-1}}]_{u/v}$$

for some numerator $\frac{u}{v}$ *and integer* $k \geqslant 3$*, then for* n*, the number of purely periodic expansions with period length* k*,* $k \mid n$*. Furthermore, if* $a_i \neq a_{k-1-i}$ *for some* $0 \leqslant i \leqslant \frac{k-1}{2}$ *(that is, the period of* $\frac{x_0}{y_0}$ *is not symmetric), then* $2k \mid n$*. In particular, for* $0 \leqslant l < k$ *(mod* k*), there are* k *expansions of the form*

$$\frac{x_l}{y_l} = [\overline{a_l, \ldots, a_{k-1}, a_0, \ldots, a_{l-1}}]_{u/v}$$

and there are k *expansions of the form*

$$\frac{x_l}{y_l} = [\overline{a_l, \ldots, a_0, a_{k-1}, \ldots, a_{l+1}}]_{u/v} .$$

Proof. Start with an expansion of $\frac{x_0}{y_0}$ with numerator $\frac{u}{v}$, and we will construct more. Suppose

$$\frac{x_0}{y_0} = [\overline{a_0, a_1, \ldots, a_{k-1}}]_{u/v} .$$

By utilizing Theorem 2.8 and Theorem 2.9,

$$\frac{uy}{v(x - a_0 y)} = \frac{x_1}{y_1} = [\overline{a_1, a_2, \ldots, a_{k-1}, a_0}]_{u/v} .$$

Continuing with the same method, we are able to construct expansions that are purely periodic that use each a_i as a leading term. This makes k expansions of this form with period length k.
Next, the application of Theorem 3.4 to $\frac{x_0}{y_0}$ gives

$$\frac{x_0'}{y_0'} = [\overline{a_{k-1}, \ldots, a_1, a_0}]_{u/v} .$$

Then, similarly to above, we can repeatedly apply Theorem 2.8 and Theorem 2.9 to find k expansions that are purely periodic with period length k, which use each a_i as leading term, but in reverse order. □

3.2 Period Length 1

With tools at hand to examine periodic continued fractions, we next give special attention to period length 1, where we can extend ideas that applied to any period length. By using (3.2), the numerators that can produce expansions with periodic length 1 are classified by solving

$$s^2 = (uq_{-1} + vp_0)^2 + 4vu = v^2 a_0^2 + 4vu. \tag{3.16}$$

Example. For numerator $\frac{7}{4}$, see that $4^2 3^2 + 4 * 7 * 4 = 256 = 16^2$. This shows the existence of a continued fraction which has period length 1, specifically with $a_0 = 3$.

Repeatedly applying examples to (3.16) shows a pattern, which brings us to the following theorem.

Theorem 3.6. *There exists rational $x = [\overline{a_0}]_{u/v}$, a purely periodic continued fraction with period length 1, if and only if the following conditions hold.*

1. $v = w^2$ *for some* $w \in \mathbb{N}$, $w^2 < u$.

2. *There exists* $m_1 > m_2 > 0$ *such that* $u = 2^n(m_1 m_2)$ *and* $w \mid 2^k m_1 - 2^{n-k} m_2$ *for some* $0 \leqslant k \leqslant n$.

Then, $a_0 = \frac{(2^k m_1 - 2^{n-k} m_2)}{w}$ *and* $x = \frac{a_0}{2} + \frac{2^k m_1 + 2^{n-k} m_2}{2w} = \frac{2^k m_1}{w}$.

Proof. Suppose that we have a continued fraction of the form $x = [\overline{a_0}]_{u/v}$. Following directly from Theorem 3.2, $v = w^2$ for some $w \in \mathbb{N}$, so we rewrite (3.16) as

$$w^4 a_0^2 + 4w^2 u = w^2(w^2 a_0^2 + 4u) = (s')^2$$

where s' is an integer. Let $s = \frac{s'}{w}$ in order to eliminate the w^2 factor, and thus there must be an integer solution to

$$w^2 a_0^2 + 4u = s^2.$$

This can be rearranged as

$$s^2 - w^2 a_0^2 = 4u,$$

and factored as

$$(s - wa_0)(s + wa_0) = 4u. \tag{3.17}$$

Suppose $u = 2^n m_1 m_2$ for some $n, m_1, m_2 \in \mathbb{N}$ where $m_1 > m_2$, $2 \nmid m_1$, and $2 \nmid m_2$. Now we have

$$(s - wa_0)(s + wa_0) = 2^{n+2} m_1 m_2.$$

Let $x = s - wa_0$ and $y = s + wa_0$, then $xy = 2^{n+2} m_1 m_2$ and $y - x = 2wa_0$. Since $2 \mid y - x$, it must be that $2 \mid x$ and $2 \mid y$. Therefore, we may suppose that $y = 2^{k+1} m_1$ and $x = 2^{n-k+1} m_2$ for some $0 \leqslant k \leqslant n$. So,

$$2wa_0 = y - x = 2^{k+1} m_1 - 2^{n-k+1} m_2,$$

and after dividing by 2 we find

$$wa_0 = 2^k m_1 - 2^{n-k} m_2.$$

Since a_0 is an integer, $w \mid 2^k m_1 - 2^{n-k} m_2$ and $a_0 = \frac{2^k m_1 - 2^{n-k} m_2}{w}$.
Finally, we will solve for x as

$$x = [\overline{a_0}]_{u/w^2} \implies x = a_0 + \frac{u/w^2}{x} \implies x^2 - a_0 x - u/w^2 = 0.$$

The the quadratic formula gives

$$x = \frac{a_0 \pm \sqrt{a_0^2 + \frac{4u}{w^2}}}{2}.$$

Substitute a_0 and u to find

$$\begin{aligned} x &= \frac{\frac{2^k m_1 - 2^{n-k} m_2}{w} \pm \sqrt{(\frac{2^k m_1 - 2^{n-k} m_2}{w})^2 + \frac{2^{n+2} m_1 m_2}{w^2}}}{2} \\ &= \frac{\frac{2^k m_1 - 2^{n-k} m_2}{w} \pm \sqrt{(\frac{2^k m_1 + 2^{n-k} m_2}{w})^2}}{2} \\ &= \frac{\frac{2^k m_1 - 2^{n-k} m_2}{w} \pm \frac{2^k m_1 + 2^{n-k} m_2}{w}}{2}. \end{aligned}$$

The subtractive case gives a negative number, so we shall only consider the additive case. Therefore,

$$x = \frac{2^k m_1 + 2^k m_1}{2w} = \frac{2^k m_1}{w}.$$

The converse can be proved by construction. For numerator $\frac{u}{v}$, suppose that $v = w^2$ for some $w \in \mathbb{N}$, $w^2 < u$. Also suppose that there exists $m_1 > m_2 > 0$ such that $u = 2^n(m_1 m_2)$ and $w \mid 2^k m_1 - 2^{n-k} m_2$ for some $0 \leqslant k \leqslant n$. Let $a_0 = \frac{2^k m_1 - 2^{n-k} m_2}{w}$ and $x = x_0 = \frac{2^k m_1}{w}$. For the next step,

$$x_1 = \frac{\frac{u}{w^2}}{x_0 - a_0} = \frac{\frac{2^n m_1 m_2}{w^2}}{\frac{2^k m_1}{w} - \frac{2^k m_1 - 2^{n-k} m_2}{w}} = \frac{\frac{2^n m_1 m_2}{w^2}}{\frac{2^{n-k} m_2}{w}} = \frac{2^k m_1}{w} = x_0.$$

Thus, we can construct $x = [\overline{a_0}]_{u/v}$, a purely periodic continued fraction with period length 1. □

Recall the discussion of numerator $\frac{7}{4}$, and apply our new result.

Example. Let $r = \frac{7}{4}$. By Theorem 3.6, $a_0 = \frac{(7-1)}{2} = 3$ and $x = \frac{3}{2} + \frac{(7+1)}{4} = \frac{7}{2}$. Indeed, we confirm with direct computation that $\frac{7}{2} = \left[\overline{3}\right]_r$. To go further, use Theorem 2.8 to find examples such as

$$\frac{1}{2} = \left[0, \overline{3}\right]_r, \qquad \frac{11}{2} = \left[5, \overline{3}\right]_r, \qquad \frac{13}{10} = \left[1, 5, 2, 17, \overline{3}\right]_r,$$

and so on.

4 Other Results and Conjectures

For continued fractions with integer numerators, certain facts can be shown such as the existence of arbitrarily long terminating expansions and infinitely many distinct periodic expansions [1, Theorem 1.11 on p.2446]. This relies on several expansion techniques [1, Lemma 1.10 on p.2446], such as for $N \geqslant 2$ and $k \geqslant 0$,

$$N = \left[\overline{(N-1)}_k, N\right]_N$$

and

$$N = \left[\overline{(N-1)}_\infty\right]_N.$$

The hope is to make an analogue for expansions with rational numerators. However, these prove to be slightly more complicated, especially considering the variety of algorithms that could be used. Several specific cases have been discovered, and left as propositions.

Proposition 4.1. *For $m, n, u, v \in \mathbb{N}$ and the max algorithm,*

(i) $\frac{u}{v} = \frac{mnv+n}{v} = [mn, mv+1]_{u/v}$, *where $v > n$.*

(ii) $\frac{u}{v} = \frac{2m-1}{m} = [1, 2, 2m-3, 2m-1]_{u/v}$, *where $m > 2$.*

(iii) $\frac{u}{v} = \frac{2m-2}{m} = [1, 2, m-3, m-1]_{u/v}$, *where $m > 4$.*

(iv) $\frac{u}{v} = \frac{6m-1}{2m} = [2, 3, 3m-2, 12m-2]_{u/v}$, *where $m > 1$.*

Proof. The construction of (i) follows

$$a_0 = \lfloor \frac{mnv+n}{v} \rfloor = \lfloor mn + \frac{n}{v} \rfloor = mn \qquad x_1 = \frac{(mnv+n)/v}{n/v} = mv+1.$$

The construction of (ii) follows

$$a_0 = \lfloor \frac{2m-1}{m} \rfloor = \lfloor 2 - \frac{1}{m} \rfloor = 1 \qquad x_1 = \frac{(2m-1)/m}{(m-1)/m} = \frac{2m-1}{m-1}$$

$$a_1 = \lfloor \frac{2m-1}{m-1} \rfloor = 2 \qquad x_2 = \frac{(2m-1)/m}{1/(m-1)} = \frac{2m^2-3m+1}{m}$$

$$a_2 = \lfloor \frac{2m^2-3m+1}{m} \rfloor = \lfloor 2m-3+\frac{1}{m} \rfloor = 2m-3 \qquad x_3 = \frac{(2m-1)/m}{1/m} = 2m-1.$$

The construction of (iii) follows

$$a_0 = \lfloor \frac{2m-2}{m} \rfloor = \lfloor 2 - \frac{2}{m} \rfloor = 1 \qquad x_1 = \frac{(2m-2)/m}{(m-2)/m} = \frac{2m-2}{m-2}$$

$$a_1 = \lfloor \frac{2m-2}{m-2} \rfloor = 2 \qquad x_2 = \frac{(2m-2)/m}{2/(m-2)} = \frac{2m^2-6m+4}{2m}$$

$$a_2 = \lfloor \frac{2m^2-6m+4}{2m} \rfloor = \lfloor m-3+\frac{2}{m} \rfloor = m-3 \qquad x_3 = \frac{(2m-2)/m}{4/(2m)} = m-1.$$

The construction of (iv) follows

$$a_0 = \lfloor \frac{6m-1}{2m} \rfloor = \lfloor 3 - \frac{1}{2m} \rfloor = 2 \qquad x_1 = \frac{(6m-1)/(2m)}{(2m-1)/(2m)} = \frac{6m-1}{2m-1}$$
$$a_1 = \lfloor \frac{6m-1}{2m-1} \rfloor = 3 \qquad x_2 = \frac{(6m-1)/(2m)}{2/(2m-1)} = \frac{12m^2-8m+1}{4m}$$
$$a_2 = \lfloor \frac{12m^2-8m+1}{4m} \rfloor = \lfloor 3m - 2 + \frac{1}{4m} \rfloor = 3m-2 \quad x_3 = \frac{(6m-1)/(2m)}{1/(4m)} = 12m-2.$$

□

Proposition 4.2. *For any $u, v \in \mathbb{N}$ and the min algorithm,*

$$\frac{u}{v} = [0, 1]_{u/v}\,.$$

Proof. Select $a_0 = \lfloor \frac{u}{v} \rfloor - \lfloor \frac{u}{v} \rfloor = 0$. Then, $x_1 = \frac{u/v}{u/v} = 1$. □

For conjectures about the behavior of continued fraction expansions with rational numerators, there are a number of interesting properties that show up that could not be proved at the time of this writing. In particular, these conjectures were made while collecting data over a large sample of expansions, and are mentioned in Appendix A.1. Following that, more conjectures could be made about the data in Appendix A.3.

5 Future Work

From the onset, there were several inquiries regarding continued fractions, and more have spawned along the way. One of the desires for this work was to find a method for proving that any expansion of a rational number terminates for a certain numerator (such as $\frac{3}{2}$). This is one of the most basic properties of simple continued fractions and appears to carry over to some cases here. See A.1 for lists of numerators that appear to have this property with the max and min algorithms. We have discovered some that work for both algorithms, which includes

$$\frac{3}{2},\ \frac{4}{3},\ \frac{7}{2},\ \frac{9}{2},\ \frac{10}{3},\ \text{and}\ \frac{14}{3}.$$

The most that can be said up to this point is that none of these will produce continued fraction expansions that are periodic with odd period length. Future work could perhaps show that every rational number will indeed terminate with these numerators. This could involve a method of showing a bound on the algorithm remainders or showing a strictly decreasing sequence.

Furthermore, it is within the realm of possibility that the list of numerators that appear to terminate all rational numbers could be added to by computing further. That is, over the set of rational numbers, calculate the expansions to a greater length than is done here. Perhaps this could show that some expansions with unknown behavior actually do terminate.

Another possibility for future work is to look into other algorithms for computing continued fraction expansions. The main focus in this work was to apply either the max or min algorithm, yet there are numerous other possibilities. For example, one could compute continued fractions by alternating each step between the min and max algorithms. This could produce patterns that were not seen here. There is also the possibility that every rational expansion could terminate if the correct algorithm is chosen. One could intelligently choose the max, min, or some other choice at each step. However, this is all speculation and requires investigation.

To conclude, there is certainly further work to be done on the subject of continued fractions. A number of interesting patterns and properties have been discovered for rational numerators, but questions remain. These discoveries could have impact on other fields of mathematics, and that is an intriguing prospect for the future.

References

[1] Anselm, M. and Weintraub, H. “A Generalization of Continued Fractions.” *Journal of Number Theory* 131 (2011): 2442-2460.

[2] Olds, C. D. *Continued Fractions*. New York: Random House, 1963.

A Data Collection

This section is included to look at some of the data and trends that have been found, with the hope of inspiring future work.

A.1 Terminating Rationals

As discussed, every simple continued fraction expansion of a rational number terminates. We have shown that this is not the case for expansions with rational numerators. However, there appears to be some numerators for which all rational expansions terminate, based on having large samples that do just that. Here we list all numerators $\frac{u}{v}$ $(u, v < 100)$ for which all expansions of $\frac{x}{y}$ $(x, y < 500)$ terminate in less than 100 steps with the

(i) max algorithm:

$\frac{3}{2}, \frac{4}{3}, \frac{5}{2}, \frac{7}{2}, \frac{8}{3}, \frac{8}{5}, \frac{9}{2}, \frac{10}{3}, \frac{10}{7}, \frac{11}{2}, \frac{11}{3}, \frac{12}{5}, \frac{12}{7}, \frac{12}{11}, \frac{14}{3}, \frac{14}{5}, \frac{15}{2}, \frac{16}{3}, \frac{17}{2}, \frac{18}{5}, \frac{18}{7}, \frac{18}{11}, \frac{19}{2}, \frac{20}{3}, \frac{20}{7}, \frac{20}{11}, \frac{21}{2}, \frac{22}{3}, \frac{23}{2}, \frac{23}{3}, \frac{24}{5}, \frac{24}{7}, \frac{24}{11}, \frac{24}{13}, \frac{26}{3}, \frac{26}{5}, \frac{27}{2}, \frac{27}{5}, \frac{28}{3}, \frac{29}{2}, \frac{29}{3}, \frac{30}{11}, \frac{30}{13}, \frac{33}{2}, \frac{33}{5}, \frac{33}{7}, \frac{34}{3}, \frac{34}{5}, \frac{34}{7}, \frac{35}{2}, \frac{35}{3}, \frac{36}{5}, \frac{36}{11}, \frac{36}{13}, \frac{38}{3}, \frac{38}{5}, \frac{39}{2}, \frac{39}{5}, \frac{39}{7}, \frac{40}{3}, \frac{40}{7}, \frac{40}{17}, \frac{42}{11}, \frac{43}{2}, \frac{44}{3}, \frac{44}{5}, \frac{45}{2}, \frac{45}{13}, \frac{47}{2}, \frac{47}{3}, \frac{48}{5}, \frac{48}{7}, \frac{48}{11}, \frac{48}{13}, \frac{48}{17}, \frac{48}{19}, \frac{50}{7}, \frac{51}{2}, \frac{52}{3}, \frac{52}{5}, \frac{52}{7}, \frac{52}{11}, \frac{53}{2}, \frac{53}{3}, \frac{54}{5}, \frac{54}{7}, \frac{54}{11}, \frac{54}{17}, \frac{55}{2}, \frac{55}{7}, \frac{56}{3}, \frac{56}{5}, \frac{57}{5}, \frac{58}{3}, \frac{60}{7}, \frac{60}{11}, \frac{60}{13}, \frac{60}{17}, \frac{60}{19}, \frac{60}{31}, \frac{62}{7}, \frac{63}{2}, \frac{63}{13}, \frac{64}{5}, \frac{64}{13}, \frac{65}{2}, \frac{65}{3}, \frac{65}{7}, \frac{66}{5}, \frac{66}{7}, \frac{66}{17}, \frac{67}{2}, \frac{68}{3}, \frac{68}{5}, \frac{68}{7}, \frac{69}{2}, \frac{69}{5}, \frac{70}{3}, \frac{70}{13}, \frac{72}{5}, \frac{72}{11}, \frac{72}{13}, \frac{72}{17}, \frac{74}{5}, \frac{75}{2}, \frac{75}{7}, \frac{76}{7}, \frac{76}{11}, \frac{77}{2}, \frac{77}{3}, \frac{78}{5}, \frac{78}{7}, \frac{80}{3}, \frac{80}{7}, \frac{80}{23}, \frac{82}{3}, \frac{84}{5}, \frac{84}{11}, \frac{84}{13}, \frac{84}{17}, \frac{84}{19}, \frac{84}{23}, \frac{86}{3}, \frac{86}{5}, \frac{87}{2}, \frac{87}{5}, \frac{88}{3}, \frac{88}{5}, \frac{90}{7}, \frac{90}{13}, \frac{90}{17}, \frac{90}{29}, \frac{90}{31}, \frac{92}{3}, \frac{93}{2}, \frac{94}{3}, \frac{95}{2}, \frac{95}{3}, \frac{95}{7}, \frac{96}{5}, \frac{96}{7}, \frac{96}{13}, \frac{96}{17}, \frac{98}{5}, \frac{99}{5}, \frac{99}{13}.$

(ii) min algorithm:

$\frac{3}{2}, \frac{4}{3}, \frac{7}{2}, \frac{9}{2}, \frac{10}{3}, \frac{14}{3}.$

Next, we examine several particular numerators listed above to see patterns in the lengths of the continued fraction expansions. For numerators that appear to make all rational numbers terminate, there seems to be several patterns.

1. The max and min algorithms produce different behavior in regard to frequency of expansion length.
2. Expansions of odd length are relatively more common than even lengths. The curves of odd and even lengths appear to differ by a factor of approximately v (where $\frac{u}{v}$ is the numerator that was used).
3. The min algorithm appears to produce expansion lengths of larger mean and larger standard deviation.

Examples of these patterns are shown in the following figures.

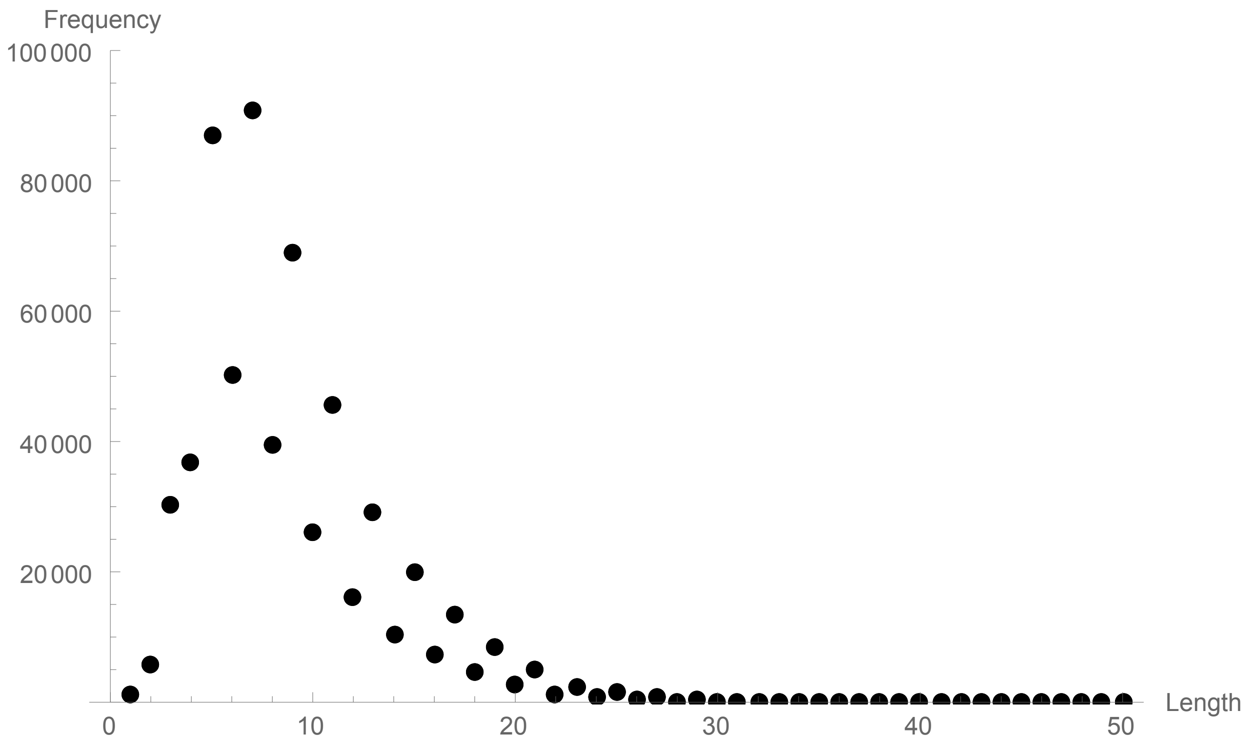

Figure 1: Terminating lengths of $\frac{x}{y}$ $(x, y < 1000)$ with numerator $\frac{3}{2}$ and the max algorithm.

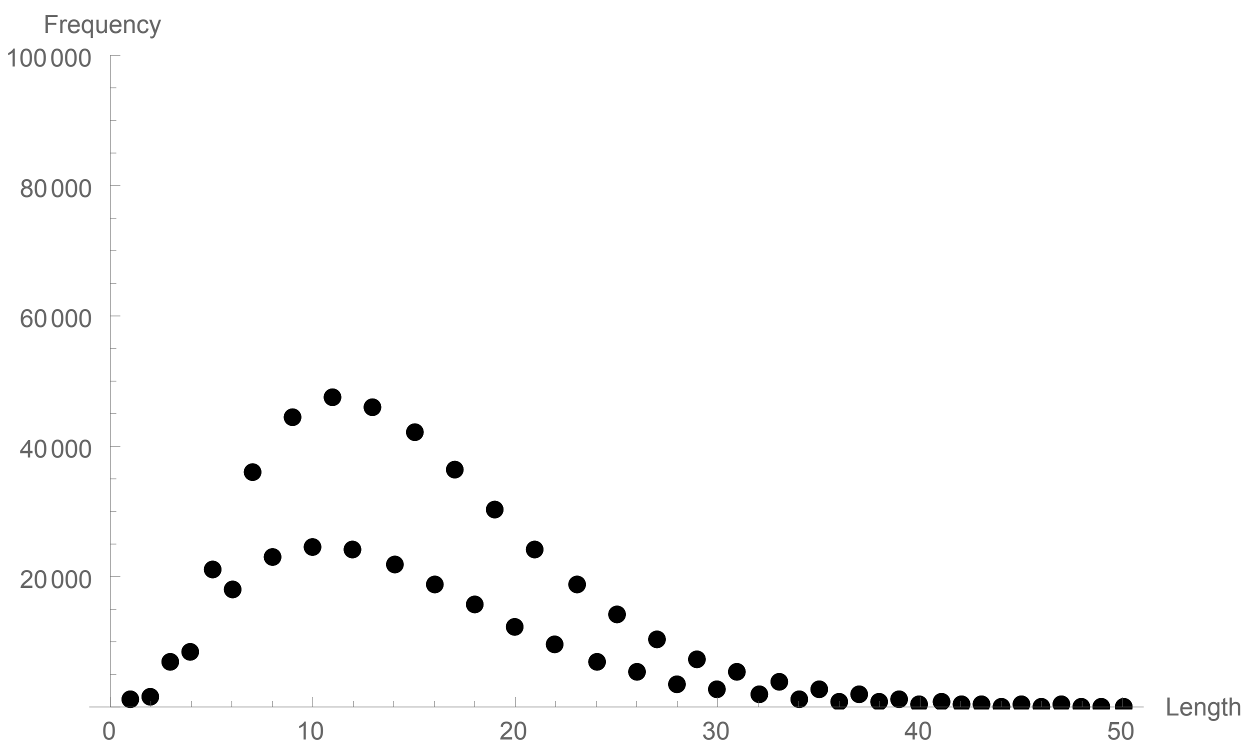

Figure 2: Terminating lengths of $\frac{x}{y}$ $(x, y < 1000)$ with numerator $\frac{3}{2}$ and the min algorithm.

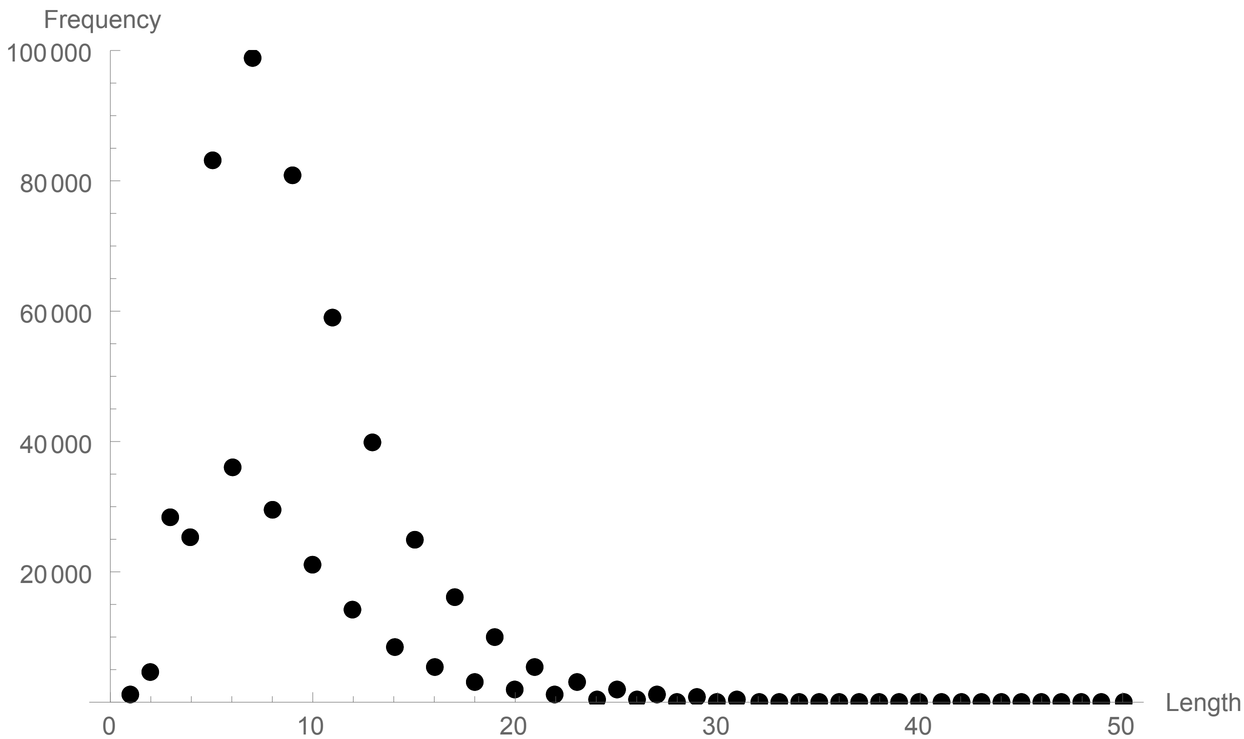

Figure 3: Terminating lengths of $\frac{x}{y}$ $(x, y < 1000)$ with numerator $\frac{4}{3}$ and the max algorithm.

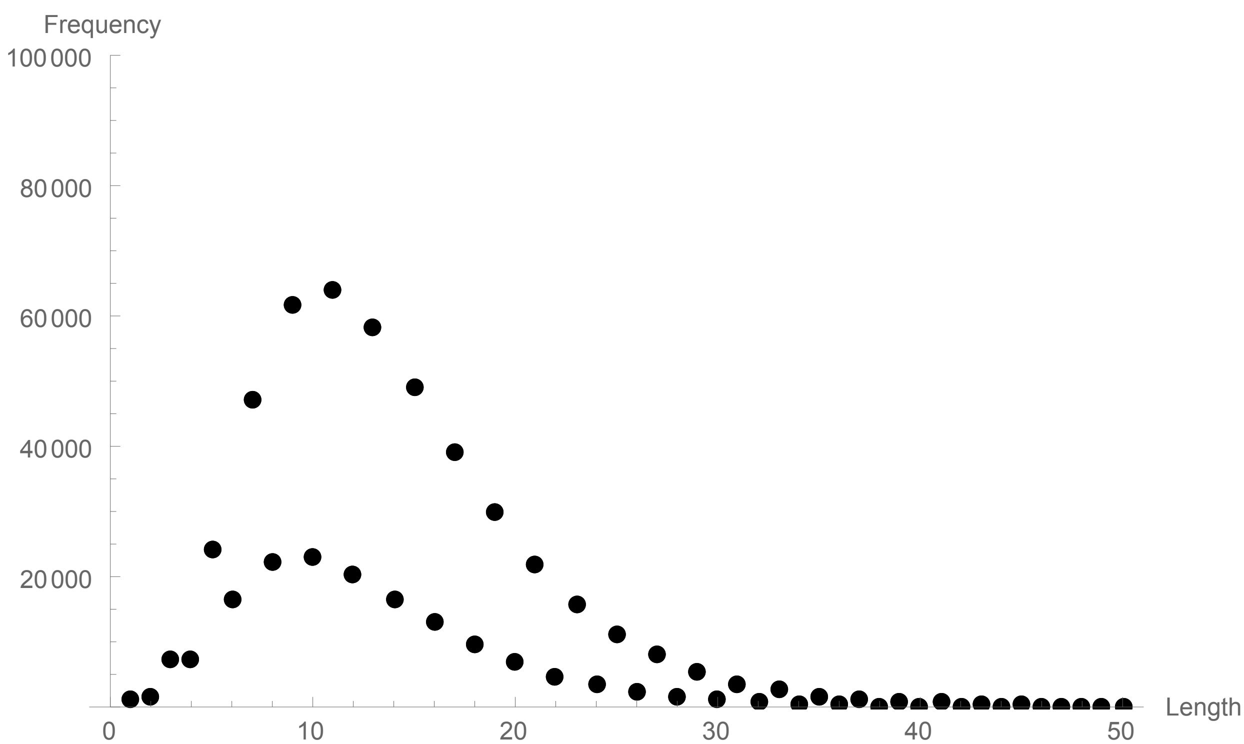

Figure 4: Terminating lengths of $\frac{x}{y}$ $(x, y < 1000)$ with numerator $\frac{4}{3}$ and the min algorithm.

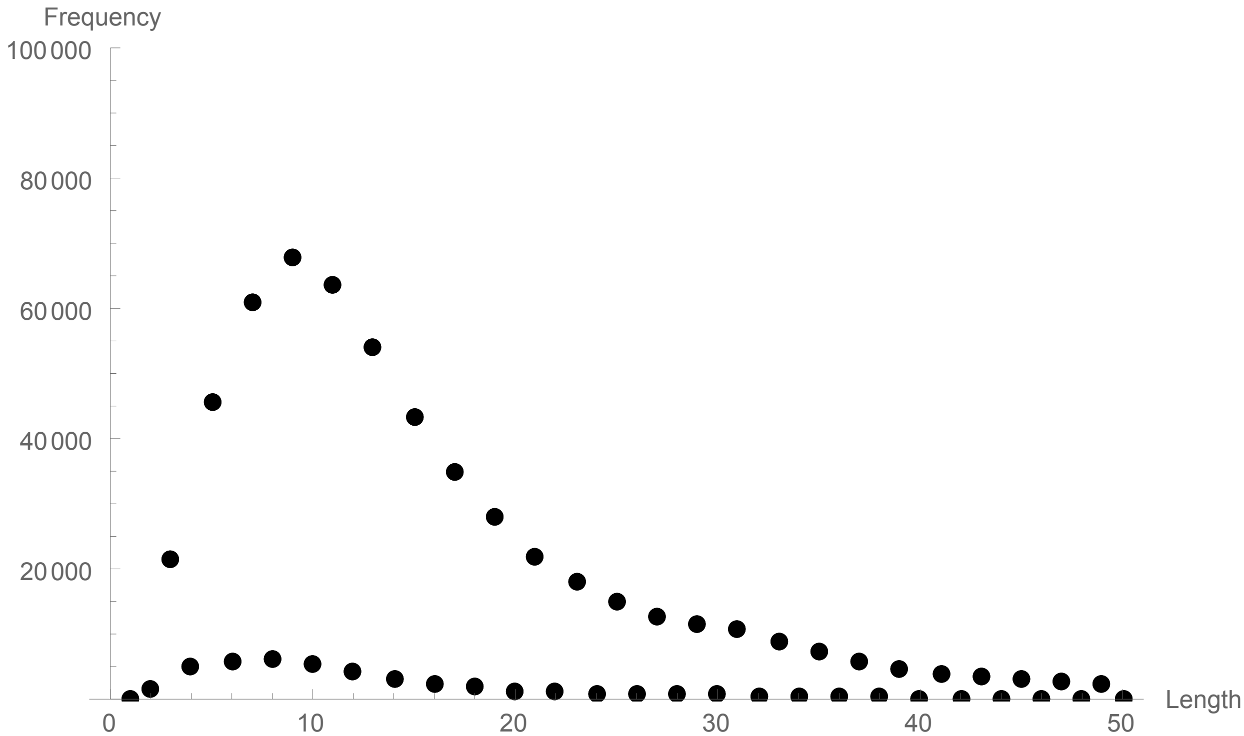

Figure 5: Terminating lengths of $\frac{x}{y}$ $(x, y < 1000)$ with numerator $\frac{99}{13}$ and the max algorithm.

A.2 Periodic Rationals

Unlike simple continued fractions, those with rational numerators may be periodic. For numerators that can be used to create periodic expansions, there are two data sets of interest. First is the frequency of lengths for those expansions that do terminate. Second is the frequency of tail lengths for the expansions that become periodic. Several examples of these are shown below.

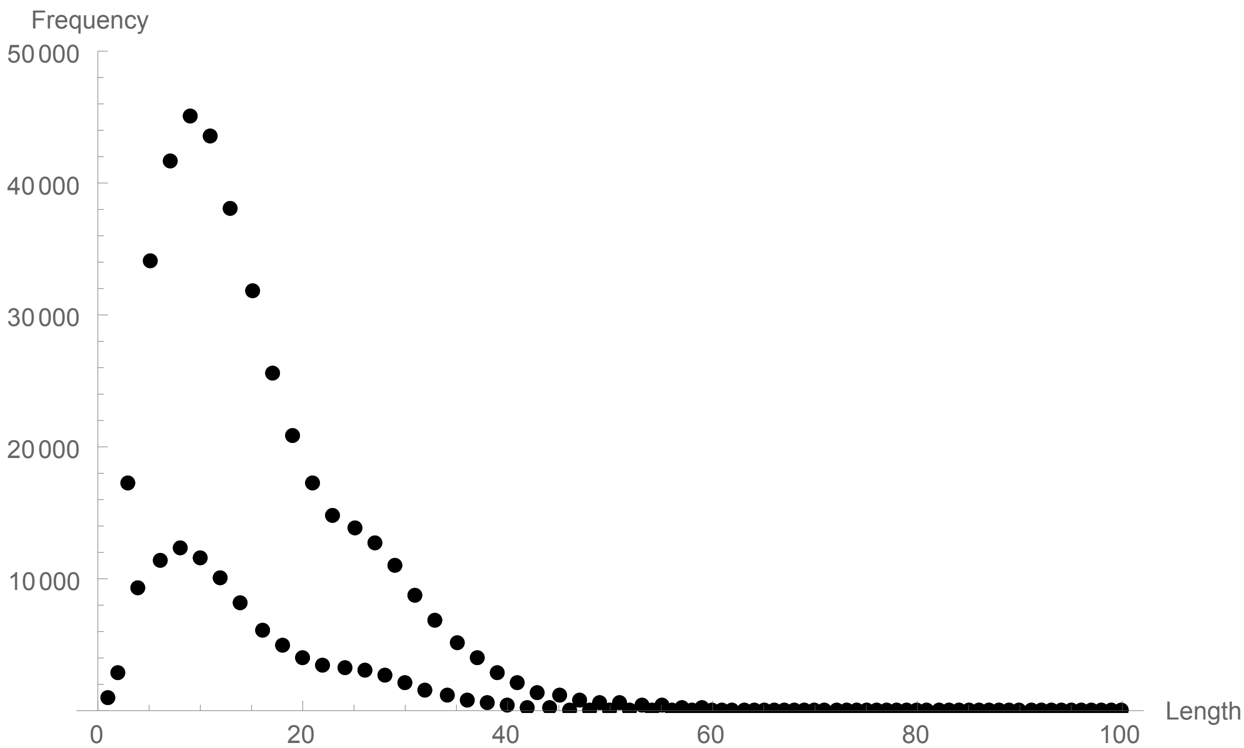

Figure 6: Terminating lengths of $\frac{x}{y}$ $(x, y < 1000)$ with numerator $\frac{7}{4}$ and the max algorithm.

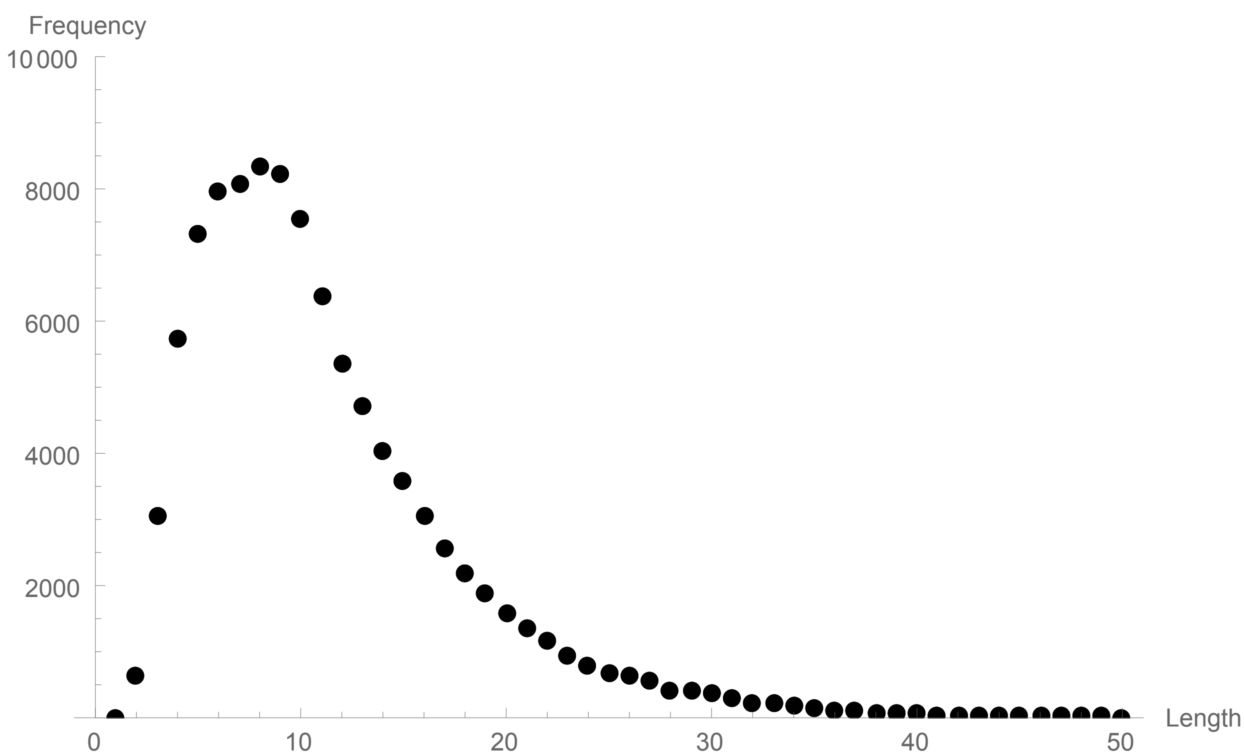

Figure 7: Tail lengths of $\frac{x}{y}$ $(x, y < 1000)$ with numerator $\frac{7}{4}$ and the max algorithm.

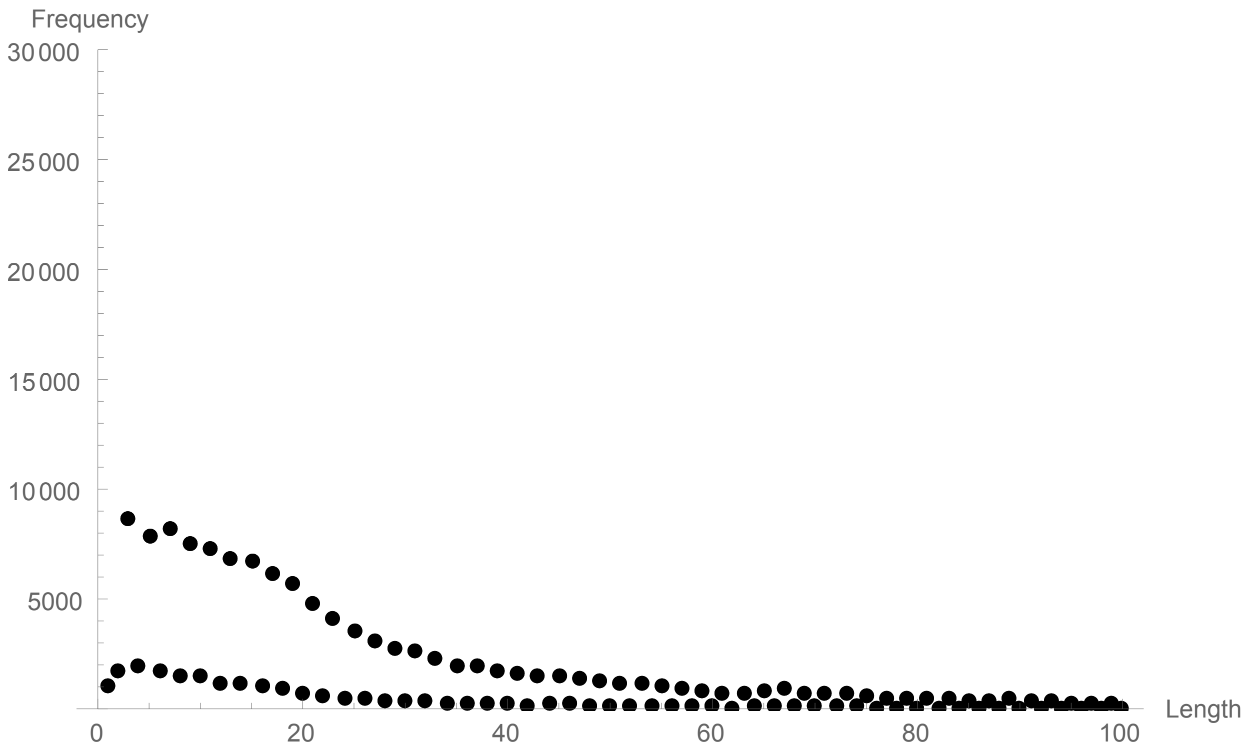

Figure 8: Terminating lengths of $\frac{x}{y}$ $(x, y < 1000)$ with numerator $\frac{19}{6}$ and the max algorithm.

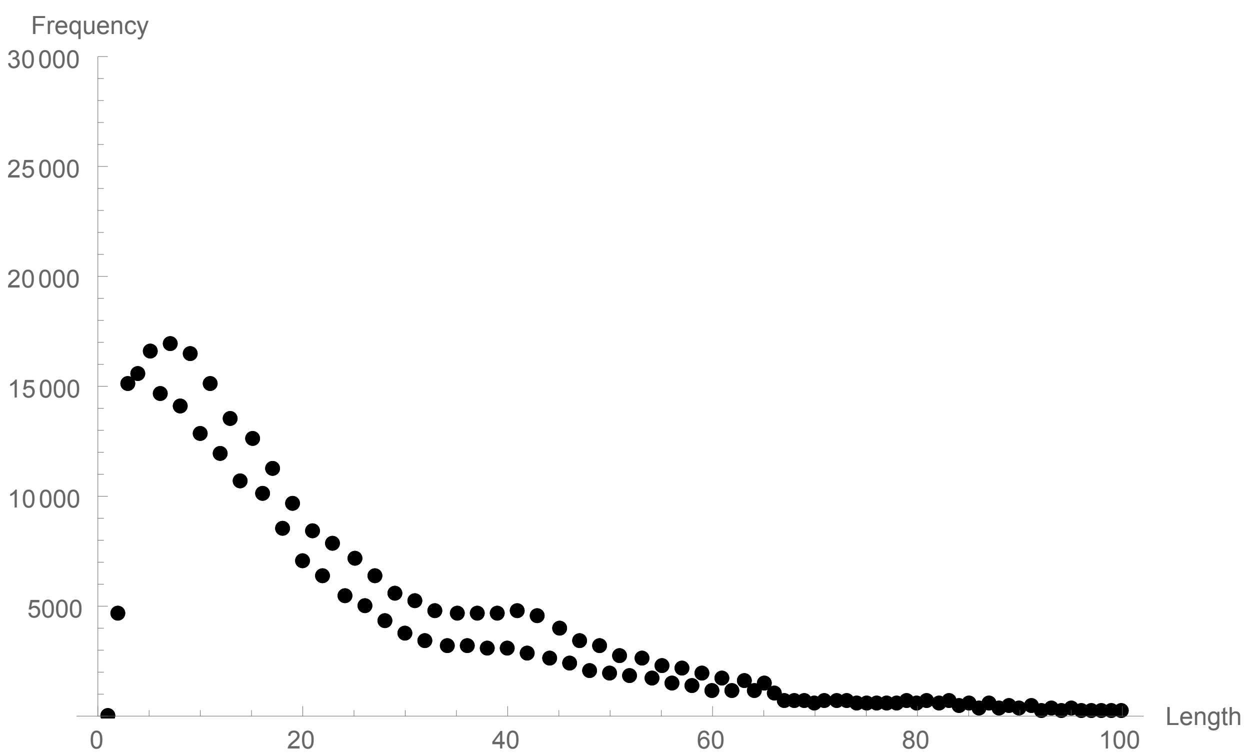

Figure 9: Tail lengths of $\frac{x}{y}$ $(x, y < 1000)$ with numerator $\frac{19}{6}$ and the max algorithm.

A.3 Continued Fraction Rational Numerators

Below is the complete dataset that was collected for a sample of continued fraction numerators. The collected data includes, for each numerator, the portion of expansions that terminate, the portion of expansions that are eventually periodic, the period lengths that were discovered, and the portion of expansions that neither terminated nor became periodic.

Table 2: Numerators $\frac{u}{v}$ $(u, v < 25)$ and the corresponding percentage of expansions that are terminating, periodic, or unknown over a sample of all $\frac{x}{y}$ $(x, y < 500$, 151831 total) in less than 200 steps with the max algorithm.

$\frac{u}{v}$	Terminating %	Periodic %	Period Lengths	Unknown %
$\frac{3}{2}$	100.0	0		0
$\frac{4}{3}$	100.0	0		0
$\frac{5}{2}$	100.0	0		0
$\frac{5}{3}$	92.71	7.290	12	0
$\frac{5}{4}$	39.01	60.95	$1, 14, 16$	0.04018
$\frac{6}{5}$	99.31	0.6935	$10, 16$	0
$\frac{7}{2}$	100.0	0		0
$\frac{7}{3}$	50.23	49.77	$2, 6$	0
$\frac{7}{4}$	83.33	16.67	$1, 3, 5, 28$	0
$\frac{7}{5}$	65.29	34.23	$8, 16, 24$	0.4782
$\frac{7}{6}$	29.32	64.73	$2, 4, 6$	5.950
$\frac{8}{3}$	100.0	0		0
$\frac{8}{5}$	100.0	0		0
$\frac{8}{7}$	50.31	49.68	8	0.006586
$\frac{9}{2}$	100.0	0		0
$\frac{9}{4}$	83.33	16.67	1	0
$\frac{9}{5}$	99.94	0.06455	38	0
$\frac{9}{7}$	85.78	14.12	8	0.09748
$\frac{9}{8}$	37.86	61.03	$2, 18$	1.112
$\frac{10}{3}$	100.0	0		0
$\frac{10}{7}$	100.0	0		0
$\frac{10}{9}$	83.21	16.61	$1, 6, 26$	0.1798
$\frac{11}{2}$	100.0	0		0
$\frac{11}{3}$	100.0	0		0
$\frac{11}{4}$	83.14	16.86	$1, 4, 12$	0.003293

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Table 2 – *Continued from previous page*

$\frac{u}{v}$	Terminating %	Periodic %	Period Lengths	Unknown %
$\frac{11}{5}$	24.70	74.56	2, 4, 6, 8, 24, 26	0.7396
$\frac{11}{6}$	58.51	41.30	4, 8	0.1884
$\frac{11}{7}$	43.65	37.49	6, 18	18.86
$\frac{11}{8}$	20.40	33.75	4, 6, 8, 32, 70	45.85
$\frac{11}{9}$	39.55	0.1917	22	60.26
$\frac{11}{10}$	8.125	41.16	2, 4, 6, 8, 12, 14, 18, 20, 24, 42, 44, 48, 52, 58	50.72
$\frac{12}{5}$	100.0	0		0
$\frac{12}{7}$	100.0	0		0
$\frac{12}{11}$	100.0	0		0
$\frac{13}{2}$	85.06	14.94	2	0
$\frac{13}{3}$	50.23	49.77	2	0
$\frac{13}{4}$	41.71	58.26	1, 2, 14	0.03293
$\frac{13}{5}$	90.84	9.119	8, 10	0.04347
$\frac{13}{6}$	27.80	68.57	2, 6, 12, 16, 24	3.637
$\frac{13}{7}$	33.30	63.53	4	3.171
$\frac{13}{8}$	30.61	18.12	2, 8, 24, 34, 40, 72	51.27
$\frac{13}{9}$	20.10	13.90	1, 2, 3, 12	66.00
$\frac{13}{10}$	12.67	17.87	8	69.45
$\frac{13}{11}$	24.23	0		75.77
$\frac{13}{12}$	5.695	17.64	2, 4, 6, 10, 12, 14	76.66
$\frac{14}{3}$	100.0	0		0
$\frac{14}{5}$	100.0	0		0
$\frac{14}{9}$	68.91	30.51	2, 8, 10, 14, 24	0.5835
$\frac{14}{11}$	64.20	0		35.80
$\frac{14}{13}$	34.96	24.14	12, 36	40.90
$\frac{15}{2}$	100.0	0		0
$\frac{15}{4}$	83.33	16.67	1	0
$\frac{15}{7}$	64.12	35.88	2	0
$\frac{15}{8}$	75.07	24.93	2	0
$\frac{15}{11}$	47.50	42.50	8, 10	10.00
$\frac{15}{13}$	78.54	14.06	8	7.400
$\frac{15}{14}$	58.72	35.26	2, 10, 60	6.021
$\frac{16}{3}$	100.0	0		0
$\frac{16}{5}$	83.01	16.99	2	0

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Table 2 – *Continued from previous page*

$\frac{u}{v}$	Terminating %	Periodic %	Period Lengths	Unknown %
$\frac{16}{7}$	50.31	49.69	4	0
$\frac{16}{9}$	78.34	21.65	1, 4	0.0006586
$\frac{16}{11}$	98.60	0		1.405
$\frac{16}{13}$	91.25	0.8463	14	7.900
$\frac{16}{15}$	29.74	42.70	2, 8, 10, 16, 20	27.55
$\frac{17}{2}$	100.0	0		0
$\frac{17}{3}$	99.96	0.04413	8	0
$\frac{17}{4}$	41.71	58.29	1, 4, 5, 6, 24	0
$\frac{17}{5}$	99.98	0		0.01515
$\frac{17}{6}$	58.49	41.21	4	0.3003
$\frac{17}{7}$	52.70	28.01	100	19.29
$\frac{17}{8}$	15.26	44.71	2, 4, 6, 10, 12, 16, 18, 22	40.03
$\frac{17}{9}$	24.98	61.76	4, 8, 16, 18, 30, 44	13.26
$\frac{17}{10}$	28.56	19.71	8, 16, 26, 34	51.73
$\frac{17}{11}$	17.32	0		82.68
$\frac{17}{12}$	7.827	7.363	4, 74	84.81
$\frac{17}{13}$	4.410	0.3030	12	95.29
$\frac{17}{14}$	6.478	0		93.52
$\frac{17}{15}$	6.075	1.037	8	92.89
$\frac{17}{16}$	2.608	4.564	1, 2, 6, 10, 12, 14, 48	92.83
$\frac{18}{5}$	100.0	0		0
$\frac{18}{7}$	100.0	0		0
$\frac{18}{11}$	100.0	0		0
$\frac{18}{13}$	99.43	0.007245	8	0.5579
$\frac{18}{17}$	87.47	9.210	70	3.323
$\frac{19}{2}$	100.0	0		0
$\frac{19}{3}$	50.23	49.77	2, 4, 6	0
$\frac{19}{4}$	83.27	16.73	1, 5, 8, 16	0
$\frac{19}{5}$	50.05	49.72	4	0.2318
$\frac{19}{6}$	23.82	69.68	2, 4, 6, 8, 12, 16, 66	6.497
$\frac{19}{7}$	90.52	0.0006586	12	9.476
$\frac{19}{8}$	25.53	26.62	4, 12, 24	47.85
$\frac{19}{9}$	9.770	20.34	1, 2, 4, 6, 8, 10, 12, 16, 22, 27, 32, 54, 76	69.89
$\frac{19}{10}$	18.57	55.70	4, 8, 10, 20	25.73

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Table 2 – *Continued from previous page*

$\frac{u}{v}$	Terminating %	Periodic %	Period Lengths	Unknown %
$\frac{19}{11}$	23.32	0		76.68
$\frac{19}{12}$	6.739	11.64	$2, 4, 48$	81.62
$\frac{19}{13}$	11.06	0		88.94
$\frac{19}{14}$	8.810	5.623	12	85.57
$\frac{19}{15}$	3.157	4.303	$4, 8$	92.54
$\frac{19}{16}$	2.323	4.417	$4, 8$	93.26
$\frac{19}{17}$	3.131	0		96.87
$\frac{19}{18}$	2.276	4.074	$2, 4, 6, 8, 10, 18$	93.65
$\frac{20}{3}$	100.0	0		0
$\frac{20}{7}$	100.0	0		0
$\frac{20}{9}$	83.38	16.62	$2, 6$	0
$\frac{20}{11}$	100.0	0		0
$\frac{20}{13}$	93.08	6.350	$10, 16, 38$	0.5723
$\frac{20}{17}$	85.65	0.02305	8	14.32
$\frac{20}{19}$	69.39	26.26	12	4.355
$\frac{21}{2}$	100.0	0		0
$\frac{21}{4}$	83.33	16.67	1	0
$\frac{21}{5}$	67.69	32.31	2	0
$\frac{21}{8}$	75.03	24.97	$2, 16$	0
$\frac{21}{10}$	59.67	40.29	$2, 26$	0.04215
$\frac{21}{11}$	99.99	0.01054	8	0.0006586
$\frac{21}{13}$	66.74	14.41	$16, 24$	18.85
$\frac{21}{16}$	16.78	16.80	$1, 3, 4, 9, 16, 36, 40$	66.42
$\frac{21}{17}$	29.51	0.1910	10	70.30
$\frac{21}{19}$	22.21	25.86	$16, 48$	51.93
$\frac{21}{20}$	18.95	34.88	$2, 6, 10, 12, 14, 26, 44, 48$	46.17
$\frac{22}{3}$	100.0	0		0
$\frac{22}{5}$	76.31	23.69	$2, 6$	0
$\frac{22}{7}$	50.31	49.69	$2, 4, 10$	0.003293
$\frac{22}{9}$	83.34	16.23	$1, 9$	0.4307
$\frac{22}{13}$	92.78	0.4847	$10, 22$	6.735
$\frac{22}{15}$	16.62	16.42	$4, 8, 12, 50$	66.95
$\frac{22}{17}$	12.18	0.3478	$4, 20$	87.47
$\frac{22}{19}$	21.16	0		78.84

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$\frac{u}{v}$	Terminating %	Periodic %	Period Lengths	Unknown %
$\frac{22}{21}$	8.895	14.76	$2, 10, 14$	76.35
$\frac{23}{2}$	100.0	0		0
$\frac{23}{3}$	100.0	0		0
$\frac{23}{4}$	57.76	42.24	$1, 4$	0.001317
$\frac{23}{5}$	99.34	0.6408	$8, 34$	0.01844
$\frac{23}{6}$	37.07	59.72	$4, 8, 32$	3.213
$\frac{23}{7}$	20.71	44.68	$6, 12$	34.62
$\frac{23}{8}$	32.69	53.53	$4, 8, 12, 32$	13.78
$\frac{23}{9}$	40.22	8.751	$4, 24$	51.02
$\frac{23}{10}$	15.36	10.45	$8, 10, 16, 40$	74.18
$\frac{23}{11}$	4.785	12.89	$2, 4, 6, 12, 24$	82.33
$\frac{23}{12}$	12.22	25.21	$4, 12, 20$	62.57
$\frac{23}{13}$	11.07	0		88.93
$\frac{23}{14}$	5.895	7.275	$6, 12, 18, 24, 26$	86.83
$\frac{23}{15}$	6.782	0.9912	$8, 16$	92.23
$\frac{23}{16}$	3.731	2.521	$4, 20$	93.75
$\frac{23}{17}$	4.071	0		95.93
$\frac{23}{18}$	3.486	0.001317	6	96.51
$\frac{23}{19}$	2.940	0		97.06
$\frac{23}{20}$	2.551	0.02635	8	97.42
$\frac{23}{21}$	2.432	0		97.57
$\frac{23}{22}$	1.619	1.855	$2, 6, 8, 20$	96.53
$\frac{24}{5}$	100.0	0		0
$\frac{24}{7}$	100.0	0		0
$\frac{24}{11}$	100.0	0		0
$\frac{24}{13}$	100.0	0		0
$\frac{24}{17}$	98.90	0.04479	16	1.056
$\frac{24}{19}$	83.93	11.05	$4, 26$	5.024
$\frac{24}{23}$	49.12	44.22	$14, 86$	6.656

Table 3: Numerators $\frac{u}{v}$ $(u, v < 25)$ and the corresponding percentage of expansions that are terminating, periodic, or unknown over a sample of all $\frac{x}{y}$ $(x, y < 500$, 151831 total) in less than 200 steps with the min algorithm.

$\frac{u}{v}$	Terminating %	Periodic %	Period Lengths	Unknown %
$\frac{3}{2}$	100.0	0		0
$\frac{4}{3}$	100.0	0		0
$\frac{5}{2}$	100.0	0.001976	8	0
$\frac{5}{3}$	96.81	0		3.190
$\frac{5}{4}$	41.69	56.72	1, 14, 16, 28, 34	1.589
$\frac{6}{5}$	99.81	0.1917	10, 16	0
$\frac{7}{2}$	100.0	0		0
$\frac{7}{3}$	41.57	45.05	4, 6, 12, 18, 48, 50, 66, 74	13.37
$\frac{7}{4}$	22.56	5.899	2, 20	71.54
$\frac{7}{5}$	38.30	15.04	10	46.66
$\frac{7}{6}$	22.31	48.33	2, 4, 6, 10, 16, 18, 22	29.36
$\frac{8}{3}$	94.87	5.116	4, 24	0.01054
$\frac{8}{5}$	77.66	0		22.34
$\frac{8}{7}$	49.67	50.28	8, 12, 22, 36	0.04545
$\frac{9}{2}$	100.0	0		0
$\frac{9}{4}$	68.73	12.56	2, 10, 80, 96	18.71
$\frac{9}{5}$	13.73	8.787	10	77.49
$\frac{9}{7}$	57.51	16.43	8, 14, 16, 52	26.06
$\frac{9}{8}$	37.17	54.06	2, 12, 18, 20, 86	8.770
$\frac{10}{3}$	100.0	0		0
$\frac{10}{7}$	66.02	0		33.98
$\frac{10}{9}$	83.24	16.49	1, 6, 10, 26	0.2687
$\frac{11}{2}$	99.92	0.07772	6	0
$\frac{11}{3}$	89.23	0.5539	12, 72, 90	10.22
$\frac{11}{4}$	13.42	12.59	2, 6, 8, 16	73.99
$\frac{11}{5}$	4.860	3.946	10, 20, 60	91.19
$\frac{11}{6}$	2.760	0.06520	12	97.17
$\frac{11}{7}$	2.999	0.6501	12	96.35
$\frac{11}{8}$	3.152	0.4057	6	96.44
$\frac{11}{9}$	4.900	0		95.10
$\frac{11}{10}$	3.514	6.750	2, 4, 6, 10, 32	89.74

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$\frac{u}{v}$	Terminating %	Periodic %	Period Lengths	Unknown %
$\frac{12}{5}$	99.98	0		0.02042
$\frac{12}{7}$	57.23	0.0006586	6	42.77
$\frac{12}{11}$	88.03	11.95	6, 30	0.02042
$\frac{13}{2}$	100.0	0		0.004610
$\frac{13}{3}$	42.97	35.70	4, 6, 12, 18, 22, 32, 90	21.33
$\frac{13}{4}$	13.58	10.57	4, 6, 8, 10, 12, 16, 24	75.85
$\frac{13}{5}$	6.137	1.929	10, 30	91.93
$\frac{13}{6}$	2.671	0.5038	4, 16	96.83
$\frac{13}{7}$	1.975	0.7798	4	97.25
$\frac{13}{8}$	1.705	1.060	2, 4	97.23
$\frac{13}{9}$	2.094	0.5025	2	97.40
$\frac{13}{10}$	2.071	0		97.93
$\frac{13}{11}$	4.952	0		95.05
$\frac{13}{12}$	2.320	6.129	2, 4, 6, 10, 14, 48	91.55
$\frac{14}{3}$	100.0	0		0
$\frac{14}{5}$	70.75	0.3484	6	28.91
$\frac{14}{9}$	11.35	3.019	2	85.63
$\frac{14}{11}$	11.27	14.98	6, 8	73.75
$\frac{14}{13}$	22.69	46.49	38, 40	30.82
$\frac{15}{2}$	99.80	0.1989	8	0
$\frac{15}{4}$	83.10	16.24	1, 2, 6, 8, 18	0.6639
$\frac{15}{7}$	15.75	0		84.25
$\frac{15}{8}$	10.35	4.606	2, 4, 8	85.04
$\frac{15}{11}$	10.12	0.001317	6	89.87
$\frac{15}{13}$	49.01	0		50.99
$\frac{15}{14}$	54.36	26.06	2, 10, 16	19.58
$\frac{16}{3}$	89.49	10.46	6, 8	0.05006
$\frac{16}{5}$	40.92	28.87	8, 12, 14, 20, 24	30.21
$\frac{16}{7}$	13.23	3.860	8, 42, 76	82.91
$\frac{16}{9}$	5.315	2.431	2, 4	92.25
$\frac{16}{11}$	6.212	0		93.79
$\frac{16}{13}$	15.53	0		84.47
$\frac{16}{15}$	20.57	37.44	2, 8, 10, 12, 16, 24, 30	41.99
$\frac{17}{2}$	95.40	4.490	10, 22	0.1074

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$\frac{u}{v}$	Terminating %	Periodic %	Period Lengths	Unknown %
$\frac{17}{3}$	75.12	0.7699	$4, 30$	24.11
$\frac{17}{4}$	13.04	8.002	$8, 12, 16, 24, 40$	78.95
$\frac{17}{5}$	6.819	1.775	10	91.41
$\frac{17}{6}$	3.084	0.1594	16	96.76
$\frac{17}{7}$	1.988	0		98.01
$\frac{17}{8}$	1.467	0.8220	$4, 16$	97.71
$\frac{17}{9}$	1.172	0		98.83
$\frac{17}{10}$	1.359	0		98.64
$\frac{17}{11}$	1.757	0		98.24
$\frac{17}{12}$	1.290	0		98.71
$\frac{17}{13}$	1.053	0		98.95
$\frac{17}{14}$	1.185	0		98.82
$\frac{17}{15}$	2.540	0.05071	$8, 10, 18$	97.41
$\frac{17}{16}$	1.449	3.462	$1, 2, 4, 6, 8, 10, 16, 26$	95.09
$\frac{19}{2}$	98.02	1.876	$8, 10$	0.1034
$\frac{19}{3}$	39.60	30.02	$8, 12, 24, 48, 90$	30.37
$\frac{19}{4}$	15.00	7.502	$2, 6, 8, 16, 20, 22$	77.50
$\frac{19}{5}$	4.585	3.570	$6, 8, 10, 20$	91.84
$\frac{19}{6}$	2.671	1.552	$10, 48$	95.78
$\frac{19}{7}$	2.469	0		97.53
$\frac{19}{8}$	1.769	0		98.23
$\frac{19}{9}$	1.208	0.3807	6	98.41
$\frac{19}{10}$	0.9623	0		99.04
$\frac{19}{11}$	1.242	0		98.76
$\frac{19}{12}$	1.081	0.4031	2	98.52
$\frac{19}{13}$	0.9458	0		99.05
$\frac{19}{14}$	1.249	0		98.75
$\frac{19}{15}$	1.133	0.0006586	10	98.87
$\frac{19}{16}$	1.465	0.9076	$4, 8$	97.63
$\frac{19}{17}$	1.668	0		98.33
$\frac{19}{18}$	1.189	1.664	$2, 6, 8, 10$	97.15
$\frac{20}{3}$	99.26	0.7344	$6, 10$	0.0006586
$\frac{20}{7}$	42.98	44.04	$8, 14, 16$	12.98
$\frac{20}{9}$	20.17	5.589	$2, 10$	74.24

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Table 3 – *Continued from previous page*

$\frac{u}{v}$	Terminating %	Periodic %	Period Lengths	Unknown %
$\frac{20}{11}$	8.207	0		91.79
$\frac{20}{13}$	9.316	0		90.68
$\frac{20}{17}$	41.65	0.3813	16	57.97
$\frac{20}{19}$	88.65	0.1515	18	11.20
$\frac{21}{2}$	99.54	0.4624	4	0
$\frac{21}{4}$	73.39	24.80	$2, 4, 8$	1.813
$\frac{21}{5}$	89.65	0		10.35
$\frac{21}{8}$	19.25	11.37	$2, 4, 8, 24, 32$	69.37
$\frac{21}{10}$	5.024	2.503	8	92.47
$\frac{21}{11}$	4.620	0		95.38
$\frac{21}{13}$	3.491	0		96.51
$\frac{21}{16}$	3.024	1.180	$1, 2$	95.80
$\frac{21}{17}$	4.926	0		95.07
$\frac{21}{19}$	26.23	0		73.77
$\frac{21}{20}$	12.75	19.39	$2, 6, 12, 20, 26$	67.85
$\frac{22}{3}$	99.84	0.1508	$6, 8$	0.005269
$\frac{22}{5}$	57.54	20.02	10	22.44
$\frac{22}{7}$	13.64	8.243	$10, 42$	78.12
$\frac{22}{9}$	9.221	1.394	$2, 12, 70$	89.38
$\frac{22}{13}$	1.995	0		98.01
$\frac{22}{15}$	2.024	1.687	8	96.29
$\frac{22}{17}$	2.970	0.7607	4	96.27
$\frac{22}{19}$	6.261	0		93.74
$\frac{22}{21}$	4.918	6.513	$2, 14$	88.57
$\frac{23}{2}$	99.73	0.1199	20	0.1495
$\frac{23}{3}$	80.09	0.2490	34	19.66
$\frac{23}{4}$	25.55	5.168	$2, 18, 20$	69.28
$\frac{23}{5}$	4.987	4.332	$8, 10, 24$	90.68
$\frac{23}{6}$	2.381	1.704	$6, 14$	95.92
$\frac{23}{7}$	2.142	0.03425	6	97.82
$\frac{23}{8}$	1.219	0.9359	$8, 18$	97.84
$\frac{23}{9}$	1.828	0.1126	$4, 20$	98.06
$\frac{23}{10}$	1.027	0		98.97
$\frac{23}{11}$	1.041	0.008562	6	98.95

Continued on next page

Table 3 – *Continued from previous page*

$\frac{u}{v}$	Terminating %	Periodic %	Period Lengths	Unknown %
$\frac{23}{12}$	0.8167	0.7423	4, 6	98.44
$\frac{23}{13}$	1.124	0		98.88
$\frac{23}{14}$	0.7660	0		99.23
$\frac{23}{15}$	1.048	0		98.95
$\frac{23}{16}$	0.7673	0.3346	8	98.90
$\frac{23}{17}$	0.9464	0		99.05
$\frac{23}{18}$	0.7416	0.3860	6	98.87
$\frac{23}{19}$	0.9096	0		99.09
$\frac{23}{20}$	0.7824	0.2233	8	98.99
$\frac{23}{21}$	1.171	0		98.83
$\frac{23}{22}$	1.113	0.5835	2	98.30
$\frac{24}{5}$	99.99	0		0.01449
$\frac{24}{7}$	46.15	44.63	8, 10, 28	9.223
$\frac{24}{11}$	15.35	0.01910	22	84.63
$\frac{24}{13}$	10.82	0.4637	4, 10	88.72
$\frac{24}{17}$	11.80	0		88.20
$\frac{24}{19}$	14.84	2.231	4	82.93
$\frac{24}{23}$	47.41	39.99	14	12.60

B Mathematica Code

The following code can be pasted as seen and executed in a Mathematica notebook. There are three functions which encompass the computations needed for this work, each of which are described in more detail using code comments.

- CFRational - Create a single continued fraction expansion.
- MultiXSingleR - Collect data about a single numerator over a sample of continued fraction expansions.
- MultiR - Collect data over a sample of numerators, each over a sample of continued fraction expansions.

```
(*
CFRational Computes a continued fraction for a rational number x/y \
with a numerator u/v.
Arguments:
```

```
  x: The numerator to compute for.
y: The denominator to compute for.
u: The numerator of the r-value to compute with.
v: The denominator of the r-value to compute with.
max: The max number of iterations to compute up to.
dec: The max decrease allowed for a_n values.
printXList: If we should print the x list at the end.
*)
CFRational[x_, y_, u_, v_, max_, dec_, printXList_] := Block[ {},
   {$MaxExtraPrecision = 1000};

   (* Start with xN=x, yN=y *)
   xN = x;
   yN = y;

   (* Start with blank lists of a_n and x_n *)
   aList = {};
   xList = {};

   For[iteration = 0, iteration < max, iteration++, Block[ {},

     (* Check if xN/yN is an integer,
    we can stop now if so *)
     If[IntegerQ[(xN/yN)], Block[{},
       AppendTo[aList, (xN/yN)];
       Break[]
       ]];

     (* Set a_n to be the floor of xN/yN *)

    aN = Floor[(xN/yN)];
     (* Check if we should decrease aN,
    make sure it is a safe value if so *)

    If[iteration == 0, Block[{},
       aNTemp = aN;
       aN = Max[aN - dec, 0];

      While[(xN/yN) - aN > (u/v) && aN < aNTemp, Block[{}, aN++]];
       ]];
     If[iteration != 0, Block[{},
       aNTemp = aN;
       aN = Max[aN - dec, 1];

      While[(xN/yN) - aN > (u/v) && aN < aNTemp, Block[{}, aN++]];
```

```
   ]];

 (* Add to our running lists *)

AppendTo[xList, (xN/yN)];
 AppendTo[aList, aN];

 (* Check if we already have this xN/yN,
it means a cycle was hit if so *)

If[xList[[Floor[Length[xList] / 2]]] == (xN/yN), Block[{},

   (* Get the periodic part of the aN list *)

  For[i = 0, i < Length[xList], i++, Block[ {},
     positions = Position[xList, xList[[i]]];

    If[positions[[1]][[1]] == 1 && aList[[1]] == 0,
     Block[{}, positions = Delete[positions, 1]]];
     If[Length[positions] >= 2, Block[{}, Break[];]];
     ]];
   If[Length[positions] >= 2, Block[{},

    periodicPart =
     Part[aList, positions[[1]][[1]] ;; positions[[2]][[1]] - 1];

     (* Drop the end of the a_n list,
    and append the periodic part *)

    aList = Drop[
      aList, -(Length[aList] - positions[[1]][[1]] + 1)];
     AppendTo[aList, periodicPart];
     Break[];
     ]];
   ]];

 (* Setup our new xN/yN value *)
 xNTemp = xN;
 xN = u*yN;
 yN = v*xNTemp - v*aN*yN;
 (* Reduce to lowest terms *)
 gcd = GCD[xN, yN];
 xN = xN/gcd;
 yN = yN/gcd;
```

```
    ]];

   If[printXList, Block[{}, Print[xList]]];
   Return[aList];
  ]

(*
MultiXSingleR does analysis on a range of x/y continued fractions for \
a single numerator u/v value.
Arguments:
  minX: The min numerator to compute for.
maxX: TThe max numerator to compute for.
minY: The min denomiator to compute for.
maxY: The max denomiator to compute for.
u: The numerator of the r-value to compute with.
v: The denominator of the r-value to compute with.
max: The max number of iterations to compute up to.
dec: The max decrease allowed for a_n values.
multiR: If we are handling multiple r values.
*)
MultiXSingleR[minX_, maxX_, minY_, maxY_, u_, v_, max_, dec_,
  multiR_] := Block[ {},
  startTime = AbsoluteTime[];

  (* Keep track of whether the CF expansions terminate, are periodic,
  or unknown *)
  endings = Array[0 &, 3];

  (* Keep track of the CF expansion with the longest period *)

  periodLengths = Array[0 &, max];

  If[! multiR, Block[{},
    (* Keep track of the longest CF expansions *)

    lengthsTerminate = Array[0 &, max];
    lengthsPeriodic = Array[0 &, max];
    longest = 0;
    longestLength = 0;

    longestPeriod = 0;
    longestLengthPeriod = 0;
    ]];

  For[x = minX, x < maxX, x++, Block[ {},
```

```
If[! multiR, Block[{},
  (* Do some quick calculations to estimate time remaining *)

        partDone = ((x - minX)/(maxX -
        minX) + ((y - minY)/(maxY - minY))/(maxX - minX));
  secondsSoFar = AbsoluteTime[] - startTime;
  secondsRemaining = ((1/(partDone + 0.001)) - 1)*secondsSoFar;
  If[secondsRemaining < 120, Block[{},

    percentPrint =
      PrintTemporary[N[partDone*100, 2],
       "%  EstimatedTimeRemaining:", N[secondsRemaining, 3],
       " seconds"];
    ], Block[{},

    percentPrint =
      PrintTemporary[N[partDone*100, 2],
       "%  EstimatedTimeRemaining:", N[secondsRemaining/60, 4],
       " minutes"];
    ]];
  ]];

For[y = minY, y < maxY, y++, Block[ {},

   (* If the x and y are coprime,
  do the CF expansion *)
   If[CoprimeQ[x, y], Block[ {},
      result = CFRational[x, y, u, v, max, dec, False];

      (*
     Check what type of ending the CF expansion had and get
     data accordingly *)
      length = Length[result];
      If[ListQ[result[[Length[result]]]], Block[{},
        (* Periodic *)
        endings[[2]]++;
        periodLength = Length[result[[Length[result]]]];
        periodLengths[[periodLength]]++;
        If[! multiR, Block[{},
          If[periodLength > longestLengthPeriod, Block[ {},
            longestLengthPeriod = periodLength;
            longestPeriod = x/y;
            ]];
          lengthsPeriodic[[length]]++;
```

```
            ]];
          ], If[length >= max, Block[{},
            (* Overmax(Unknown) *)

           endings[[3]]++;
            ], Block[{},
            (* Terminating *)

           endings[[1]]++;
            If[! multiR, Block[{},
               lengthsTerminate[[length]]++;

               If[length > longestLength, Block[ {},
                 longestLength = length;
                 longest = x/y;
                 ];
                ]];
             ]]];
         ];
        ]];

    ]];
   NotebookDelete[percentPrint];
   ];
  ]
 (* If we want data printed, do so now *)
 If[! multiR, Block[{},
   Print["r=", u, "/", v];
   Print["Terminated: ", endings[[1]]];
   Print["Periodic: ", endings[[2]]];
   Print["Unknown(overflow): ", endings[[3]], "\n"];
   Print["Terminating CF lengths: ", lengthsTerminate];
   Print["Longest:", longest, ", with length ", longestLength, "\n"];
   Print["Perioidic CF lengths: ", lengthsPeriodic, "\n"];
   Print["Lengths of periods: ", periodLengths];
   Print["Longest Period:", longestPeriod, " with length ",
    longestLengthPeriod];
   ]];
AppendTo[terminated, {u/v, endings[[1]]}];
AppendTo[periodic, {u/v, endings[[2]]}];
periodLengths = DeleteDuplicates[periodLengths];
uniqueLengths = Length[periodLengths] - 1;
AppendTo[uniquePeriods, {u/v, uniqueLengths}];
AppendTo[unknown, {u/v, endings[[3]]}];
]
```

```
(*
MultiR does analysis on a range of numerator u/v values.
Arguments:
  minU: The min numerator of the r-value to compute with.
maxU: The max numerator of the r-value to compute with.
minV: The min denomiator of the r-value to compute with.
maxV: The max denomiator of the r-value to compute with.
maxXY: The max value for x and y to compute continued fractions for \
x/y.
max: The max number of iterations to compute up to.
decString: The string to determine the max decrease for aN values \
(e.g., "max" or "min").
*)
MultiR[minU_, maxU_, minV_, maxV_, maxXY_, max_, decString_] :=
 Block[ {},

  (* Make lists to track behavior *)
  terminated = {};
  periodic = {};
  uniquePeriods = {};
  unknown = {};

  (* Loop over the desired range of u/v values *)

  For[u = minU, u < maxU, u++, Block[ {},
    (* Print the current u value for status updates *)

    For[v = minV, v < maxV, v++, Block[ {},
       statusPrint = PrintTemporary["Current:", u, "/", v];

       (* Set up the dec value *)
       dec = 0;
       If[decString == "max", Block[{}, dec = 0]];
       If[decString == "min", Block[{}, dec = Floor[u/v]]];
       (* Loop over x values if gcd(u,v)=1 and u/v > 1 *)

       If[CoprimeQ[u, v] && ! IntegerQ[u/v] && (u/v) > 1, Block[ {},
          MultiXSingleR[1, maxXY, 1, maxXY, u, v, max, dec, True];
          ]];
       NotebookDelete[statusPrint];
       ]];

    ];
   Print["Terminated:", Sort[terminated, #1[[2]] > #2[[2]] &]];
```

```
 Print["Periodic:", Sort[periodic, #1[[2]] > #2[[2]] &]];
 Print["Unique Periods:",
  Sort[uniquePeriods, #1[[2]] > #2[[2]] &]];
 Print["Unknown:", Sort[unknown, #1[[2]] > #2[[2]] &]];
 ]
]
```