PERIODIC SOLUTIONS TO DIFFERENCE EQUATIONS

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Chapter 1: Introduction

In the area of nonlinear difference equations, one fundamental question deals with the existence of periodic solutions over the integers. For example, the difference equation

$$a_n = \begin{cases} \dfrac{(a_{n-1} + a_{n-2})}{5}, & \text{when 5 divides } a_{n-1} + a_{n-2}, \\ a_{n-1} + a_{n-2}, & \text{otherwise,} \end{cases}$$

has a periodic solution starting with $a_0 = 9$ and $a_1 = 11$, namely 9, 11, 4, 3, 7, 2, 9, 11 … In this project, I looked at a generalization of this and studied the system

(1.1) $$a_n = \begin{cases} x(a_{n-1} + a_{n-2}), & \text{if } x(a_{n-1} + a_{n-2}) \text{ is an integer,} \\ a_{n-1} + a_{n-2}, & \text{otherwise,} \end{cases}$$

where x is a rational number. The problem was to find values of x which would allow these periodic solutions to appear.

This type of problem has been looked at by others within the mathematics community. In [5], a different version of this system,

$$a_n = \lceil c a_{n-1} \rceil - a_{n-2},$$

is studied for values of $|c| < 2$ with $c \neq 0, \pm 1$. In [4], the system

$$a_n = \begin{cases} \dfrac{\alpha a_{n-1} + \beta a_{n-2}}{2}, & \text{if } a_{n-1} + a_{n-2} \text{ is even,} \\ \gamma a_{n-1} + \delta a_{n-2}, & \text{otherwise,} \end{cases}$$

with $\alpha, \beta, \gamma, \delta = \pm 1$, is studied. The authors were able to use a duality lemma to reduce the number of equations being studied from 16 to 8. One of these particular systems,

$$a_n = \begin{cases} \dfrac{a_{n-1} - a_{n-2}}{2}, & \text{if } a_{n-1} + a_{n-2} \text{ is even,} \\ a_{n-1} + a_{n-2}, & \text{otherwise,} \end{cases}$$

is talked about more in [6] with some conjectures given. Other examples can be found within the mathematical literature.

Dr. John Greene had done some research into values of x which allow a periodic solution to (1.1). The approach he was using was to search for products of the matrices A and B that have one as an eigenvalue, where

(1.2) $$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} x & x \\ 1 & 0 \end{pmatrix}.$$

To shift from (1.1) to the use of A and B involved moving from our second order system to a first order system, namely:

For vectors $\boldsymbol{v}_n = \begin{pmatrix} y_n \\ z_n \end{pmatrix}$, with integer entries

(1.3) $$\boldsymbol{v}_n = \begin{cases} B\boldsymbol{v}_{n-1}, & \text{if } x(y_{n-1} + z_{n-1}) \text{ is an integer,} \\ A\boldsymbol{v}_{n-1}, & \text{otherwise,} \end{cases}$$

where A and B are as in (1.2).

So, in our example with $x = \frac{1}{5}$ let $\boldsymbol{v}_0 = \begin{pmatrix} 11 \\ 9 \end{pmatrix}$. The periodic solution 9, 11, 4, 3, 7, 2, 9, 11 … for system (1.1) corresponds to the periodic solution to (1.3) of $\boldsymbol{v}_0 = \begin{pmatrix} 11 \\ 9 \end{pmatrix}, \boldsymbol{v}_1 = \begin{pmatrix} 4 \\ 11 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix} 7 \\ 3 \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \boldsymbol{v}_5 = \begin{pmatrix} 9 \\ 2 \end{pmatrix}, \boldsymbol{v}_6 = \begin{pmatrix} 11 \\ 9 \end{pmatrix}, \cdots$. Since $\boldsymbol{v}_6 = A\boldsymbol{v}_5, \boldsymbol{v}_5 = A\boldsymbol{v}_4, \boldsymbol{v}_4 = B\boldsymbol{v}_3$, and so on, we see that $\boldsymbol{v}_0 = A^2BAB^2\boldsymbol{v}_0$. That is, $\boldsymbol{v}_0$ is an eigenvector of A^2BAB^2 with eigenvalue 1. This leads us to the following theorem:

Theorem 2.1: For every periodic solution to (1.1) with period k, there is a corresponding matrix W, which is a product of A's and B's, that has an eigenvector $\boldsymbol{v}_0$, with eigenvalue 1. The entries of $\boldsymbol{v}_0$ give the initial conditions for the periodic solution.

Initially, we were only interested in positive values of x which lead to a periodic solution. Dr. Greene had exhaustively checked all products of A's and B's through strings of length 12 and had found $\frac{1}{2}, \frac{1}{3}$, and $\frac{1}{5}$ to be the only positive values of x which allowed periodic solutions. We were interested in whether or not there were other possible positive values of x which would lead to periodic solutions. Along the way we also became interested in the negative x-values and the patterns that seemed to occur.

This paper deals mainly with how the products of A's and B's were found and the rational x-values that correspond to these products. We exhaustively checked all products of A and B through strings of length 24 (reproducing those through strings of length 12 to check for accuracy) and found that patterns of x-values were appearing early on. These patterns occurred mainly among negative x-values, which is what prompted the search for all values of x through strings of length 24.

The organization of this paper is as follows: In Chapter 2, we will present theorems that were used to reduce the number of products that had to be examined. Studying strings of length 24 would have involved 2^{24} possibilities or about 17 million possibilities. Using some of these theorems allowed us to decrease this number to about 700,000. We also include some theoretical results that are needed in Chapter 3.

Using the theoretical considerations from Chapter 2, we searched for possible values of x. The results of this search are given in Chapter 3. We also looked at the patterns that seemed to be appearing as we collected our data and include proofs for those patterns that we were able to prove to be true. Chapter 4 will include questions that I still have about this topic as well as what more could be done in this area of research. This will be followed by appendices that include the *Mathematica* code we used as well as a list of the results.

Chapter 2: Theoretical Considerations

As in the introduction, let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} x & x \\ 1 & 0 \end{pmatrix}$. Then relying on Theorem 1.1, the goal of this project was to find all values of x for which some product of at most 24 A's and B's had one as an eigenvalue. The information presented in this chapter was essential in helping with our search. The use of the function $f_W(x)$, defined below, was critical in our search and led quickly to looking at reversals and necklaces to cut the amount of work required. This chapter contains theoretical material about $f_W(x)$ used in our search. We also include some needed background on the Fibonacci and Lucas numbers that is useful in Chapter. 3.

We were only interested in products of these matrices that had one as an eigenvalue, so we started with the basics: the characteristic polynomial of a 2x2 matrix. Now, one is an eigenvalue for X if and only if $\det(X - I) = 0$. For general $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\det(X - I) = 1 - (a + d) + (ad - bc)$$
$$= 1 - \mathrm{tr}(X) + \det(X).$$

Now, let's say W is some product of A's and B's. We define a function $f_W(x)$ by

(2.1) $$f_W(x) = -\det(W - I) = \mathrm{tr}(W) - 1 - \det(W).$$

Since $-f_W(x)$ is the result of setting $z = 1$ in the characteristic polynomial $\det(W - zI)$, W will have one as an eigenvalue for exactly those x for which $f_W(x) = 0$. Looking at this equation we see that -1 is not dependent on the order of the A's and B's. Also, $\det(W)$ is not dependent on the order of the A's and B's since $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$ [3, p.

467]. Therefore, in our equation, the order of the A's and B's only matters when dealing with the tr(W).

A linear algebra fact that proved to be useful is that cyclic permutations (also known as necklaces) of matrix products have the same trace [3, p. 110]. This meant we would not have to look at cyclic permutations of A's and B's, reducing the number of cases by approximately a factor of n when dealing with strings of length n. The 2x2 case is included below as a theorem, and a proof follows. (The 2x2 case is all that is used in this paper, but the theorem holds true for mxm matrices as well.)

Theorem 2.1: Let $A_1, A_2, \cdots, A_n$ be 2x2 matrices. Then

$$\mathrm{tr}(A_1\, A_2 \cdot \cdots \cdot A_n) = \mathrm{tr}(A_n\, A_1 \cdot \cdots \cdot A_{n-1}).$$

Proof: Let $A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then

$$A_1 A_2 = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$

so

$$\mathrm{tr}(A_1 A_2) = ae + bg + cf + dh.$$

Also,

$$A_2 A_1 = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}$$

has

$$\mathrm{tr}(A_2 A_1) = ea + fc + gb + hd.$$

Therefore $\mathrm{tr}(A_1 A_2) = \mathrm{tr}(A_2 A_1)$. In general, for matrices $A_1, A_2, \cdots, A_n$, let

$$B = A_1\, A_2 \cdot \cdots \cdot A_{n-1}$$

Then by the above calculations,

$$\mathrm{tr}(BA_n) = \mathrm{tr}(A_n B)$$

or

$$\mathrm{tr}(A_1\, A_2 \cdot \cdots \cdot A_n) = \mathrm{tr}(A_n\, A_1 \cdot \cdots \cdot A_{n-1}).$$

Thus the trace is invariant under cyclic permutations. ■

Corollary 2.2: Let $A_1, A_2, \cdots, A_n$ be 2x2 matrices of the form $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} x & x \\ 1 & 0 \end{pmatrix}$. If $W_1 = A_1\, A_2 \cdot \cdots \cdot A_n$ and $W_2 = A_n\, A_1 \cdot \cdots \cdot A_{n-1}$, then $f_{W_1}(x) = f_{W_2}(x)$.

Proof: By (2.1) and Theorem 2.1,

$$\begin{aligned} f_{W_1}(x) &= \mathrm{tr}(A_1\, A_2 \cdot \cdots \cdot A_n) - 1 - \det(A_1\, A_2 \cdot \cdots \cdot A_n) \\ &= \mathrm{tr}(A_n\, A_1 \cdot \cdots \cdot A_{n-1}) - 1 - \det(A_n\, A_1 \cdot \cdots \cdot A_{n-1}) \\ &= f_{W_2}(x). \quad ■ \end{aligned}$$

While looking at small cases of strings of A's and B's, it seemed apparent that two strings of A's and B's which were reversals of each other also had the same eigenvalues. We defined a reversal to mean reversing the order of the string. For example, A^3B^2AB and BAB^2A^3 are reversals of each other. Moreover, if W' is a reversal of W, it seemed that $f_{W'}(x) = f_W(x)$. We used A^3B^2AB as an example to check whether this was even sometimes true.

We let $W = A^3B^2AB$ and $W' = BAB^2A^3$. Then

$$W = \begin{pmatrix} 6x^3 + 10x^2 + 5x & 6x^3 + 7x^2 \\ 4x^3 + 6x^2 + 3x & 4x^3 + 4x^2 \end{pmatrix}$$

and

$$f_W(x) = 11x^3 + 14x^2 + 5x - 1.$$

Similarly,

$$W' = \begin{pmatrix} 10x^3 + 11x^2 & 6x^3 + 7x^2 \\ 5x^2 + 8x & 3x^2 + 5x \end{pmatrix}$$

and

$$f_{W'}(x) = 11x^3 + 14x^2 + 5x - 1.$$

So we see that $f_W(x) = f_{W'}(x)$ for at least this case. It appeared that $f_W(x)$ always equaled $f_{W'}(x)$, but to show this, we would first need the following theorem.

Theorem 2.3: For any 2x2 matrix W, $W^2 = \mathrm{tr}(W) \cdot W - \det(W) \cdot I$. In other words, every 2x2 matrix satisfies its own characteristic equations [3, p. 509].

Proof: The matrix $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has characteristic equation $\lambda^2 - \mathrm{tr}(W)\lambda + \det(W) = 0$ so we have $W^2 - \mathrm{tr}(W)W + \det(W)I$

$$= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} - (a + d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} - \begin{pmatrix} a^2 + ad & ab + db \\ ac + cd & ad + d^2 \end{pmatrix} + \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $W^2 - \mathrm{tr}(W)W + \det(W)I = 0$ so $W^2 = \mathrm{tr}(W)W - \det(W)I$. ■

Consequently, for any 2x2 matrix W, $W^2 = cW + dI$ for some c, d. Now assume that for all strings of length six and less, reversals have the same trace. Then A^3B^2AB can be rewritten as AA^2B^2AB and BAB^2A^3 can be rewritten as BAB^2A^2A. We then let $A^2 = cA + dI$ and substitute this into both products giving us $cAAB^2AB + dAB^2AB$ and $cBAB^2AA + dBAB^2A$. Matrices AAB^2AB and BAB^2AA are reversals of each other and are strings of length six. Also, AB^2AB and BAB^2A are also reversals of each other and are strings of length five. So,

$$\mathrm{tr}(A^3B^2AB) = c \cdot \mathrm{tr}(AAB^2AB) + d \cdot \mathrm{tr}(AB^2AB)$$

and

$$\mathrm{tr}(BAB^2A^3) = c \cdot \mathrm{tr}(BAB^2AA) + d \cdot \mathrm{tr}(BAB^2A).$$

By our hypothesis, AAB^2AB and BAB^2AA have the same trace and AB^2AB and BAB^2A have the same trace, so $\mathrm{tr}(A^3B^2AB) = \mathrm{tr}(BAB^2A^3)$. This method can then be generalized to products of n matrices.

Theorem 2.4: Let W be a product of 2x2 matrices A and B, say $W = V_1 \cdot V_2 \cdot \cdots \cdot V_n$, where V_i is either A or B. Let $W' = V_n \cdot V_{n-1} \cdot \cdots \cdot V_1$. Then $f_W(x) = f_{W'}(x)$.

Proof: With $f_W(x) = \mathrm{tr}(W) - 1 - \det(W)$ and $f_{W'}(x) = \mathrm{tr}(W') - 1 - \det(W')$. Note that, $\det(W) = \det(W')$, since the determinant is not dependent on the order of the A's and B's. Thus we only have to show that $\mathrm{tr}(W) = \mathrm{tr}(W')$. We do this using induction on n where $n = 1$ is a trivial case and $n = 2$ follows from Theorem 2.1.

Therefore we assume that this is true for all products of A's and B's with fewer than n matrices. Then given $W = V_1 \cdot V_2 \cdot \cdots \cdot V_n$, we will first look at the case in which W does not contain two consecutive A's or B's. This means that the A's and B's alternate which gives us two cases to look at. The first of these is either $ABAB \ldots AB$ or $BABA \ldots BA$. These are just cyclic permutations of each other and thus have the same trace as already proved above. The second case is either $ABAB \ldots BA$ or $BABA \ldots AB$, which are their own reversals and therefore have the same trace as well.

Now let's consider the case in which W does contain two or more consecutive A's or two or more consecutive B's. Here we will apply the fact that $M^2 = cM + dI$ in order to rewrite W as a linear combination of fewer products. If $V_i = V_{i+1}$, then $W = W_1 V_i^2 W_2$ where $W_1 = V_1 \cdot V_2 \cdot \cdots \cdot V_{i-1}$ and $W_2 = V_{i+2} \cdot V_{i+3} \cdot \cdots \cdot V_n$. Also, $W' = W_2' V_i^2 W_1'$. Now using $V_i^2 = cV_i + dI$, we get

$$W = cW_1 V_i W_2 + dW_1 W_2$$

and

$$W' = cW_2' V_i W_1' + dW_2' W_1'.$$

Also,

$$W_2' V_i W_1' = (W_1 V_i W_2)'$$

and

$$W_2' W_1' = (W_1 W_2)'.$$

Now, $W_1 V_i W_2$ and $W_2' V_i W_1'$ are strings of length $n-1$ and $W_1 W_2$ and $W_2' W_1'$ are strings of length $n-2$. Therefore, by induction, $W_1 V_i W_2$ and $W_2' V_i W_1'$ have the same trace since they are reversals of each other and $W_1 W_2$ and $W_2' W_1'$ have the same trace since they are also reversals of each other. Thus,

$$\begin{aligned}
\mathrm{tr}(W) &= \mathrm{tr}(W_1 V_i^2 W_2) \\
&= c \cdot \mathrm{tr}(W_1 V_i W_2) + d \cdot \mathrm{tr}(W_1 W_2) \\
&= c \cdot \mathrm{tr}(W_2' V_i W_1') + d \cdot \mathrm{tr}(W_2' W_1')
\end{aligned}$$

$$= \mathrm{tr}(W_2' V_i^2 W_1')$$

$$= \mathrm{tr}(W'). \quad \blacksquare$$

Fibonacci numbers were used at different times throughout this project and are defined as follows:

$$F_0 = 0, F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2}, n \geq 2.$$

This gives the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, …. We also found Lucas numbers to be useful. Lucas numbers are defined as:

$$L_0 = 2, L_1 = 1, \text{ and } L_n = L_{n-1} + L_{n-2}, n \geq 2,$$

leading to the sequence 2, 1, 3, 4, 7, 11, 18, 29, ….

Some identities associated with these two sets of numbers which were useful to us include the following from [1, p. 59]:

(2.1) $$F_m L_n = F_{m+n} + (-1)^n F_{m-n}$$

(2.2) $$L_m F_n = F_{m+n} + (-1)^{n+1} F_{m-n}.$$

Other items that were useful to us include the following lemmas and theorem.

Lemma 2.5: If $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then $A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ [1, p. 65].

Proof: If $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then $A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $A^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$. Assume that

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

for some n. Then

$$A^{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} F_{n+1} + F_n & F_n + F_{n-1} \\ F_{n+1} & F_n \end{pmatrix}.$$

But

$$F_{n+1} + F_n = F_{n+2} \text{ and } F_n + F_{n-1} = F_{n+1}$$

so $A^{n+1} = \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}$. Thus $A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$. ■

Lemma 2.6 (Cassini's Identity): $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ [1, p. 57].

Proof: Given $A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$, $F_{n+1}F_{n-1} - F_n^2 = \det(A^n) = (det\ A)^n = (-1)^n$. ■

Theorem 2.7: Let M and N be 2x2 matrices.

a. If M and N both have $\boldsymbol{v}$ as an eigenvector with eigenvalue 1, then MN also has $\boldsymbol{v}$ as an eigenvector with eigenvalue 1.

b. If M and N both have $\boldsymbol{v}$ as an eigenvector with eigenvalue -1, then MN also has $\boldsymbol{v}$ as an eigenvector, but with eigenvalue 1.

Proof:

a. By definition, $M\boldsymbol{v} = \boldsymbol{v}$ and $N\boldsymbol{v} = \boldsymbol{v}$. Thus,

$$(MN)\boldsymbol{v} = M(N\boldsymbol{v})$$

$$= M\boldsymbol{v}$$

$$= \boldsymbol{v}.$$

b. By definition, $M\boldsymbol{v} = -\boldsymbol{v}$ and $N\boldsymbol{v} = -\boldsymbol{v}$. So,

$$\begin{aligned}(MN)\boldsymbol{v} &= M(N\boldsymbol{v}) \\ &= M \cdot (-\boldsymbol{v}) \\ &= -M\boldsymbol{v} \\ &= \boldsymbol{v}. \quad \blacksquare\end{aligned}$$

Using these theorems and lemmas we are able to now look only at the necklaces of the products of matrices A and B. Also, we are able to ignore the reversals of all cases that we do investigate. This will also help reduce the number of cases that we are actually interested in.

Chapter 3: Results

Using *Mathematica*, we exhaustively searched all necklaces of A's and B's up to strings of length 24, finding all the rational zeroes of $f_W(x)$ for these cases. An example of the code we used can be found in Appendix 1, but an explanation of how the code works follows.

Combinatorica is a package within *Mathematica* which performs functions from combinatorics and graph theory. Loading this package allowed us to then use the ListNecklaces function to find only the cyclic necklaces of A's and B's rather than all possible permutations. 0's were used to represent A's and 1's were used to represent B's for simplicities sake during this process. The sequences of 0's and 1's were each individually labeled for ease of reference purposes. The next step in the code multiplies the appropriate A's and B's together. For example, if f[i]={0,0,0,1,1,1,0,0,0,1,1,1,0,1,1,1}, the code would multiply $A.A.A.B.B.B.A.A.A.B.B.B.A.B.B.B$ and store the resulting matrix. In the same loop, $f_W(x)$ is found, the rational solutions to this polynomial are found, and the solutions are stored. The final loop looks at these values and prints out only those that are rational solutions along with their corresponding necklace value.

While first starting to collect our data, we noticed a rather large number of cases in which $x = 0$ and $x = -1$. We knew for any string W of length n with a A's and b B's that $\det(W) = (-1)^n x^b$. Therefore,

$$f_W(x) = \mathrm{tr}(W) - 1 - \det(W) = \mathrm{tr}(W) - 1 - (-1)^n x^b.$$

Thus, whenever the trace had a constant term of 1, this 1 would cancel with the -1. This would leave all the remaining terms with at least one x in them, giving $x = 0$ as a solution to $f_W(x)$. To prevent having to look through so many cases, we disregard cases in which $x = 0$.

We also noticed early on that many of the cases in which $x = -1$ had combinations of AB^2 and B^3 in them. When $x = -1$, AB^2 and B^3 have $\binom{0}{1}$ as an eigenvector with an eigenvalue of one. Thus by Theorem 2.7, any product using only AB^2 and B^3 has an eigenvalue of one when $x = -1$. Because of the large number of these combinations, we choose to disregard cases in which $x = -1$ as well.

Next, we organized the solutions by the number of B's in the string. Table 3.1 shows x-values for strings with only one B.

Table 3.1

Product	**x-value**
B	½
AB	0
A^2B	0
A^3B	-1/4
A^4B	-2/9
A^5B	-1/3
A^6B	-7/22
A^7B	-4/11
A^8B	-5/14
A^9B	-3/8
$A^{10}B$	-54/145
$A^{11}B$	-11/29
$A^{12}B$	-143/378
$A^{13}B$	-8/21
$A^{14}B$	-94/247
$A^{15}B$	-29/76
$A^{16}B$	-986/2585
$A^{17}B$	-21/55
$A^{18}B$	-2583/6766
$A^{19}B$	-76/199
$A^{20}B$	-1691/4428
$A^{21}B$	-55/144
$A^{22}B$	-17710/46369
$A^{23}B$	-199/521

Early on in the research, while examining these values, we noticed that the numbers in both the numerator and denominator seemed to be close to Fibonacci numbers. From Lemma 2.5 we know that $A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$, and thus $A^nB = \begin{pmatrix} xF_{n+1} + F_n & xF_{n+1} \\ xF_n + F_{n-1} & xF_n \end{pmatrix}$. From there we found that

$$f_W(x) = xF_{n+2} + F_n - 1 - (-1)^{n+1}x = (F_{n+2} - (-1)^{n+1})x + F_n - 1.$$

Setting this equal to zero, we found that

$$x = -\frac{F_n - 1}{F_{n+2} + (-1)^n}.$$

Therefore, for $W = A^n B$, there is always a value of x allowing W to have one as an eigenvalue.

Looking more closely at these values we also noticed that when n was odd, there seemed to be some sort of cancellation happening. Table 3.2 contains only these values.

Table 3.2

Product	**x-value**
AB	0
A^3B	-1/4
A^5B	-1/3
A^7B	-4/11
A^9B	-3/8
$A^{11}B$	-11/29
$A^{13}B$	-8/21
$A^{15}B$	-29/76
$A^{17}B$	-21/55
$A^{19}B$	-76/199
$A^{21}B$	-55/144
$A^{23}B$	-199/521

We soon noticed that those of the form $A^{4n+1}B$ had Fibonacci numbers as their numerators and denominators, while those of the form $A^{4n-1}B$ had Lucas numbers as their numerators and denominators.

We can prove that these are real patterns. Starting with those cases of the form $A^{4n+1}B$, we obtained the numbers $0, -\frac{1}{3}, -\frac{3}{8}, -\frac{8}{21}, \cdots$ which gives in general, $-\frac{F_{2n}}{F_{2n+2}}$, rather than our expected $-\frac{F_{4n+1}-1}{F_{4n+3}-1}$. We then identified that by (2.1)

$$F_{2n}L_{2n+1} = F_{4n+1} - F_{-1}$$
$$= F_{4n+1} - 1.$$

Similarly,

$$F_{2n+2}L_{2n+1} = F_{4n+3} - F_1$$
$$= F_{4n+3} - 1.$$

Thus, we have

$$-\frac{F_{4n+1} - 1}{F_{4n+3} - 1} = -\frac{F_{2n}L_{2n+1}}{F_{2n+2}L_{2n+1}}$$
$$= -\frac{F_{2n}}{F_{2n+2}}.$$

Now, using a similar argument for those cases of the form $A^{4n-1}B$, we were getting $-\frac{L_{2n-1}}{L_{2n+1}}$ rather than $-\frac{F_{4n-1}-1}{F_{4n+1}-1}$. Using (2.2), we identified

$$L_{2n-1}F_{2n} = F_{4n-1} - F_{-1}$$
$$= F_{4n-1} - 1,$$

and

$$L_{2n+1}F_{2n} = F_{4n+1} - F_1$$
$$= F_{4n+1} - 1.$$

Therefore,

$$-\frac{F_{4n-1} - 1}{F_{4n+1} - 1} = -\frac{L_{2n-1}F_{2n}}{L_{2n+1}F_{2n}}$$

$$= -\frac{L_{2n-1}}{L_{2n+1}}.$$

Next we looked at string with two B's. Table 3.3 shows all cases where $f_W(x)$ has rational zeroes x.

Table 3.3

String Length	Product	x-value
2	B^2	½
3	AB^2	-1, 1/3
4	$(AB)^2$	0, -2/3
5	A^2BAB	0, -4/7
6	A^3BAB	-1/3
6	$(A^2B)^2$	0, -1
8	$(A^3B)^2$	-1/2, -1/4
10	A^5BA^3B	-3/8
10	$(A^4B)^2$	-4/7, -2/9
12	$(A^5B)^2$	-1/3, -3/7
14	A^7BA^5B	-8/21
14	$(A^6B)^2$	-9/20, -7/22
16	$(A^7B)^2$	-2/5, -4/11
18	A^9BA^7B	-21/55
18	$(A^8B)^2$	-5/14, -11/27
20	$(A^9B)^2$	-7/18, -3/8
22	$A^{11}BA^9B$	-55/144
22	$(A^{10}B)^2$	-56/143, -54/145
24	$(A^{11}B)^2$	-5/13, -11/29

Looking at these x-values, there seem to be two patterns. The first is for W of the form $(A^nB)^2$ and the second for W of the form $A^{2n+1}BA^{2n-1}B$. We first looked at the $(A^nB)^2$ case. Table 3.4 shows just these cases.

Table 3.4

Product	x-value from A^nB	New x-value
B^2	½	-
$(AB)^2$	0	-2/3
$(A^2B)^2$	0	-1
$(A^3B)^2$	-1/4	-1/2
$(A^4B)^2$	-2/9	-4/7
$(A^5B)^2$	-1/3	-3/7
$(A^6B)^2$	-7/22	-9/20
$(A^7B)^2$	-4/11	-2/5
$(A^8B)^2$	-5/14	-11/27
$(A^9B)^2$	-3/8	-7/18
$(A^{10}B)^2$	-54/145	-56/143
$(A^{11}B)^2$	-11/29	-5/13

Here, we confirmed our suspicion that one of the x-values was the same as the x-value in the A^nB case. We just needed to find where the other x-value was coming from. For this case,

$$f_{W^2}(x) = -\det(W^2 - I) = -\det(W - I)\det(W + I)$$

where W is some product of A's and B's. Now, $-\det(W - I) = f_W(x)$, which gives us the old x-value, so the new one is coming from $\det(W + I)$ and

$$\det(W + I) = \mathrm{tr}(W) + 1 + \det(W).$$

If we let

$$g_W(x) = \mathrm{tr}(W) + 1 + \det(W),$$

then

$$f_{W^2}(x) = f_W(x)g_W(x).$$

Setting $g_W(x)$ equal to zero and solving, we have

$$x = -\frac{F_n + 1}{F_{n+2} - (-1)^n}.$$

Therefore in the case of $(A^nB)^2$, the corresponding x-values are

$$x = -\frac{F_n - 1}{F_{n+2} + (-1)^n} \text{ and } x = -\frac{F_n + 1}{F_{n+2} - (-1)^n}.$$

The second case of two B's is of the form $A^{2n+1}BA^{2n-1}B$. Table 3.5 gives only these cases.

Table 3.5

Product	**x-value**
A^3BAB	-1/3
A^5BA^3B	-3/8
A^7BA^5B	-8/21
A^9BA^7B	-21/55
$A^{11}BA^9B$	-55/144

Using *Maple* we checked further cases of these values to determine if there really was a pattern or if it just seemed to be early on. We found that the pattern really did exist and thus we tried to determine what the x-value was. Since

$$A^nB = \begin{pmatrix} xF_{n+1} + F_n & xF_{n+1} \\ xF_n + F_{n-1} & xF_n \end{pmatrix}$$

and

$$A^mB = \begin{pmatrix} xF_{m+1} + F_m & xF_{m+1} \\ xF_m + F_{m-1} & xF_m \end{pmatrix}.$$

The product of these is

$$A^mBA^nB = \begin{pmatrix} x^2F_{m+1}F_{n+2} + xF_{m+2}F_{n+1} + F_mF_n & x^2F_{m+1}F_{n+2} + xF_mF_{n+1} \\ x^2F_mF_{n+2} + xF_{m+1}F_{n+1} + F_{m-1}F_n & x^2F_mF_{n+2} + xF_{m-1}F_{n+1} \end{pmatrix}.$$

Then,

$$\begin{aligned} \mathrm{tr}(A^mBA^nB) &= x^2F_{m+1}F_{n+2} + xF_{m+2}F_{n+1} + F_mF_n + x^2F_mF_{n+2} + xF_{m-1}F_{n+1} \\ &= x^2(F_{m+1}F_{n+2} + F_mF_{n+2}) + x(F_{m+2}F_{n+1} + F_{m-1}F_{n+1}) + F_mF_n \end{aligned}$$

$$= x^2F_{n+2}(F_{m+1}+F_m) + xF_{n+1}(F_{m+2}+F_{m-1}) + F_mF_n$$

$$= x^2F_{m+2}F_{n+2} + xF_{n+1}(F_{m+1}+F_m+F_{m-1}) + F_mF_n$$

$$= x^2F_{m+2}F_{n+2} + xF_{n+1}(F_{m+1}+F_{m+1}) + F_mF_n$$

$$= x^2F_{m+2}F_{n+2} + 2xF_{m+1}F_{n+1} + F_mF_n.$$

Using this formula, we have

$$\mathrm{tr}(A^{2n+1}BA^{2n-1}B) = x^2F_{2n+3}F_{2n+1} + 2xF_{2n+2}F_{2n} + F_{2n+1}F_{2n-1}$$

and

$$\det(A^{2n+1}BA^{2n-1}B) = x^2.$$

Therefore

$$f_W(x) = x^2F_{2n+3}F_{2n+1} + 2xF_{2n+2}F_{2n} + F_{2n+1}F_{2n-1} - 1 - x^2$$

$$= x^2(F_{2n+3}F_{2n+1} - 1) + 2xF_{2n+2}F_{2n} + F_{2n+1}F_{2n-1} - 1.$$

Applying Lemma 2.6,

$$F_{2n+3}F_{2n+1} = F_{2n+2}^2 + (-1)^{2n+2} = F_{2n+2}^2 + 1.$$

Similarly,

$$F_{2n+1}F_{2n-1} = F_{2n}^2 + (-1)^{2n} = F_{2n}^2 + 1.$$

Therefore,

$$f_W(x) = x^2F_{2n+2}^2 + 2xF_{2n+2}F_{2n} + F_{2n}^2 = (xF_{2n+2} + F_{2n})^2.$$

Setting this equal to zero and solving, we find that

$$x = -\frac{F_{2n}}{F_{2n+2}}.$$

Next we looked at strings with three B's, which are shown in Table 3.6.

Table 3.6

String Length	Product	x-value
3	B^3	½
6	A^2BAB^2	1/5
8	$A^3BA^2B^2$	-1/2
12	A^5BA^3BAB	-1/3
12	$A^4BA^3BA^2B$	-1/7
12	$(A^3B)^3$	-1/4
15	$(A^4B)^3$	-2/9
16	$A^9B(A^2B)^2$	-3/4
16	$A^8BA^3BA^2B$	-1/2
16	$A^7BA^4BA^2B$	-2/5
18	$A^8BA^4BA^3B$	-1/3
18	$A^7B(A^4B)^2$	-4/15
18	$A^6BA^5BA^4B$	-17/59
18	$(A^5B)^3$	-1/3
20	$A^9BA^5BA^3B$	-3/8
21	$(A^6B)^3$	-7/22
24	$A^{11}BA^6BA^4B$	-5/13
24	$A^8BA^7BA^6B$	-37/107
24	$(A^7B)^3$	-4/11

Again, when looking at this table, we noticed that certain patterns were appearing. The first of these involved products $A^{2n+1}BA^{n+1}BA^{n-1}B$ which are shown in Table 3.7.

Table 3.7

n-value	Product	x-value
1	$A^3BA^2B^2$	-1/2
2	A^5BA^3BAB	-1/3
3	$A^7BA^4BA^2B$	-2/5
4	$A^9BA^5BA^3B$	-3/8
5	$A^{11}BA^6BA^4B$	-5/13

To explain this table, we refer back to Theorem 2.7 and give two examples of how this works. If we look at row two of Table 3.7, with $x = -\frac{1}{3}$, $M = A^5B$, $N = A^3BAB$ then $\boldsymbol{v} = \binom{2}{1}$ is an eigenvector with eigenvalue 1 for both M and N. Then $MN = A^5BA^3BAB$ and when $x = -\frac{1}{3}$, $\boldsymbol{v} = \binom{2}{1}$ is an eigenvector for MN with eigenvalue 1.

Looking at row three of Table 3.7, if $x = -\frac{2}{5}$, $M = A^7B$ and $N = A^4BA^2B$, then $\boldsymbol{v} = \binom{3}{2}$ is an eigenvector with eigenvalue -1 for both M and N. Therefore $MN = A^7BA^4BA^2B$ and when $x = -\frac{2}{5}$, $\boldsymbol{v} = \binom{3}{2}$ is an eigenvector for MN with eigenvalue 1.

The next pattern that was noticed is shown in Table 3.8.

Table 3.8

Product	x-value
A^2BAB^2	1/5
$A^4BA^3BA^2B$	-1/7
$A^6BA^5BA^4B$	-17/59
$A^8BA^7BA^6B$	-37/107

Here the exponents on the A's have the pattern (2, 1, 0), (4, 3, 2), (6, 5, 4), and (8, 7, 6). This seemed very suspicious to us and using *Maple* again, we checked to see if this pattern continued. The pattern is real and much work was done on finding the x-value for this pattern. It is too in depth for this paper, but for more on this subject see [2].

Seeing these infinite patterns in the cases of one B, two B's, and three B's sparked our curiosity as to whether or not there would be infinite patterns for the case of four B's or five B's, or even six B's. Again, using *Maple* we checked to see if patterns could be found for these. It was determined that there were infinite families for the cases of four B's and six B's and the formulas for the x-values of these can also be found in [2].

Chapter 4: Conclusion

One question that was particularly interesting to me, but never answered, is whether or not there are more positive values of x which produce a periodic solution. We found that through strings of length 24, the only positive values which produce periodic solutions are $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$. Would other positive values be found in strings of length greater than 24? We suspect that $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$ are the only positive solutions, but were not able to prove this.

As far as negative values of x, we found what we referred to as infinite families while doing our research [2] and have speculated that it's possible that these are the only infinite families that give solutions to (1.1). Again, we were unable to verify this, and it would be something that could be examined more closely on a future project.

It would also be interesting to try to write a code for *Mathematica*, or a similar program, that allows this information to run faster. When looking at strings of length 24 with 12 B's, the code being used took about two days to run. It may be possible to write a code that would be faster and thus allow a person to look at longer strings. This may reveal more patterns or lead to a greater understanding of what is going on.

One final point of interest is that while examining the infinite families, we came across some rather interesting Fibonacci and Lucas identities.

$$F_{4n+2}F_{4n+1}F_{4n} - 1 = (F_{4n}F_{2n-1} + F_{4n-1}F_{2n+1})(F_{4n+1}L_{2n-1} + F_{4n}L_{2n+1}),$$

$$F_{4n+2}F_{4n+1}F_{4n} + 1 = (F_{4n}L_{2n-1} + F_{4n-1}L_{2n+1})(F_{4n+1}F_{2n-1} + F_{4n}F_{2n+1}),$$

$$F_{4n+4}F_{4n+3}F_{4n+2} - 1 = (F_{4n+2}L_{2n} + F_{4n+1}L_{2n+2})(F_{4n+4}F_{2n} + F_{4n+2}F_{2n+1}),$$

$$F_{4n+4}F_{4n+3}F_{4n+2} + 1 = (F_{4n+2}F_{2n} + F_{4n+1}F_{2n+2})(F_{4n+4}L_{2n} + F_{4n+2}L_{2n+1}).$$

These are discussed more in [2], but it's possible that further investigation of the infinite families may lead to more of these types of identities.

Appendix 1: *Mathematica* Code

This is an example of the *Mathematica* code that we used to find all rational solutions to $f_W(x)$. The code was used to find necklaces for strings of length n, $4 \leq n \leq 24$, with the number of B's ranging from one to n. The code then finds $f_W(x)$ for each combination and solves for all rational solutions. This particular code is a string of length 16 with 9 B's.

```
<<"Combinatorica`"
n:=16
neck=ListNecklaces[n, {0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1}, Cyclic]
a={{1,1},{1,0}}
b={{x,x},{1,0}}
For [i=1, i<Length[neck] +1, f[i_]:=neck[[i]]; i++]
For[l=1,l<Length[neck],l++,
For[k=1, k≤n,k++, If[f[l][[k]]==0,mult=mult.a, mult=mult.b];];
        values[l]=Reduce[1-Tr[mult]+Det[mult]==0,x,Rationals];
                mult:=IdentityMatrix[2]]
For[i=1,i≤Length[neck],i++,If[Head[values[i]]==Or||Head[values[i]]==Equal,Print[i,":
",values[i],"  ",neck[[i]]]]]
```

Appendix 2: Results by Number of *B*'s

B's	**String Length**	**Product**	**X-Value**	**Why?**
1	1	B	½	A^nB
	2	AB	0	A^nB
	3	A^2B	0	A^nB
	4	A^3B	-1/4	A^nB
	5	A^4B	-2/9	A^nB
	6	A^5B	-1/3	A^nB
	7	A^6B	-7/22	A^nB
	8	A^7B	-4/11	A^nB
	9	A^8B	-5/14	A^nB
	10	A^9B	-3/8	A^nB
	11	$A^{10}B$	-54/145	A^nB
	12	$A^{11}B$	-11/29	A^nB
	13	$A^{12}B$	-143/378	A^nB
	14	$A^{13}B$	-8/21	A^nB
	15	$A^{14}B$	-94/247	A^nB
	16	$A^{15}B$	-29/76	A^nB
	17	$A^{16}B$	-986/2585	A^nB
	18	$A^{17}B$	-21/55	A^nB
	19	$A^{18}B$	-2583/6766	A^nB
	20	$A^{19}B$	-76/199	A^nB
	21	$A^{20}B$	-1691/4428	A^nB
	22	$A^{21}B$	-55/144	A^nB
	23	$A^{22}B$	-17710/46369	A^nB
	24	$A^{23}B$	-199/521	A^nB
2	2	B^2	½	$(A^nB)^2$
	3	AB^2	-1, 1/3	
	4	$(AB)^2$	0, -2/3	$(A^nB)^2$
	5	A^2BAB	0, -4/7	
	6	A^3BAB	-1/3	$A^{2n+1}BA^{2n-1}B$
	8	$(A^3B)^2$	-1/2, -1/4	$(A^nB)^2$
	10	A^5BA^3B	-3/8	$A^{2n+1}BA^{2n-1}B$
	10	$(A^4B)^2$	-4/7, -2/9	$(A^nB)^2$
	12	$(A^5B)^2$	-1/3, -3/7	$(A^nB)^2$
	14	A^7BA^5B	-8/21	$A^{2n+1}BA^{2n-1}B$
	14	$(A^6B)^2$	-9/20, -7/22	$(A^nB)^2$

B's	String Length	Product	X-Value	Why?
	16	$(A^7B)^2$	-2/5, -4/11	$(A^nB)^2$
	18	A^9BA^7B	-21/55	$A^{2n+1}BA^{2n-1}B$
	18	$(A^8B)^2$	-5/14, -11/27	$(A^nB)^2$
	20	$(A^9B)^2$	-7/18, -3/8	$(A^nB)^2$
	22	$A^{11}BA^9B$	-55/144	$A^{2n+1}BA^{2n-1}B$
	22	$(A^{10}B)^2$	-56/143, -54/145	$(A^nB)^2$
	24	$(A^{11}B)^2$	-5/13, -11/29	$(A^nB)^2$
3	3	B^3	-1, ½	powers
	6	A^2BAB^2	-1, 1/5	$A^{4n+2}BA^{4n+1}BA^{4n}B$*
	8	$A^3BA^2B^2$	-1/2	$A^{2n+1}BA^{n+1}BA^{n-1}B$
	12	A^5BA^3BAB	-1/3	$A^{2n+1}BA^{n+1}BA^{n-1}B$
	12	$A^4BA^3BA^2B$	-1/7	$A^{4n+4}BA^{4n+3}BA^{4n+2}B$*
	12	$(A^3B)^3$	-1/4	powers
	15	$(A^4B)^3$	-2/9	powers
	16	$A^9B(A^2B)^2$	-3/4	
	16	$A^8BA^3BA^2B$	-1/2	
	16	$A^7BA^4BA^2B$	-2/5	$A^{2n+1}BA^{n+1}BA^{n-1}B$
	18	$A^8BA^4BA^3B$	-1/3	
	18	$A^7B(A^4B)^2$	-4/15	
	18	$A^6BA^5BA^4B$	-17/59	$A^{4n+2}BA^{4n+1}BA^{4n}B$*
	18	$(A^5B)^3$	-1/3	powers
	20	$A^9BA^5BA^3B$	-3/8	$A^{2n+1}BA^{n+1}BA^{n-1}B$
	21	$(A^6B)^3$	-7/22	powers
	24	$A^{11}BA^6BA^4B$	-5/13	$A^{2n+1}BA^{n+1}BA^{n-1}B$
	24	$A^8BA^7BA^6B$	-37/107	$A^{4n+4}BA^{4n+3}BA^{4n+2}B$*
	24	$(A^7B)^3$	-4/11	powers
4	4	B^4	½	powers
	6	$(AB^2)^2$	-1, 1/3	powers
	8	$(A^2B^2)^2$	-1/2	$(A^{2n+2}BA^{2n}B)^2$*
	8	$(AB)^4$	0, -2/3	powers
	10	$(A^2BAB)^2$	0, -4/7	powers
	12	$(A^4B^2)^2$	-1, -1/9	$(A^{2n+4}BA^{2n}B)^2$*
	12	$(A^3BAB)^2$	-1/3	powers
	16	$A^5BA^3B(A^2B)^2$	-3/4	
	16	$A^5BA^2BA^3BA^2B$	-1/2	
	16	$(A^4BA^2B)^2$	-2/5	$(A^{2n+2}BA^{2n}B)^2$*
	16	$(A^3B)^4$	-1/2, -1/4	powers

B’s	String Length	Product	X-Value	Why?
	18	$A^{11}BA^3B^3$	-1/3	
	18	$A^7BA^4BA^3B^2$	-1/3	
	18	$(A^5B)^2A^3BAB$	-1/3	
	20	$(A^6BA^2B)^2$	-9/16, -1/4	$(A^{2n+4}BA^{2n}B)^2$*
	20	$(A^5BA^3B)^2$	-3/8	powers
	20	$(A^4B)^4$	-4/7, -2/9	powers
	24	$A^{11}BA^2BA^5\ BA^2B$	-1/5	
	24	$A^9BA^4BA^3BA^4B$	-1/3	
	24	$A^7B(A^4B)^3$	-2/7	
	24	$(A^6BA^4B)^2$	-5/13	$(A^{2n+2}BA^{2n}B)^2$*
	24	$(A^5B)^4$	-3/7, -1/3	powers
5	5	B^5	½	powers
	16	$A^6BA^3B^3A^2B$	-1/2	
	16	$(A^3B)^3A^2B^2$	-1/2	
	18	$A^7BA^3BA^3B^3$	-1/3	
	18	$A^5B(A^3BAB)^2$	-1/3	
	20	$A^6BA^2BA^5B^2A^2B$	-1/4	
	20	$(A^5B)^3B^2$	-1/4	
	20	$(A^3B)^5$	-1/4	powers
	24	$A^{11}BA^2BA^3BA^2BAB$	-3/13	
	24	$A^9BA^7BA^3B^3$	-1/3	
	24	$A^9BA^5B^3A^5B$	-1/5	
	24	$A^9BA^4BA^3B^2A^3B$	-1/3	
	24	$A^9BA^2BA^5BA^2BAB$	-1/5	
	24	$A^8BA^5B(A^2B)^3$	-1/2	
	24	$A^7BA^4BA^5B^2A^3B$	-1/3	
	24	$A^7BA^4BA^3BA^4BAB$	-1/3	
	24	$A^7B(A^3B)^4$	-1/3	
	24	$A^6BA^3BA^5BA^3BA^2B$	-1/2	
	24	$(A^5B)^3A^3BAB$	-1/3	
6	6	B^6	-1, ½	powers
	9	$(AB^2)^3$	-1, 1/3	powers
	12	$(A^2BAB^2)^2$	-1, 1/5	powers
	12	$(AB)^6$	0, -2/3	powers
	15	$(A^2BAB)^3$	0, -4/7	powers
	16	$A^6BA^2B^4A^2B$	-1/2	
	16	$A^5BA^2B^3A^3B^2$	-1/2	

B's	String Length	Product	X-Value	Why?
	16	$(A^3B)^2(A^2B^2)^2$	-1/2	
	16	$(A^3BA^2B^2)^2$	-1/2	powers
	18	$A^4BA^3BA^4B^2AB^2$	-1/3	
	18	$(A^3BAB)^3$	-1/3	powers
	20	$A^5BA^2BA^2B^2A^5B^2$	-1/10	
	20	$(A^5BA^2B^2)^2$	-1/4	
	24	$A^9BA^6B^4A^3B$	-1/2	
	24	$A^9BA^3BA^3BA^3B^3$	-1/3	
	24	$A^8BA^3BA^2BA^3B^2A^2B$	-1/2	
	24	$A^7BA^4BA^3B^2A^3BAB$	-1/3	
	24	$A^7BA^3B^3A^7BAB$	-1/3	
	24	$A^6BA^3BA^2BA^2BA^3BA^2B$	-1/2	
	24	$A^6BA^2BA^3BA^5B^2A^2B$	-1/2	
	24	$(A^5B)^2(A^3BAB)^2$	-1/3	
	24	$(A^5B)^2(A^2B)^4$	-1/2	
	24	$A^5BA^3BA^2BA^5B^2A^3B$	-1/2	
	24	$(A^4BA^3BA^2B)^2$	-7/11, -1/7	
	24	$(A^3B)^6$	-1/2, -1/4	powers
7	7	B^7	½	powers
	16	$A^5BA^2B^4A^2B^2$	-1/2	
	16	$A^3B(A^2B^2)^3$	-1/2	
	24	$A^9BA^3B^4A^3BA^2B$	-1/2	
	24	$A^9B(A^2B)^3A^2B^3$	-1/2	
	24	$A^8BA^3BAB^2ABA^4B^2$	-1/3	
	24	$A^8BA^3B^4A^6B^2$	-1/2	
	24	$A^7BA^4B(AB)^3A^2BAB$	-2/7	
	24	$A^7BA^4B^3A^4B(AB)^2$	-1/3	
	24	$A^7B(A^3B)^2(AB)^4$	-1/3	
	24	$A^7BAB(A^3B)^2A^3B^3$	-1/3	
	24	$A^7BAB^3A^4BA^5B^2$	-1/3	
	24	$A^6BA^5B^2(A^2B)^2A^2B^2$	-1/2	
	24	$A^6B(A^3B)^2A^3B^3A^2B$	-1/2	
	24	$A^6BA^3BA^2BA^3BA^3B^3$	-1/2	
	24	$A^6BA^2BA^3BA^2BA^2B^2A^2B$	-1/2	
	24	$A^6B^2(A^3B)^2A^2BA^3B^2$	-1/2	
	24	$A^5B(A^4B^2)^2ABA^3B$	-1/3	
	24	$A^5B(A^3BA^3B^2)^2$	-1/3	
	24	$A^5BA^3BA^2BA^5B^2A^2B^2$	-1/2	

B’s	String Length	Product	X-Value	Why?
	24	$A^5B(A^3BAB)^3$	-1/3	
	24	$A^5BA^2BA^3BA^2BA^3B^2A^2B$	-1/2	
	24	$A^5BA^2BA^3B(A^2B)^2A^3B^2$	-1/2	
	24	$(A^3B)^5A^2B^2$	-1/2	
8	8	B^8	½	powers
	12	$(AB^2)^4$	-1, 1/3	powers
	16	$(A^2B^2)^4$	-1/2	powers
	16	$(AB)^8$	0, -2/3	powers
	20	$(A^2BAB)^4$	-4/7, 0	powers
	24	$A^{11}BA^3B^6A^2B$	-1/2	
	24	$A^{11}BAB^2(AB)^2AB^2AB$	-1/3	
	24	$A^8BA^3B^4A^3BA^2B^2$	-1/2	
	24	$A^7B^2A^4BAB^2ABA^3B^2$	-1/3	
	24	$A^6B(A^3B)^2A^2B^4A^2B$	-1/2	
	24	$A^6BA^3BA^2B^2A^3B^3A^2B$	-1/2	
	24	$A^6BA^3B^3A^2BA^3B^2A^2B$	-1/2	
	24	$A^6BA^2B^2A^3BA^3B^3A^2B$	-1/2	
	24	$(A^6BA^2B^3)^2$	-1/2	
	24	$(A^6BABAB^2)^2$	-1/17	
	24	$(A^5B)^2A^2B^2A^2BA^2B^3$	-1/2	
	24	$(A^5B)^2A^2B^2A^2B^3A^2B$	-1/2	
	24	$A^5BA^5B^2A^3B(AB)^2AB^2$	-1/3	
	24	$A^5BA^3B(A^2B^2A^2B)^2$	-1/2	
	24	$A^5BA^2BA^3BA^2B(A^2B^2)^2$	-1/2	
	24	$A^5BA^2B^2A^2BA^5BA^2B^3$	-1/2	
	24	$A^5BA^2B^2A^2BA^3BA^2B^2A^2B$	-1/2	
	24	$A^5BA^2B^3(A^3B)^2A^3B^2$	-1/2	
	24	$(A^4B^2)^4$	-1/9	
	24	$(A^3B)^4(A^2B^2)^2$	-1/2	
	24	$(A^3B)^3A^2B^2A^3BA^2B^2$	-1/2	
	24	$(A^3B)^2A^2B^2(A^3B)^2A^2B^2$	-1/2	
	24	$(A^3BAB)^4$	-1/3	powers
9	9	B^9	-1, ½	powers
	18	$(A^2BAB^2)^3$	-1, 1/5	powers
	24	$A^{11}B^2A^2BA^2B^6$	-1/2	
	24	$A^8BA^3B^6(A^2B)^2$	-1/2	
	24	$A^6BA^3BA^2B^2A^2B^4A^2B$	-1/2	

B's	String Length	Product	X-Value	Why?
	24	$A^6BA^2B^2A^3BA^2B^4A^2B$	-1/2	
	24	$A^6B(A^2B^2)^2A^3B^3A^2B$	-1/2	
	24	$A^6BA^2B^4A^5BA^2B^3$	-1/2	
	24	$A^5BA^3BA^2B^3A^5B^4$	-1/2	
	24	$A^5BA^3B^3A^3BA^2BA^2B^3$	-1/2	
	24	$A^5BA^2B^2A^3BA^2B^3A^3B^2$	-1/2	
	24	$A^5BA^2B^3A^3BA^3B^2A^2B^2$	-1/2	
	24	$A^5BA^2B^3A^3B^2A^2BA^3B^2$	-1/2	
	24	$A^5BA^2B^4A^2BA^3BA^3B^2$	-1/2	
	24	$(A^3B)^3(A^2B^2)^3$	-1/2	
	24	$(A^3B)^2A^2B^2A^3B(A^2B^2)^2$	-1/2	
	24	$(A^3B)^2(A^2B^2)^2A^3BA^2B^2$	-1/2	
	24	$(A^3BA^2B^2)^3$	-1/2	
10	10	B^{10}	½	powers
	15	$(AB^2)^5$	-1, 1/3	powers
	20	$(AB)^{10}$	-2/3, 0	powers
	24	$A^8BA^2B^6A^2BA^2B^2$	-1/2	
	24	$A^6B(A^2B^2)^2A^2B^4A^2B$	-1/2	
	24	$A^5BA^2B^2A^3BA^2B^4A^2B^2$	-1/2	
	24	$A^5BA^2B^3A^3B^2(A^2B^2)^2$	-1/2	
	24	$(A^5BA^2B^4)^2$	-1/2	
	24	$A^5BA^2B^4A^2BA^3B^2A^2B^2$	-1/2	
	24	$A^5BA^2B^4A^2B^2A^2BA^3B^2$	-1/2	
	24	$A^5B^2A^3BA^3B^3A^3B^4$	-1/2	
	24	$A^5B^2A^2BA^3B^4A^2BA^2B^2$	-1/2	
	24	$(A^3B)^2(A^2B^2)^4$	-1/2	
	24	$A^3B(A^2B)^2A^2B^2A^3B^3A^2B^2$	-1/2	
	24	$A^3BA^2B^2A^3B(A^2B^2)^3$	-1/2	
	24	$[A^3B(A^2B^2)^2]^2$	-1/2	
11	11	B^{10}	½	powers
	24	$A^{11}B^4A^2B^7$	-1/2	
	24	$A^6B^4A^5B^3A^2B^4$	-1/2	
	24	$A^5BA^2B^4(A^2B^2)^3$	-1/2	
	24	$A^3B(A^2B^2)^5$	-1/2	
12	12	B^{12}	-1, ½	powers
	18	$(AB^2)^6$	-1, 1/3	powers

B's	String Length	Product	X-Value	Why?
	24	$(A^2BAB^2)^4$	1/5	powers
	24	$(A^2B^2)^6$	-1/2	powers
	24	$(AB)^{12}$	-2/3, 0	powers
13	13	B^{13}	½	powers
	24	$A^6B^5A^3B^2A^2B^6$	-1/5	
14	14	B^{14}	½	powers
	21	$(AB^2)^7$	-1, 1/3	powers
15	15	B^{15}	-1, ½	powers
16	16	B^{16}	½	powers
	24	$(AB^2)^8$	1/3	powers
17	17	B^{17}	½	powers
18	18	B^{18}	-1, ½	powers
19	19	B^{19}	½	powers
20	20	B^{20}	½	powers
21	21	B^{21}	-1, ½	powers
22	22	B^{22}	½	powers
23	23	B^{23}	½	powers
24	24	B^{24}	½	powers

* See [2]

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