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BIJECTIONS RELATED TO STATISTICS ON WORDS

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Combinatorial proofs of the identities

$$\sum_{w\in M} q^{\mathrm{Inv}(w)} = \begin{bmatrix} n_1+n_2+\cdots+n_k \\ n_1, n_2, \ldots, n_k \end{bmatrix} = \sum_{w\in M} q^{Z(w)}$$

are given and bijections are constructed between the sets

$$\{w\in M \mid \mathrm{Inv}(w)=m\}, \qquad \{w\in M \mid \mathrm{Maj}(w)=m\}, \qquad \{w\in M \mid Z(w)=m\},$$

where M is the collection of all multiset permutations with n_1 1's, n_2 2's, $\ldots$, n_k k's, $\mathrm{Inv}(w)$ is the inversion number of w, $\mathrm{Maj}(w)$ is its major index and $Z(w)$ is the z-index of w.

1. Introduction

Let $M = M(n_1, n_2, \ldots, n_k)$ be the collection of all multiset permutations with n_1 1's, n_2 2's, $\ldots$, n_k k's. If $w \in M$, write $w = a_1 a_2 \cdots a_n$, where $n = n_1 + n_2 + \cdots + n_k$. The inversion number, $\mathrm{Inv}(w)$, is the number of ordered pairs (i, j) such that $1 \leqslant i < j \leqslant n$ and $a_i > a_j$. The major index, $\mathrm{Maj}(w)$, is the sum of all j such that $a_j > a_{j+1}$.

In [9], MacMahon showed that

$$\sum q^{\mathrm{Inv}(w)} = \sum q^{\mathrm{Maj}(w)}, \tag{1.1}$$

the summation ranging over all $w \in M$. As a consequence he derived that for fixed m,

$$|\{w\in M \mid \mathrm{Inv}(w)=m\}| = |\{w\in M \mid \mathrm{Maj}(w)=m\}|, \tag{1.2}$$

MacMahon's proof of (1.1) was indirect, he showed that each side of (1.1) was in fact equal to the q-multinomial coefficient

$$\begin{bmatrix} n \\ n_1, n_2, \ldots, n_k \end{bmatrix}_q .$$

To do this he used an inductive argument for $\mathrm{Inv}(w)$ in [9] and a clever combinatorial argument for $\mathrm{Maj}(w)$ in [8]. Foata gave a direct combinatorial proof of (1.2) in [3].

In their proof of Andrews' q-Dyson conjecture, Zeilberger and Bressoud [2] introduced a new statistic on M, $Z(w)$, which satisfies the same generating function relation as Maj and Inv,

$$\sum q^{Z(w)} = \begin{bmatrix} n \\ n_1, n_2, \ldots, n_k \end{bmatrix}_q . \tag{1.3}$$

They proved (1.3) by induction and asked for a combinatorial proof similar to Foata's for (1.2).

In this paper we give a combinatorial proof of (1.3) and use it to construct bijections between the three sets

$$\begin{aligned} &\{w \in M \mid Z(w) = m\}, \qquad \{w \in M \mid \mathrm{Inv}(w) = m\}, \\ &\{w \in M \mid \mathrm{Maj}(w) = m\}. \end{aligned} \tag{1.4}$$

In particular, our bijection between the last two of these will differ from Foata's bijection.

This paper is organized as follows: Notation and combinatorial preliminaries are discussed in Section 2. In Section 3 we prove (1.3) and give a combinatorial proof that

$$\sum q^{\mathrm{Inv}(w)} = \begin{bmatrix} n \\ n_1, n_2, \ldots, n_k \end{bmatrix}_q . \tag{1.5}$$

Bijections between the sets in (1.4) are given in Section 4 and we conclude with some remarks in Section 5.

2. Notation and combinatorial preliminaries

If A is a statement, define $N(A)$ by

$$N(A) = \begin{cases} 1, & \text{if } A \text{ is true}, \\ 0, & \text{if } A \text{ is false}. \end{cases} \tag{2.1}$$

With this notation,

$$\mathrm{Inv}(w) = \sum_{1 \le i < j \le n} N(a_j < a_i), \tag{2.2}$$

and

$$\mathrm{Maj}(w) = \sum_{j=1}^{n-1} j N(a_{j+1} < a_j). \tag{2.3}$$

Pairs (a_i, a_j), where $i < j$ but $a_j < a_i$ are called *inversions*. Pairs (a_j, a_{j+1}), where $a_{j+1} < a_j$ are called *descents*.

Given $w \in M$, let w_{ij} be the subword of w formed by deleting all letters a_m such that $a_m \neq i$ or j. If $w = 312432314$,

$$\begin{aligned} &w_{12} = 1221, \\ &w_{13} = 31331, \qquad w_{23} = 32323, \\ &w_{14} = 1414, \qquad w_{24} = 2424, \qquad w_{34} = 34334. \end{aligned}$$

The Z-index of w is defined to be the sum of the major indices of all 2-letter subwords, w_{ij}, of w. That is,

$$Z(w)=\sum_{1\le i<j\le k}\mathrm{Maj}(w_{ij}). \tag{2.4}$$

For our example, $\mathrm{Inv}(w)=13$, $\mathrm{Maj}(w)=17$ and $Z(w)=18$.

In this paper we will follow notational conventions contained in [5] and [6] and suggested in [10]. For completeness, we include the description of these conventions here.

A partition, λ, with m parts is a nonincreasing sequence of m nonnegative integers, called parts, $\lambda=:(\lambda(1),\lambda(2),\ldots,\lambda(m))$, $\lambda(1)\ge\lambda(2)\ge\cdots\ge\lambda(m)\ge 0$. In particular we will allow parts of size 0. Given a set T of partitions and an interval I, T_k^I will denote the set of all partitions with exactly k parts, each part size in interval I. The lack of a postscript or subscript will imply no restriction on part size or number of parts. We will use P to denote the set of all partitions and PD to denote partitions with distinct parts. Given a partition λ, define $\|\lambda\|$ to be the sum of the parts of λ and $|\lambda|$ to be the number of parts in λ. Let $\emptyset$ denote the partition with no parts and set $\|\emptyset\|=0$.

Define the weight of a partition λ by $\mathrm{wt}(\lambda)=q^{\|\lambda\|}$ and the sign of λ by $\mathrm{sgn}(\lambda)=(-1)^{|\lambda|}$. If T is a set of partitions, define $\mathrm{wt}(T)$ by

$$\mathrm{wt}(T)=\sum_{\lambda\in T}\mathrm{wt}(\lambda)=\sum_{\lambda\in T}q^{\|\lambda\|}, \tag{2.5}$$

and $\mathrm{wt}(T(-1))$ by

$$\mathrm{wt}\,(T(-1))=\sum_{\lambda\in T}\mathrm{wt}(\lambda)\mathrm{sgn}\,\lambda=\sum_{\lambda\in T}q^{\|\lambda\|}(-1)^{|\lambda|}.$$

For sets S, T of partitions, $\mathrm{wt}(S\times T)=\mathrm{wt}(S)\mathrm{wt}(T)$, etc. We will make use of the facts

$$\mathrm{wt}(P_n^{[0,\infty)})=\frac{1}{(q)_n}=\mathrm{wt}(P^{[0,n)}), \tag{2.6}$$

and

$$\mathrm{wt}(\mathrm{PD}^{[1,n]}(-1))=(q)_n, \tag{2.7}$$

where $(q)_n=(1-q)(1-q^2)\cdots(1-q^n)$.

A composition, α, with m parts is a sequence of m nonnegative integers, $\alpha=:(\alpha(1),\alpha(2),\ldots,\alpha(m))$, $\alpha(i)\ge 0$.

If $\varphi:A\to B$ is a bijection of sets, we say φ is weight-preserving (WP) if for each $a\in A$, $\mathrm{wt}(a)=\mathrm{wt}(\varphi(a))$. If $\varphi:A(-1)\to A(-1)$ is a weight-preserving involution such that for all $a\in A$, $\varphi(a)=a$, or $\mathrm{sgn}(\varphi(a))=-\mathrm{sgn}(a)$, we say φ is a sign-reversing weight-preserving involution (SRWP). A WP-signed bijection $(\theta,\psi;\varphi)$ between signed sets $A(-1)$ and $B(-1)$ consists of SRWP-involutions θ and ψ on A and B with fixed sets A' and B', and a WP-bijection φ from A' to B'. If such a bijection exists, it follows that $\mathrm{wt}(A(-1))=\mathrm{wt}(B(-1))$.

3. Properties on Inv, Maj and Z

The goal of this section is to give a combinatorial proof that

$$\frac{1}{(q)_n}\sum_{w\in M} q^{S(w)}=\frac{1}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_k}},\tag{3.1}$$

where $S(w)$ is any of the three statistics Inv, Maj or Z. To do this, we define a weight function on words $w\in M$ by $\mathrm{wt}(w)=q^{S(w)}$ and produce a WP-bijection between the sets $P_n^{[0,\infty)}\times M$ and $P_n^{[0,\infty)}\times P_{n_2}^{[0,\infty)}\times\cdots\times p_{n_k}^{[0,\infty)}$. The result in (3.1) will then follow in view of (2.6).

From (3.1) it follows that

$$\sum q^{S(w)}=\begin{bmatrix} n \\ n_1, n_2, \ldots, n_k\end{bmatrix}_q=\frac{(q)_n}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_k}}.\tag{3.2}$$

In Section 4 we will combinatorially derive (3.2) from (3.1) and use the involution principle of Garsia and Milne [4] to derive bijections between the sets in (1.4).

Given $\lambda\in P_n^{[0,\infty)}$ and $w\in M$, construct partitions $\lambda_1', \lambda_2', \ldots, \lambda_k'$, where $\lambda_i'\in P_{n_i}^{[0,\infty)}$ by using w to sort λ. The procedure for sorting λ according to w is to place λ below w in a two-line notation,

$$\begin{pmatrix} a_1 & a_2 & a_2 & \cdots & a_n \\ \lambda(1) & \lambda(2) & \lambda(3) & \cdots & \lambda(n)\end{pmatrix},$$

and then to define λ_m' to be the parition consisting of all those parts of λ below letters a_i with $a_i=m$. Thus if $\lambda=443222100$ and $w=312432314$, we have

$$\begin{pmatrix} 3&1&2&4&3&2&3&1&4 \\ 4&4&3&2&2&2&1&0&0\end{pmatrix}\Rightarrow(40, 32, 421, 20).$$

Clearly, $\lambda_i'\in P_{n_i}^{[0,\infty)}$ as desired, and $\|\lambda\|=\sum_{i=1}^{k}\|\lambda_i'\|$.

For each of the statistics S, we now associate a k-tuple of partitions based on w. For Inv, define $(\pi_1, \pi_2, \ldots, \pi_k)$ by

$$\begin{aligned}\pi_i(j) = {} & \text{the number of letters } a_l \text{ to the right}\\ & \text{of the } j\text{th } i \text{ in } w \text{ for which, } a_l<i.\end{aligned}\tag{3.3}$$

for example, $312432314\Rightarrow(00, 11, 421, 40)$. For Maj, let $(\mu_1, \ldots, \mu_k)$ be defined by

$$\begin{aligned}\mu_i(j) = {} & \text{the number of descents that occur}\\ & \text{in } w \text{ to the right of the } j\text{th } i.\end{aligned}\tag{3.4}$$

For 312432314, the descents to the right of the second 3 are (32) and (31) so $\mu_3(2)=2$ and $(\mu_1, \mu_2, \mu_3, \mu_4)=(30, 31, 421, 30)$. Finally, for Z, let

$(\nu_1, \nu_2, \ldots, \nu_k)$ be defined by

$$\begin{aligned} &\text{(a)} \quad \nu_i(n_i) = \text{the number of distinct letters to the right of the } n_i\text{th } i \text{ in } w \text{ which are smaller than } i. \\ &\text{(b)} \quad \nu_i(j) - \nu_i(j+1) = \text{the number of distinct letters in } w \text{ between the } j\text{th } i \text{ and the } (j+1)\text{st } i,\ 1 \leqslant j \leqslant n_i - 1. \end{aligned} \tag{3.5}$$

In this case, for $w = 312432314$, $\nu_1(1) - \nu_1(2) = 3$, $\nu_1(2) = 0$ so $\nu_1 = 30$; $\nu_2(1) - \nu_2(2) = 2$, $\nu_2(2) = 1$ so $\nu_2 = 31$, etc. and $(\nu_1, \nu_2, \nu_3, \nu_4) = (30, 31, 521, 30)$.

Lemma 3.6. *With π, μ and ν described as above,*

(a) $\operatorname{Inv}(w) = \|\pi_1\| + \cdots + \|\pi_k\|$,

(b) $\operatorname{Maj}(w) = \|\mu_1\| + \cdots + \|\mu_k\|$,

(c) $Z(w) = \|\nu_1\| + \cdots + \|\nu_k\|$.

Proof. Parts (a) and (b) follow from a straightforward calculation.

In case (b) for example,

$$\begin{aligned} \sum_{i=1}^{k} \|\nu_i\| &= \sum_{i=1}^{k} \sum_{j=1}^{n_i} \mu_i(j) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{n_i} (\text{number of descents to the right of the } j\text{th } i \text{ in } w) \\ &= \sum_{l=1}^{n-1} (\text{number of descents to the right of } a_l) \\ &= \sum_{l=1}^{n-1} \sum_{j=l}^{n-1} N(a_j > a_{j+1}) = \sum_{j=1}^{n-1} jN(a_j > a_{j+1}) \\ &= \operatorname{Maj}(w) \quad \text{as desired.} \end{aligned}$$

Part (a) is similar but less complicated. For part (c),

$$Z(w) = \sum_{1 \leqslant i < j \leqslant k} \operatorname{Maj}(w_{ij}).$$

With the binary word w_{ij}, associate a pair of partitions (μ_{ij}, μ_{ji}) as in the discussion preceding the lemma. By part (b), $\|\mu_{ij}\| + \|\mu_{ji}\| = \operatorname{Maj}(w_{ij})$. The partition μ_{ij} may also be defined by the conditions

$$\begin{aligned} &\text{(a)} \quad \mu_{ij}(n_i) = \begin{cases} 1, & \text{if the last } i \text{ in } w_{ij} \text{ is followed by at least one } j \text{ and } j < i, \\ 0, & \text{otherwise} \end{cases} \\ &\text{(b)} \quad \mu_{ij}(l) - \mu_{ij}(l+1) = \begin{cases} 1, & \text{if there is at least one } j \text{ between the } l\text{th } i \text{ and the } (l+1)\text{st } i \text{ in } w_{ij}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.7}$$

The condition (3.7) holds because in a binary word, these are precisely the conditions needed to give rise to descents.

Now,

$$Z(w)=\sum_{1\leq i<j\leq k}\mathrm{Maj}(w_{ij})=\sum_{1\leq i<j\leq k}(\|\mu_{ij}\|+\|\mu_{ji}\|)=\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k}\|\mu_{ij}\|.$$

From (3.7) it follows that

$$\sum_{\substack{j=1\\ j\neq i}}^{k}(\mu_{ij}(l)-\mu_{ij}(l+1))=v_i(l)-v_i(l+1),$$

for $1\leq l\leq n_i-1$, and

$$\sum_{\substack{j=1\\ j\neq i}}^{k}\mu_{ij}(n_i)=v_i(v_i).$$

Consequently,

$$\sum_{\substack{j=1\\ j\neq i}}^{k}\mu_{ij}(l)=v_i(l)\quad\text{for all } i \text{ and } l.$$

Hence,

$$\begin{aligned}Z(w)&=\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k}\|\mu_{ij}\|=\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k}\sum_{l=1}^{n_i}\mu_{ij}(l)\\&=\sum_{i=1}^{k}\sum_{l=1}^{n_i}v_i(l)=\sum_{i=1}^{k}\|v_i\|\end{aligned}$$

as desired. This completes the proof of the lemma. □

Corollary 3.8. *Define maps Φ_1, Φ_2 and Φ_3 from $P_n^{[0,\infty)}\times M$ to $P_{n_1}^{[0,\infty)}\times P_{n_2}^{[0,\infty)}\times\cdots\times P_{n_k}^{[0,\infty)}$ by*

(i) $\Phi_1(\lambda, w)=(\lambda_1'+\pi_1,\ldots,\lambda_k'+\pi_k)$,

(ii) $\Phi_2(\lambda, w)=(\lambda_1'+\mu_1,\ldots,\lambda_k'+\mu_k)$,

(iii) $\Phi_3(\lambda, w)=(\lambda_1'+v_1,\ldots,\lambda_k'+v_k)$,

where the addition of partitions is done componentwise. Then $\mathrm{wt}(\lambda, w)=\mathrm{wt}(\Phi_i(\lambda, w))$ *for $i=1$, 2 or 3. That is, each Φ_i is a weight preserving map.*

To establish the result in (3.1), we must show that each Φ_i is in fact a bijection. We will do this by constructing inverse maps Ψ_1, Ψ_2 and Ψ_3 from $P_{n_1}^{[0,\infty)}\times\cdots\times P_{n_k}^{[0,\infty)}$ to $P_n^{[0,\infty)}\times M$. Our proof of (3.1) for the case $S(w)=\mathrm{Maj}(w)$ is due to MacMahon. He gave a map equivalent to our Φ_2 and its inverse Ψ_2 in [8]. See [1] for an account in notation similar to this paper. We will restrict our attention to Inv and Z in what follows.

Lemma 3.9. *Given $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in P_{n_1}^{[0,\infty)} \times \cdots \times P_{n_k}^{[0,\infty)}$, there is a unique word $\tilde{w}$ on the alphabet $\{0, 1, \ldots, k\}$ such that*

(i) *$\tilde{w}$ contains n_1 1's, ..., n_k k's,*

(ii) *$\tilde{w}$ begins with a nonzero letter,*

(iii) *$\lambda_i(j)$ = the number of letters to the right of the jth i in $\tilde{w}$ which are smaller than i.*

Proof. For the uniqueness of $\tilde{w}$, suppose $\tilde{w}_1$ and $\tilde{w}_2$ are two different words satisfying properties (i)–(iii). Find the first place reading from right to left at which $\tilde{w}_1$ and $\tilde{w}_2$ disagree—say this occurs in the $(i+1)$st letter from the end. We can write $\tilde{w}_1 = a_1a_2 \cdots a_{m_1-i}a_{m_1-i+1} \cdots a_{m_1}$ and $\tilde{w}_2 = b_1b_2 \cdots b_{m_2-i}b_{m_2-i+1} \cdots b_{m_2}$, where $a_{m_1-i+1} = b_{m_2-i+1}, \ldots, a_{m_1} = b_{m_2}$, but $a_{m_1-i} \neq b_{m_2-i}$. We may take $a_{m_1-i} = l > b_{m_2-i}$. Suppose r l's have occurred among the letters $a_{m_i-i+1}, \ldots, a_{m_1}$. Then $\lambda_l(n_l - r)$ is the number of letters in $\tilde{w}_1$ to the right of a_{m_1-i} which are smaller than l. But in $\tilde{w}_2$, b_{m_2-i} is the right of the $(n_l - r)$th l so the number of letters in $\tilde{w}_2$ to the right of the $(n_l - r)$th l which are smaller than l is greater for $\tilde{w}_2$ than for $\tilde{w}_1$. Consequently, (iii) fails for $\tilde{w}_2$ and we have a contradiction.

The existence of $\tilde{w}$ is guaranteed by the following algorithm for constructing it.

(i) Put down place holders for the letters in $\tilde{w}$. (It is easy to see that $\max_i (\lambda_i(1) + n_i + \cdots + n_k)$ place holders are needed.)

(ii) Place a k in the position such that there are $\lambda_k(n_k)$ empty spaces between it and the right hand end.

(iii) Continue placing k's, then $(k-1)$'s, ..., 1's so that n_i i's are contained in $\tilde{w}$. Place the jth i, so that it comes $\lambda_i(n_i - j + 1)$ empty spaces from the right hand end of $\tilde{w}$.

(iv) Fill all remaining empty spaces with 0's.

Clearly the word $\tilde{w}$ found in this way satisfies properties (i)–(iii). □

As an example, let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (31, 531, 95, 42)$. For this case, $\max_i (\lambda_i(1) + n_i + \cdots + n_k) = 13$ so $\tilde{w}$ will contain 13 letters. We place 4's first:

$$\underline{\ }\,\underline{\ }\,\underline{\ }\,\underline{\ }\,\underline{\ }\,\underline{\ }\,\underline{\ }\,\underline{4}\,\underline{\ }\,\underline{\ }\,\underline{4}\,\underline{\ }\,\underline{\ }\,,$$

leaving two empty spaces to the right of the first 4 and four to the right of the second 4. Next, we place 3's:

$$\underline{3}\,\underline{\ }\,\underline{\ }\,\underline{\ }\,\underline{\ }\,\underline{3}\,\underline{4}\,\underline{\ }\,\underline{\ }\,\underline{4}\,\underline{\ }\,\underline{\ }\,,$$

then 2's:

$$\underline{3}\,\underline{\ }\,\underline{2}\,\underline{\ }\,\underline{\ }\,\underline{3}\,\underline{2}\,\underline{4}\,\underline{\ }\,\underline{\ }\,\underline{4}\,\underline{2}\,\underline{\ }\,,$$

then the 1's:

$$\underline{3}\,\underline{\ }\,\underline{2}\,\underline{1}\,\underline{\ }\,\underline{3}\,\underline{2}\,\underline{4}\,\underline{\ }\,\underline{1}\,\underline{4}\,\underline{2}\,\underline{\ }\,,$$

and finally 0's to obtain $\tilde{w} = 3021032401420$.

Define a map $\Psi_1 : P_{n_1}^{[0,\infty)} \times \cdots \times P_{n_k}^{[0,\infty)} \to P_n^{[0,\infty)} \times M$ as follows: given $(\lambda_1, \ldots, \lambda_k)$, form $\tilde{w}$ as in Lemma 3.9. Let $w = a_1 a_2 \cdots a_n$, where a_i is the ith positive letter in $\tilde{w}$ and define λ by $\lambda(i)$ = the number of zeros in $\tilde{w}$ to the right of a_i. for example, given $(31, 531, 95, 42)$, $\tilde{w} = 3021032401420$ so $w = 321324142$ and $\lambda = 433222111$.

Lemma 3.10. *With Ψ_1 defined as above, $\Psi_1 \circ \Phi_1 = \mathrm{id}$ on $P_n^{[0,\infty)} \times M$ and $\Phi_1 \circ \Psi_1 = \mathrm{id}$ on $P_{n_1}^{[0,\infty)} \times \cdots \times P_{n_k}^{[0,\infty)}$. Thus Φ_1 is a bijection.*

Proof. Given $(\lambda, w) \in P_n^{[0,\infty)} \times M$, we must show that

$$\Psi_1(\Phi_1(\lambda, w)) = (\lambda, w).$$

Construct a word $\tilde{w}$ as follows: place $\lambda(n)$ zeros to the right of a_n, $\lambda(n-1) - \lambda(n)$ zeros between a_{n-1} and $a_n, \ldots, \lambda(1) - \lambda(2)$ zeros between a_1 and a_2. Then $\tilde{w}'$ contains n_i i's for $1 \leqslant i \leqslant k$ and $\tilde{w}'$ begins with a positive letter.

Now suppoe the jth i in w is a_m. By construction, the number of zero to the right of a_m in $\tilde{w}'$ is $\lambda(m)$. Consequently, the number of letters to the right of a_m in $\tilde{w}'$ which are smaller than i is $\lambda(m) - \pi_i(j) = \lambda_i'(j) + \pi_i(j) = \lambda_i(j)$, where $\Phi_1(\lambda, w) = (\lambda_1, \ldots, \lambda_k)$. Thus by uniqueness, $\tilde{w}' = \tilde{w}$ and $\Psi(\lambda_1, \ldots, \lambda_k) = (\lambda, w)$ as desired. The proof that $\Phi_1 \circ \Psi_1 = \mathrm{id}$ is similar. $\square$

As a corollary to Lemma 3.10, we have established (3.1) for the case $S(w) = \mathrm{Inv}(w)$. Finally, we must produce an inverse, Ψ_3, for Φ_3. We proceed as we did for Inv.

Lemma 3.11. *Given a k-tuple of compositions, $(\alpha_1, \ldots, \alpha_k)$, where α_i has n_i parts, there is a unique integer $t \geqslant 0$ and a unique multiset permutation $\tilde{w}$ on the alphabet $\{-t, -(t-1), \ldots, -1, 1, 2, \ldots, k\}$ such that:*

(i) *$\tilde{w}$ begins with a positive letter,*
(ii) *for each $i > 0$, i occurs in $\tilde{w}$ n_i times,*
(iii) *the letters $-t, -(t-1), \ldots, -1$ each occur once in $\tilde{w}$ and in increasing order,*
(iv) *for $1 \leqslant j \leqslant n_i - 1$, the number of distinct letters between the* jth *i and the* $(j+1)$st *i is $\alpha_i(j)$,*
(v) *the number of distinct letters between the* n_ith *i and the end of $\tilde{w}$ which are smaller than i is $\alpha_i(n_i)$.*

Proof. As was the case with Lemma 3.9, we show that two different words cannot satisfy (i)–(v) and then produce an algorithm for constructing $\tilde{w}$. Suppose $\tilde{w}$ and $\tilde{w}'$ are two different words which satisfy (i)–(v). The last positive letter in $\tilde{w}$ and $\tilde{w}'$ is uniquely determined. In fact, if r is the smallest of the numbers $\alpha_i(n_i)$, let l be the largest number such that $\alpha_l(n_l) = r$. Then $\tilde{w}$ and $\tilde{w}'$ must both end with the string $l, -r, -(r-1), \ldots, -1$ by condition (v). As in Lemma 3.9, suppose $\tilde{w} = a_1 \cdots a_{m_1-i} a_{m_1-i+1} \cdots a_{m_1}$ and $\tilde{w}' = b_1 \cdots b_{m_2-i} b_{m_2-i+1} \cdots b_{m_2}$, where $a_{m_1-i} > b_{m_2-i}$ but $a_{m_1-i+1} = b_{m_2-i+1}, \ldots, a_{m_1} = b_{m_2}$. We have three cases to consider.

Case 1. a_{m_1-i} and b_{m_2-i} are both negative letters

Clearly condition (iii) cannot be satisfied for both $\tilde{w}$ and $\tilde{w}'$.

Case 2. a_{m_1-i} is positive but b_{m_2-i} is negative

Let $a_{m_1-i} = l$ and say j l's have occured to the right of a_{m_1-i}. Since each negative letter occurs only once in $\tilde{w}$, the number of distinct letters between the $(n_l - j)$th l and the $(n_l - j + 1)$st l will be different in $\tilde{w}$ and $\tilde{w}'$ violating condition (iv) or condition (v).

Case 3. Both a_{m_1-i} and b_{m_2-i} are positive

If $a_{m_1-i} = l_1$ and $b_{m_2-i} = l_2$, find the closest l_1 or l_2 to this position which is to the right of a_{m_1-i}. If this letter is an l_1, then between the $(n_{l_1} - j)$th l_1 and $(n_{l_1} - j + 1)$st l_1 there will not be an l_2 in $\tilde{w}$ but there will be in $\tilde{w}'$ contradicting condition (iv). Similarly, there is a contradiction if this letter is an l_2. Finally, if there is no l_1 or l_2 to the right of a_{m_1-i}, condition (v) will fail for $\tilde{w}$ or $\tilde{w}'$. This exhausts the cases.

To establish the existence of $\tilde{w}$, we give the following algorithm for constructing it. Given $(\alpha_1, \alpha_2, \ldots, \alpha_k)$,

(i) Put down place holders for the letters of $\tilde{w}$.

(ii) (Initialization) For $i = 1, \ldots, k$, place an i underneath the $(\alpha_i(n_i) + 1)$st position counting from right to left.

(iii) (Initial placement) Place in the right most position of $\tilde{w}$:

$$\begin{cases} (-1), & \text{if there is no letter below this position,} \\ i, & \text{if } i \text{ is the largest letter listed below this position.} \end{cases}$$

(iv) (Adjustments) After placing a letter, i, in a position, shift one space to the left

(a) every other letter below this position,

(b) every letter $j < i$ which has not occurred previously in $\tilde{w}$,

(c) any letter j which has appeared in $\tilde{w}$ such that more than one i has been placed since the last j.

If the lth i has just been placed, put another i (if $l < n_i$) under the position $\alpha_i(n_i - l) + 1$ spaces to the left of the given one.

(v) (Next placement) Place a latter in the next empty spot according to the rules:

(a) If there are no letters below the given spot, place $-(m+1)$, where $-m$ was the last negative letter placed in $\tilde{w}$.

(b) Of the entries below the spot, place the one which was most recently placed earlier. If none of the entries has previously been placed, the largest available letter should be placed.

If all positive letters have been used, stop. Otherwise, go to step (iv). □

For example, if $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (23, 31, 620, 63)$, we put down place holders

```
-- -- -- -- -- -- -- -- -- -- -- -- -- .
                            1     2  3
                            4
```

In step (iii), the 3 is placed in the last position. After making the adjustment from step (iv) we have:

```
                                     3
-- -- -- -- -- -- -- -- -- -- -- -- -- .
                         1  4  2
                            3
```

There is nothing below the second spot so its gets filled by (-1). No adjustments are necessary so 2 fills the next spot and after adjustments,

```
                               2 -1  3
-- -- -- -- -- -- -- -- -- -- -- -- -- .
                   2  1     4
                            3
```

The 3 has previously occurred so it gets placed next. Continuing, we have:

```
                            3  2 -1  3
-- -- -- -- -- -- -- -- -- -- -- -- -- ,
       3           2     4
                   1

                         4  3  2 -1  3
-- -- -- -- -- -- -- -- -- -- -- -- -- ,
    4  3        1  2

                   2 -2  4  3  2 -1  3
-- -- -- -- -- -- -- -- -- -- -- -- -- ,
    4  3     1

             1 -3  2 -2  4  3  2 -1  3
-- -- -- -- -- -- -- -- -- -- -- -- -- ,
    4  3
    1

       3 -4  1 -3  2 -2  4  3  2 -1  3
-- -- -- -- -- -- -- -- -- -- -- -- -- ,
    4
    1
```

and finally,

```
 4  1  3 -4  1 -3  2 -2  4  3  2 -1  3
-- -- -- -- -- -- -- -- -- -- -- -- -- .
```

We can now define $\Psi_3 : P_{n_1}^{[0,\infty)} \times \cdots \times P_{n_k}^{[0,\infty)} \to P_n^{[0,\infty)} \times M$. Given $(\lambda_1, \ldots, \lambda_k)$, form a k-tuple of compositions, $(\alpha_1, \ldots, \alpha_k)$ by setting $\alpha_i(n_i) = \lambda_i(n_i)$, $\alpha_i(j) =$

$\lambda_i(j)-\lambda_i(j+1)$ for $1\le j\le n_i-1$. Construct the word $\tilde{w}$ associated with $(\alpha_1,\ldots,\alpha_k)$ in Lemma 3.11. Let $w=a_1\cdots a_n$, where a_i is the ith positive letter in $\tilde{w}$. Define $\Psi_3(\lambda_1,\ldots,\lambda_k)$ to be (λ, w), with $\lambda=\lambda(1)\lambda(2)\cdots\lambda(n)$, where $\lambda(i)$ is the absolute value of the first negative letter to the right of a_i in $\tilde{w}$. Thus, if $(\lambda_1,\ldots,\lambda_k)=(53, 41, 820, 93)$, $(\alpha_1,\ldots,\alpha_k)=(23, 31, 620, 63)$. By the previous example, $\tilde{w}=413\text{-}41\text{-}32\text{-}2432\text{-}13$, $w=413124323$, and $\lambda=444321110$. So $\Psi_3(53, 41, 820, 93)=(444321110, 413124323)$.

Lemma 3.12. *With Ψ_3 as above, $\Psi_3\circ\Phi_3=\mathrm{id}$ on $P_n^{[0,\infty)}\times M$ and $\Phi_3\circ\Psi_3=\mathrm{id}$ on $P_{n_1}^{[0,\infty)}\times\cdots\times P_{n_k}^{[0,\infty)}$. Thus, Φ_3 is a bijection.*

The proof of Lemma 3.12 is similar to that of Lemma 3.10.

4. Bijections between words with a fixed statistic

Having verified (3.1) for $S=\mathrm{Maj}$, Inv and Z, we now give a signed bijection to show that

$$\sum_{w\in M} q^{S(w)}=\begin{bmatrix} n \\ n_1, n_2,\ldots,n_k\end{bmatrix}_q. \tag{4.1}$$

For $\Phi=\Phi_1$, Φ_2 or Φ_3, extend $\Phi: P_n^{[0,\infty)}\times M\to P_{n_1}^{[0,\infty)}\times\cdots\times P_{n_k}^{[0,\infty)}$ to

$\mathrm{id}\times\Phi:\mathrm{PD}^{[1,n]}(-1)\times P_n^{[0,\infty)}\times M\to\mathrm{PD}^{[1,n]}(-1)\times P_{n_1}^{[0,\infty)}\times\cdots\times P_{n_k}^{[0,\infty)}$.

By (2.7) and (2.6),

$$\mathrm{wt}(\mathrm{PD}^{[1,n]}(-1)\times P_{n_1}^{[0,\infty)}\times\cdots\times P_{n_k}^{[0,\infty)})=\frac{(q)_n}{(q)_{n_1}\cdots(q)_{n_k}}=\begin{bmatrix} n \\ n_1,\ldots,n_k\end{bmatrix}_q.$$

Lemma 4.2. *There is a signed bijection $\mathrm{PD}^{[1,n]}(-1)\times P^{[1,n]}\leftrightarrow\{(\emptyset,\emptyset)\}$.*

Proof. We must construct a SRWP-involution, ι, with only $(\emptyset,\emptyset)$ as a fixed point. We define ι as follows: given $(\lambda,\mu)\in\mathrm{PD}^{[1,n]}(-1)\times P^{[1,n]}$, select the smallest part from among the parts of λ and μ. If this part belongs to λ, move it to μ. If it does not belong to λ, move it to λ. Clearly this map has the desired properties. □

Now let t be the WP-bijection between $P_n^{[0,\infty)}$ and $P^{[1,n]}$ defined as follows: Given $\lambda\in P_n^{[0,\infty)}$, let λ' be the partition obtained by deleting all parts of size 0 from λ. Then $t(\lambda)$ is the partition corresponding to the transpose of the Ferrer's diagram for λ'. For the inverse, given $\mu\in P^{[1,n]}$, let λ' be the partition corresponding to the transposition of the Ferrer's diagram for μ. Then $t^{-1}(\mu)$ is the partition obtain by adjoining enough parts of size 0 to λ' to make a total of n parts.

Since by Lemma 4.2,

$$\iota\circ(\mathrm{id}\times t):\mathrm{PD}^{[1,n]}(-1)\times P_n^{[0,\infty)}\leftrightarrow\{(\emptyset,\emptyset)\},$$

it follows that

$$\iota\circ(\mathrm{id}\times t\times\mathrm{id})\circ(\mathrm{id}\times\Phi^{-1}):\mathrm{PD}^{[1,n]}(-1)\times P_{n_1}^{[0,\infty)}\times P_{n_1}^{[0,\infty)}\times\cdots\times P_{n_k}^{[0,\infty)}$$
$$\leftrightarrow\emptyset\times\emptyset\times M\leftrightarrow M,$$

as desired.

Theorem 4.3. *There are bijections, denoted* IM, IZ, MI, MZ, ZI *and* ZM *from* $M(n_1,\ldots,n_k)$ *to itself with the property that if* $S_1(w)=m$, *then* $S_2(S_1S_2(w))=m$, *where* S_i *is any of* Inv, Maj *or* Z.

Proof. The map

$$T_i:\mathrm{PD}^{[1,n]}(-1)\times P_{n_1}^{[0,\infty)}\times\cdots\times P_{n_k}^{[0,\infty)}\to\mathrm{PD}^{[1,n]}(-1)\times P_{n_1}^{[0,\infty)}\times\cdots\times P_{n_k}^{[0,\infty)}$$

defined by

$$T_i=(\mathrm{id}\times\Phi_i)\circ(\mathrm{id}\times t\times\mathrm{id})\circ(\iota\times\mathrm{id})\circ(\mathrm{id}\times t\times\mathrm{id})\circ(\mathrm{id}\times\Phi_i^{-1})$$

is a SRWP-involution with fixed set $(\mathrm{id}\times\Phi_i^{-1})(\emptyset\times\emptyset\times M)$ for $i=1,2,3$, where

$$\mathrm{wt}(w)=\begin{cases}q^{\mathrm{Inv}(w)} & \text{for } i=1,\\ q^{\mathrm{Maj}(w)}, & \text{for } i=2,\\ q^{Z(w)}, & \text{for } i=3.\end{cases}$$

Since the elements in the fixed sets of T_1, T_2 and T_3 are all positive, by the involution principle of Garsia and Milne [4], there exist WP-bijections between them. These bijections extend by way of Φ_1, Φ_2 and Φ_3 to the desired WP-bijections from M to M. □

In practice, the bijections of Theorem 4.3 can be given explicitly. For example, suppose we wish to construct the bijection IZ. In this case, given a word, w, with $\mathrm{Inv}(w)=m$, we want a word $w'=\mathrm{IZ}(w)$ such that $Z(w')=m$. We have $\mathrm{id}\times\Phi_1(\emptyset,\emptyset,w)=(\emptyset,\lambda_1,\ldots,\lambda_k)$ for some partitions $\lambda_1,\ldots,\lambda_k$ with $\mathrm{Inv}(w)=\|\lambda_i\|+\cdots+\|\lambda_k\|$. If $(\emptyset,\lambda_1,\ldots,\lambda_k)$ is a fixed point of T_3 (where T_3 corresponds to the Φ_3 of Section 3), then $\mathrm{IZ}(w)=w'$, where $(\emptyset,\emptyset,w')=(\mathrm{id}\times\Phi_3^{-1})(\emptyset,\lambda_1,\ldots,\lambda_k)$. If $(\emptyset,\lambda_1,\ldots,\lambda_k)$ is not a fixed point of T_3, according to the involution principle, there is a smallest positive number, l, such that $(T_1\circ T_3)^l(\emptyset,\lambda_1,\ldots,\lambda_k)$ is a fixed point of T_3. In this case, $\mathrm{IZ}(w)=w'$, where $(\emptyset,\emptyset,w')=(\mathrm{id}\times\Phi_3^{-1})((T_1\circ T_3)^l(\emptyset,\lambda_1,\ldots,\lambda_k))$.

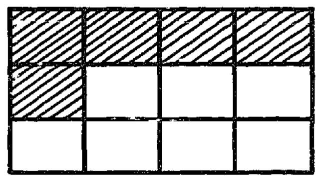
For example,let $w=2311\in M(2,1,1)$. We have $\mathrm{Inv}(2311)=4$. Also, $(\mathrm{id}\times\Phi_1)(2311)=(\emptyset;00,2,2)$, $T_3(\emptyset;00,2,2)=(2;00,1,1)$ so $(\emptyset;00,2,2)$ was not a fixed point of T_3. Now $T_1T_3(\emptyset;00,2,2)=(\emptyset;10,2,1)$ and $(T_1T_3)^2(\emptyset;00,2,2)=(\emptyset;20,1,1)$. Since $(\emptyset;20,1,1)$ is a fixed point of T_3, we have $(\mathrm{id}\times\Phi_3^{-1})\cdot(\emptyset;20,1,1)=(\emptyset,\emptyset,1231)$, and thus $\mathrm{IZ}(2311)=1231$.

5. Remarks

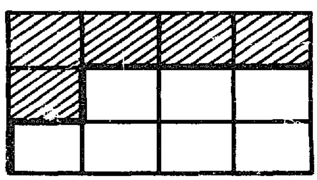
It is well known that

$$\mathrm{wt}[P_m^{[0,n]}] = \begin{bmatrix} m+n \\ n \end{bmatrix}_q. \tag{5.1}$$

The techniques of the previous section allow for a quick combinatorial proof of this fact. There is a well known WP-bijection $\varphi : P_m^{[0,n]} \to M(m,n)$, where $\mathrm{wt}(w) = \mathrm{Inv}(w)$, defined as follows: Form the Ferrer's diagram of $\lambda \in P_m^{[0,n]}$. This diagram fits inside an $m \times n$ rectangle. for example,

$4+1+0 \longleftrightarrow$ 

Now trace the path from the lower left hand corner to the upper right hand corner of this rectangle which follows the boundary of λ. For our example the path is:



Convert this path to a binary word by the rule that a vertical edge corresponds to a 0 and a horizontal edge to a 1. Thus, $4+1+0 \leftrightarrow 0101110$. Consequently, by composing this bijection with the appropriate signed bijection from Section 4 we have

$$P_m^{[0,n]} \leftrightarrow M(m,n) \leftrightarrow \mathrm{PD}^{[1,m+n]}(-1) \times P_m^{[0,\infty)} \times P_n^{[0,\infty)},$$

and (5.1) follows. The bijection φ can be found in [1]. An alternate combinatorial proof, due to Franklin can be found in [11, p. 269].

The essential idea in the algorithm of Lemma 3.9 can be found in [7, p. 12], although in this case, w is restricted to being a permutation.

The maps ι and t have occurred in the literature several times, see [10] for example. It may be pointed out that the map ι can be replaced by the map ι', in which the largest part rather than the smallest part is moved in (λ, μ). This leads to a different set of six bijections in Theorem 4.3. For ex-, ample using ι, $\mathrm{IZ}(31214231421) = 43312214211$ whereas if ι' is used instead, $\mathrm{IZ}(31214231421) = 34123242111$.

In [4], Foata gave a bijection between words with inversion number m, and words with major index m. His bijection did not use the involution principal and is different from both of the maps IM using ι and IM using ι'. It would be interesting to find bijections similar to Foata's bijection or to find alternative descriptions of our bijections similar to Foata's bijection. One characteristic of Foata's bijection is that if $w \to w'$ where $w = a_1a_2 \ldots a_n$ and $w' = b_1b_2 \ldots b_n$, then $a_n = b_n$. This result also holds for the bijections of Theorem 4.3 regardless of whether they are built up using ι or ι'. This is an easy corollary of the following result.

Theorem 5.2. *Let $w \in M(n_1, \ldots, n_k)$, $\lambda \in P_n^{[0,\infty)}$ and let Φ_1, Φ_2 and Φ_3 be the bijections of Section 3. Then if $\Phi_i^{-1}(\Phi_j(\lambda, w)) = (\lambda', w')$, the last letter in w' is the same as the last letter in w.*

Proof. The result of Theorem 5.2 will follow if we can show that given $(\lambda_1, \ldots, \lambda_k) \in P_{n_1}^{[0,\infty)} \times \cdots \times P_{n_k}^{[0,\infty)}$, and $(\lambda, w) = \Phi_i^{-1}(\lambda_1, \ldots, \lambda_k)$, the last letter in w is independent of i. In fact, if $l = \min_i \lambda_i(n_i)$, then $a_n = m$, where m is the largest value such that $\lambda_m(n_m) = l$. This is an immediate consequence of the algorithms for constructing $\tilde{w}$ in Lemma 3.9 and Lemma 3.11, in the cases of Φ_1 and Φ_3. For Φ_2, this fact follows from MacMahon's algorithm. Since we have not given MacMahon's algorithm, we present an alternative proof for this case. Given that there exists (λ, w) such that $\Phi_3(\lambda, w) = (\lambda_1, \ldots, \lambda_k)$, recall that $(\lambda_1, \ldots, \lambda_k) = (\lambda_1' + \mu_1, \ldots, \lambda_k' + \mu_k)$, where $(\lambda_1', \ldots, \lambda_k')$ is λ sorted by w and $(\mu_1, \ldots, \mu_k)$ satisfies $\mu_i(j) =$ the number of descents in w to the right of the jth i. If $a_n = m$, then $\mu_m(n_m) = 0$ since there are no descents to the right of a_n. Since $\lambda_m'(n_m) = \lambda(n)$ is the smallest part in λ, $\lambda_m(n_m)$ is certainly minimal. If $l > m$ and $l = a_r$ for some $r < n$, then $\lambda_l(n_l) > \lambda_m(n_m)$ since $\lambda_l'(n_l) = \lambda(r) \geq \lambda(n)$ and $\mu_l(n_l) > 0 = \mu_m(n_m)$ due to the fact that at least one descent must occur between a_r and a_n if $a_r < a_n$. thus m is maximal with respect to the minimality of $\lambda_i(n_i)$. $\square$

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