A generalization of continued fractions

John Greene and Jesse Schmiege  
Department of Mathematics and Statistics  
University of Minnesota Duluth  
Duluth, MN 55812  
USA  
jgreene@d.umn.edu  
schm3241@d.umn.edu

Abstract

A paper by Maxwell Anselm and Steven Weintraub investigated a generalization of classic continued fractions, where the “numerator” 1 is replaced by an arbitrary positive integer. Here, we generalize further to the case of an arbitrary real number $z \geq 1$. We focus mostly on the case where $z$ is rational but not an integer. Extensive attention is given to periodic expansions and expansions for $\sqrt{n}$, where we note similarities and differences between the case where $z$ is an integer and when $z$ is rational. When $z$ is not an integer, it need no longer be the case that $\sqrt{n}$ has a periodic expansion. We give several infinite families where periodic expansions of various types exist.

1 Introduction

Let $z$ be a positive real number. In this paper, we consider continued fractions of the form

$$a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{a_3 + \cdots}}}$$

where $a_0$ is a nonnegative integer and $a_1, a_2, a_3, \ldots$ are positive integers. We denote such a continued fraction by $[a_0, a_1, a_2, \ldots]_z$, and following Anselm and Weintraub [1], we refer to this as a $\text{cf}_z$ expansion. Such continued fraction expansions where $z$ is a positive integer have been investigated before. Edward Burger and his co-authors showed that there are infinitely many positive integers $z$ for which $\sqrt{n}$ has a periodic expansion with period 1. A similar result, but for quasi-periodic continued fractions was obtained by Komatsu [7]. A more general, comprehensive study of $\text{cf}_z$ expansions with $z$ a positive integer was conducted Anselm and Weintraub [1]. One result of Anselm and Weintraub is that if $z \geq 2$ is an integer, then every real $x$ has infinitely many $\text{cf}_z$ expansions [1, Theorem 1.8]. More recently, Dajani,
Kraaikamp and Wekken [4] interpreted \( c_f_z \) expansions with positive integer \( z \) in terms of dynamical systems to produce an ergodic proof of [1, Theorem 1.8]. This work was extended and further generalized by Dajani, Kraaikamp and Langeveld [4].

In this paper, we ask what happens if one drops the condition that \( z \) be an integer. We mostly follow Anselm and Weintraub [1] as we investigate general \( c_f_z \) expansions where \( z \) is only assumed to be a positive real number. We begin with a discussion of the general case (usually satisfying \( z \geq 1 \)) in Sections 2 and 3. Our main focus, however, is the case where \( z \) is a rational number, with some attention to the case where \( z \) is a quadratic irrational as well.

When \( z \) is an integer, it is shown by Anselm and Weintraub [1] that many well-known properties of simple continued fractions are preserved, most notably that every rational number has a finite \( c_f_z \) expansion and every quadratic irrational has a periodic \( c_f_z \) expansion. When \( z \) is rational, but not an integer, both of these properties can fail. Formula (3.1) gives an example where the unique \( c_f_z \) expansion of \( \frac{7}{4} \) is periodic, and Conjecture 27 gives an example of a rational \( x \) and rational \( z \) for which the \( c_f_z \) expansion of \( x \) appears to be aperiodic. General properties of \( c_f_z \) expansions with rational \( z \) are given in Section 4.

The notion of a reduced quadratic surd is important in the theory of simple continued fractions. Anselm and Weintraub modified the definition of a reduced quadratic surd [1, Definition 2.12] so as to apply to integers \( z > 1 \). We must again modify this definition to make it applicable to our more general setting. In Section 5, we introduce the notion of a pseudo-conjugate for a number \( x \) with a periodic \( c_f_z \) expansion and use this to develop the appropriate definition of a reduced surd. The properties of a reduced surd are developed in Section 5 as well. Finally, these properties are applied to expansions for \( \sqrt{n} \) in Section 6, and several infinite families of periodic expansions of \( \sqrt{n} \) are given.

\section{Continued fractions as rational functions}

If we view \( a_0, \ldots, a_n \) and \( z \) as being indeterminates, then the finite continued fraction \([a_0, a_1, \ldots, a_n]_z\) is a rational function in these variables. Many of the results from Anselm and Weintraub [1] are special cases of formulas of Perron’s [10] and carry over with little or no modification to this rational function setting. In this section, we give the most important properties of \([a_0, a_1, \ldots, a_n]_z\) as a rational function.

\textbf{Lemma 1.} As rational function identities, we have

\begin{align}
[a_0, a_1, \ldots, a_n]_z &= [a_0, a_1, \ldots, a_{k-1}, [a_k, a_{k+1}, \ldots, a_n]_z]_z, \\
[a_0, a_1, \ldots, a_n]_z &= [a_0, a_1, \ldots, a_{n-2}, a_{n-1} + z/a_n]_z \\
[a_0, ya_1, a_2, ya_3, \ldots, xa_n]_{yz} &= [a_0, a_1, \ldots, a_n]_z,
\end{align}

where \( x = 1 \), if \( n \) is even, \( y \) if \( n \) is odd.
Given a sequence $a_0, a_1, a_2, \ldots$, define polynomials $p_n$ and $q_n$ recursively by

$$
\begin{align*}
p_{-1} &= 1, \\
p_0 &= a_0, \\
p_n &= a_np_{n-1} + zp_{n-2}, & \text{for } n \geq 1,
\end{align*}
$$

(2.4)

$$
\begin{align*}
q_{-1} &= 0, \\
q_0 &= 1, \\
q_n &= a_nq_{n-1} + zq_{n-2}, & \text{for } n \geq 1.
\end{align*}
$$

(2.5)

As in [1], and more generally in [10] we have the following.

**Theorem 2.** For polynomials $p_n$ and $q_n$ so defined,

$$
\begin{align*}
p_nq_{n-1} - p_{n-1}q_n &= (-1)^{n-1}z^n, \\
p_nq_{n-2} - p_{n-2}q_n &= (-1)^n a_n z^{n-1}, \\
p_n \frac{q_n}{q_n} &= [a_0, a_1, \ldots, a_n], \\
[a_0, a_1, \ldots, a_n, x]_z &= \frac{p_n x + zp_{n-1}}{q_n x + zq_{n-1}}.
\end{align*}
$$

(2.6)

(2.7)

(2.8)

(2.9)

We will write $C_n$ for $\frac{p_n}{q_n}$, and following the usual conventions, as given, say, by Hardy and Wright [5, Chapter X] or Olds [8, P. 231], we refer to the variables $a_0, a_1, \ldots$ as the partial quotients of the cf, expansion and the $C_n$ as the convergents of the expansion.

We view each $p_k$ as being a polynomial in $a_0, \ldots, a_k$, and $z$. We find it useful to think of $q_k$ as a polynomial in $a_0, \ldots, a_k$ and $z$ even though it does not depend on $a_0$. With this perspective, we have certain symmetry properties of $p_n$ and $q_n$ with respect to their variables and each other.

**Theorem 3.** The following relationships exist among $p_n$ and $q_n$.

(a) $q_n(a_0, a_1, \ldots, a_n) = p_{n-1}(a_1, \ldots, a_n)$,

(b) $p_n(a_0, a_1, \ldots, a_n) = a_0q_n(a_0, a_1, \ldots, a_n) + zq_{n-1}(a_1, \ldots, a_n)$,

(c) $q_n(a_0, a_1, \ldots, a_n) = q_n(a_{n+1}, a_n, \ldots, a_1)$,

(d) $p_n(a_0, a_1, \ldots, a_n) = p_n(a_n, a_{n-1} \ldots, a_0)$.

**Proof.** Part (a) is a direct consequence of the recurrences for $q_n$ and $p_n$. That is, $q_n$ satisfies the same recurrence as $p_{n-1}$, but with indices for the $a$’s augmented by 1. The other formulas are straightforward inductions. We content ourselves with demonstrating part (d). Assuming the rest of the formulas,

$$
\begin{align*}
p_n(a_0, \ldots, a_n) &= a_0q_n(a_0, a_1, \ldots, a_n) + zq_{n-1}(a_1, \ldots, a_n) \\
&= a_0p_{n-1}(a_1, \ldots, a_n) + zp_{n-2}(a_2, \ldots, a_n) \\
&= a_0p_{n-1}(a_n, \ldots, a_1) + zp_{n-2}(a_n, \ldots, a_2) \\
&= p_n(a_n, a_{n-1} \ldots, a_0).
\end{align*}
$$

\[\square\]
Theorem 4. Viewing $p_n$ and $q_n$ as polynomials in $a_0, \ldots, a_n, z$, we have

(a) every coefficient in each polynomial is 1.

(b) When viewed as polynomials in $z$, $\deg(p_n) = \lceil \frac{n}{2} \rceil$, $\deg(q_n) = \lfloor \frac{n}{2} \rfloor$.

(c) If $m = \lceil \frac{n}{2} \rceil$, the coefficient of $z^{m-k}$ in $p_n$ is a homogeneous polynomial in $a_0, \ldots, a_n$ of degree $2k + 1$ when $n$ is even, and $2k$ when $n$ is odd. This polynomial can be explicitly described: if it has degree $j$, then it consists of the sum of all terms of the form $a_{i_1}a_{i_2} \cdots a_{i_j}$ with $i_1 < i_2 < \cdots < i_j$, with $i_1$ even, $i_2$ odd, $i_3$ even, and so on.

(d) If $m = \lfloor \frac{n}{2} \rfloor$, the coefficient of $z^{m-k}$ in $q_n$ is a homogeneous polynomial in $a_1, \ldots, a_n$ of degree $2k$ when $n$ is even, and $2k - 1$ when $n$ is odd. Such a polynomial of degree $j$ is of the sum of all terms of the form $a_{i_1}a_{i_2} \cdots a_{i_j}$ with $i_1 < i_2 < \cdots < i_j$, with $i_1$ odd, $i_2$ even, $i_3$ odd, and so on.

For example,

$p_5 = z^3 + (a_0a_1 + a_0a_3 + a_0a_5 + a_2a_3 + a_2a_5 + a_4a_5)z^2 + (a_0a_1a_2a_3 + a_0a_1a_2a_5 + a_0a_1a_4a_5 + a_0a_3a_4a_5 + a_2a_3a_4a_5)z + a_0a_1a_2a_3a_4a_5$

$q_5 = (a_1 + a_3 + a_5)z^2 + (a_1a_2a_3 + a_1a_2a_5 + a_1a_4a_5 + a_3a_4a_5)z + a_1a_2a_3a_4a_5.$

The proof of Theorem 4 is a straightforward induction. We note that as polynomials in $z$, $p_n$ is monic if $n$ is odd and $q_n$ is monic if $n$ is even.

3 Representation, convergence and uniqueness issues

In this section we let $z, a_0, a_1, \ldots$ be real numbers. Usually, $a_0$ will be a nonnegative integer and $a_k$ will be a positive integer for $k \geq 1$. We point out that for simple continued fractions, and when $z$ is an integer, $p_n$ and $q_n$ are integer sequences for all $n \geq 0$ when the $a$'s are all integers. In the case where $z$ is not an integer, however, $p_n$ and $q_n$ will usually not be integers.

The following is useful.

Lemma 5. Let $a_0$ be a nonnegative real number and let $a_1, a_2, a_3, \ldots, a_m$ be positive real numbers. If a sequence $\{x_k\}$ with $x_k \neq a_k$ for all $k < m$ can be defined by $x_0 = x$, $x_k = \frac{z}{x_{k-1} - a_{k-1}}$, then for each $n \leq m$,

$$x = [a_0, a_1, \ldots, a_{n-1}, x_n]_z.$$
This lemma is a direct consequence of formula (2.2). Following [5, 8] we refer to the \( x_n \) in Lemma 5 as the \( n \)'th complete quotient for \( x \).

We now investigate convergence issues. For general real sequences \( \{a_n\} \) we have the following.

**Theorem 6.** If \( z > 0 \), \( a_0 \geq 0 \) and \( a_k \geq 1 \) for all \( k \geq 1 \) then

\[
[a_0, a_1, a_2 \ldots]_z = \lim_{n \to \infty} [a_0, a_1, a_2, \ldots, a_n]_z
\]

exists.

**Proof.** We follow the usual proof that infinite simple continued fractions converge. Let

\[
C_n = [a_0, a_1, a_2, \ldots, a_n]_z = \frac{p_n}{q_n}
\]

By (2.7),

\[
C_n - C_{n-2} = \frac{(-1)^n a_n z^{n-1}}{q_n q_{n-2}},
\]

so \( C_n > C_{n-2} \) whenever \( n \) is even, and \( C_n < C_{n-2} \) when \( n \) is odd. Thus, the even \( C \)'s form an increasing sequence and the odd \( C \)'s form a decreasing sequence. By (2.6), \( C_{2n} < C_{2n+1} \) for each \( n \) so \( C_0 < C_{2n} < C_{2n+1} < C_1 \), meaning each subsequence is bounded, and so convergent.

By (2.6),

\[
C_n - C_{n-1} = \frac{(-1)^{n-1} z^n}{q_n q_{n-1}}.
\]

Thus, it remains to show that \( q_n q_{n-1} \) goes to infinity faster than \( z^n \). The slowest growth for \( q_n \) occurs when all the \( a \)'s are 1, in which case \( q_n = q_{n-1} + z q_{n-2} \) for all \( n \geq 2 \). An easy induction shows that for all \( n \geq 0 \), \( q_n \geq (1 + z)^{\lfloor \frac{n}{2} \rfloor} \), from which the result follows. \( \square \)

Thus, all finite and infinite continued fraction expansions represent real numbers. With no restrictions on the \( a_k \), all reals can be represented as \( cf_z \) expansions so from this point on, we restrict \( a_0 \) to be a nonnegative integer and \( a_k \) to be a positive integer for all \( k \geq 1 \).

**Theorem 7.** Every positive real number has at least one \( cf_z \) expansion if and only if \( z \geq 1 \).

**Proof.** First, if \( 0 < z < 1 \) then no real \( x \) with \( z < x < 1 \) can be represented as a \( cf_z \) expansion.

This is because the convergents \( C_n \) of Theorem 6 are increasing for even \( n \), and decreasing for odd \( n \). Thus any \( y = [a_0, a_1, a_2, \ldots]_z \) must satisfy \( a_0 \leq y \leq a_0 + \frac{z}{a_1} \). Consequently, if \( y < 1 \) then \( a_0 \) must be 0, forcing \( y \leq \frac{z}{a_1} \leq z \).

Next, suppose that \( z \geq 1 \) and let \( x \) be a positive real number. Following [1] we construct a \( cf_z \) expansion for \( x \) as follows: Let \( x_0 = x \) and for \( n \geq 0 \), given \( x_n \), let \( a_n \) be a positive integer satisfying \( x_n - z \leq a_n \leq x_n \). If \( x_n - a_n = 0 \) stop. Otherwise, set \( x_{n+1} = \frac{z}{x_n - a_n} \) and continue. By construction, \( 0 \leq x_n - a_n < z \) so if \( x_n - a_n \neq 0 \) then \( x_{n+1} > 1 \). Thus, for each \( x_n \) there will be a valid choice for \( a_n \). This associates with \( x \) the \( cf_z \) expansion \([a_0, a_1, a_2, \ldots]_z\),
where the length of this expansion is finite if any \( x_n = a_n \), and infinite otherwise. We claim that
\[
x = [a_0, a_1, a_2, \ldots]_z.
\]
To prove this, we first note that by Lemma 5 and the construction of the \( x_n \),
\[
x = [a_0, a_1, a_2, \ldots, a_{n-1}, x_n]_z.
\]
Consequently, by formula (2.9),
\[
x - [a_0, a_1, a_2, \ldots, a_{n-1}]_z = \frac{p_{n-1}x_n + zp_{n-2}q_{n-1}x_n + zq_{n-2}}{q_{n-1}(q_{n-1}x_n + zq_{n-2})}.
\]
That is,
\[
|x - [a_0, a_1, a_2, \ldots, a_{n-1}]_z| < \frac{z^n}{q_{n-1}^2},
\]
and the left hand side of this expression goes to 0 as \( n \) goes to infinity (as in the proof of Theorem 6), completing the proof. \( \square \)

As a consequence, we have the following.

**Corollary 8.** If \( x = [a_0, a_1, a_2, a_3, \ldots]_z \) then with the \( x_k \) as defined in Lemma 5 we have
\[
x_n = [a_n, a_{n+1}, a_{n+2}, \ldots]_z.
\]
In particular, if the cf\(_z\) expansion of \( x \) has at least two terms, then
\[
[a_1, a_2, a_3, \ldots]_z = \frac{z}{x - a_0}.
\]
Moreover, if \( x = [a_0, a_1, a_2, \ldots, a_{n-1}, y]_z \) for some real \( y \geq 1 \) and \( y = [b_0, b_1, \ldots]_z \), with \( b_0 \geq 1 \) then
\[
x = [a_0, a_1, a_2, \ldots, a_{n-1}, b_0, b_1, \ldots]_z,
\]
an infinite version of formula (2.1).

There are uniqueness considerations with such expansions since there may be several possible choices for \( a_n \) satisfying \( x_n - z \leq a_n \leq x_n \). One canonical choice is \( a_n = [x_n] \), the largest possible choice for \( a_n \). We call \( a_n \) the **maximal choice** if \( a_n = [x_n] \). The expansion in which the maximal choice is always made is called the **maximal expansion**. We note that when some \( a_n = [x_n] \), the resulting \( x_{n+1} \), should it exist, is as large as possible. In particular, in this case, \( x_{n+1} > z \). The other extreme is to select \( a_n = \max(1, [x_n - z]) \). If we do this for all \( n \), the resulting expansion is called the **minimal expansion**. There are significant differences between the case where \( z \) is an integer and where it is not. For example, Lemma 1.7 in the paper by Anselm and Weintraub [1] is problematic when \( z \) is not an integer.
Lemma 9. Let \( z > 1 \) and suppose that \( x \) is not an integer.

(a) The maximal \( c_f \) expansion of \( x \) will have the form \( x = [a_0, a_1, a_2, \ldots]_z \), where for all \( i \geq 1 \), \( a_i \geq \lfloor z \rfloor \), and if the expansion terminates with last term \( a_n \), then \( a_n > z \).

(b) Let \( x = [a_0, a_1, a_2, \ldots]_z \), and suppose that \( a_i \geq \lfloor z \rfloor \), for all \( i \geq 1 \) and if the expansion terminates with \( a_n \), then \( a_n > z \). Then this expansion coincides with the maximal expansion.

Proof. For the first part, as mentioned above, if \( a_{k-1} = \lfloor x_{k-1} \rfloor \), then \( x_k \) should it exist, satisfies \( x_k > z \), so \( a_k = \lfloor x_k \rfloor \geq \lfloor z \rfloor \). If the expansion terminates, then \( x_n \) is an integer so \( a_n = x_n > z \).

For the second part, if the expansion terminates with \( a_n > z \) then \( x_n = a_n > z \) as well. Thus, \( x_n = \frac{z}{x_{n-1} - a_{n-1}} > z \) implies that \( a_{n-1} = \lfloor x_{n-1} \rfloor \). Whenever \( a_k \) is not the last term in the expansion, \( x_k > a_k \geq \lfloor z \rfloor \), and again \( x_k > z \), implying that \( a_{k-1} = \lfloor x_{k-1} \rfloor \). This shows that for all \( i \geq 0 \) for which \( x_i \) exists, \( a_i = \lfloor x_i \rfloor \), and the expansion is given by the max algorithm.

When \( z \) is an integer, the two conditions in Lemma 9 coincide and we have the characterization of the maximal expansion given by Anselm and Weintraub [1]. However, when \( z \) is not an integer, these two conditions are different, so Lemma 9 fails to give a characterization in this case. The following examples show that neither condition characterizes a maximal expansion. Letting \( z = \frac{3}{2} \), first take \( x = [1, 1, 3]_z = \frac{23}{11} \). Then the \( a_i \) satisfy the conditions of the first part of Lemma 9 but the maximal expansion of \( \frac{23}{11} \) is \( [2, 16, 3]_z \). Next, if \( x = \frac{1 + \sqrt{7}}{2} \) then the max algorithm gives \( x = [1, 1, 1, \ldots]_z \), showing that the conditions in the second part of the lemma are not necessary. Both of these examples can be generalized. If \( z \) is not an integer, then consider \( x = [a, a, k]_z \), where \( a = \lfloor z \rfloor \). We have

\[
x = a + \frac{z}{a + \frac{k}{z}}.
\]

If we select \( k \) large enough that \( a + \frac{z}{k} < z \) then the maximal choice for \( a_0 \) is \( a + 1 \) instead of \( a \). If we select \( x = [a, a, a, \ldots]_z = \frac{a + \sqrt{a^2 + 4k}}{2} \), then it is not hard to show that \( x = x_n \) for all \( n \) and that \( \lfloor x \rfloor = a \). That is, \( [a, a, a, \ldots]_z \) will be the maximal expansion of \( x \) even though this expansion does not satisfy part (b) of the lemma. Call the maximal expansion of \( x \) a proper maximal expansion if the expansion satisfies part (b) of Lemma 3.5. That is, \( x \) has a proper maximal expansion if \( a_i \geq \lfloor z \rfloor \) for all \( i \geq 1 \), and in the case where the expansion terminates with \( a_n \), that \( a_n > z \).

We have the following weak condition.

Lemma 10. Given a sequence \( \{a_n\} \), with associated numerators and denominators \( p_n \) and \( q_n \) as in Theorem 2, then \( a_n > z \) for all \( n \geq 1 \) if and only if for each \( n \geq 0 \) the maximal expansion of \( \frac{p_n}{q_n} \) is \( [a_0, a_1, \ldots, a_n]_z \).
Proof. Lemma 3.5(a) gives the necessity of the condition. For sufficiency, we have

$$
\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]_z \\
= [a_0, a_1, \ldots, a_{n-1} + z/a_n]_z.
$$

If \( a_n \leq z \) then the maximal choice for the \((n - 1)\)st quotient will be larger than \( a_{n-1} \). The result now follows by induction. \( \Box \)

We also have the following obvious result.

Lemma 11. Suppose the maximal expansion of \( x \) is \([a_0, a_1, a_2, \ldots]_z\). Then for each \( n \), the maximal expansion of \( x_n \) is \([a_n, a_{n+1}, a_{n+2}, \ldots]_z\).

We now address the uniqueness of \( \operatorname{cf}_z \) expansions.

Theorem 12. Let \( x \) be a positive real number.

(a) If \( 0 < z < 1 \) and \( x = [a_0, a_1, a_2, \ldots]_z \) then this expression is unique.

(b) If \( z = 1 \) then \( x \) has a unique \( \operatorname{cf}_z \) expansion if it is irrational and exactly two expansions if it is rational.

(c) If \( z \geq 2 \) then every positive real number \( x \) has infinitely many \( \operatorname{cf}_z \) expansions.

Proof. Case (b) is well-known [5, 8, 9, 10]. For (a), let \( z < 1 \) and suppose that \( x = [a_0, a_1, a_2, \ldots]_z \). We show that each \( a_k \) is uniquely determined. Since \( a_0 < x \leq a_0 + \frac{z}{a_1} \), we have \( 0 \leq x - a_0 \leq \frac{z}{a_1} < 1 \). Thus \( a_0 = \lfloor x \rfloor \), so \( a_0 \) is unique. By Corollary 8, 

$$
\frac{z}{x - a_0} = [a_1, a_2, a_3, \ldots]_z,
$$

and the argument just given inducts to show that all \( a_k \) are uniquely determined.

For (c), suppose that \( z \geq 2 \) and \( x > 0 \). If \( x = m \), an integer, we may write \( x = [m - 1, z]_z \), and find the \( \operatorname{cf}_z \) expansion of \( z \) with the max-algorithm. More generally, if the maximal expansion of \( x \) is \([a_0, a_1, \ldots, a_n]_z\), then by Lemma 3.5, \( a_n > z \geq 2 \). Thus, we may write \( x = [a_0, a_1, \ldots, a_n - 1, z]_z \), and again expand \( z \). If the expansion of \( z \) terminates, we may iterate on the last partial quotient, allowing for infinitely many expansions of \( x \). Thus, replacing \( x \) by some \( x_k \), if necessary, we are reduced to the case where the maximal expansion for \( x \) does not terminate, \( x = [a_0, a_1, a_2, \ldots]_z \), and \( a_k \geq \lfloor z \rfloor \geq 2 \) for all \( k \geq 1 \). Consequently, for any \( n \geq 1 \), we may write \( x = [a_0, a_1, \ldots, a_{n-1}, x_n]_z \), with \( x_n > 2 \). We can derive from this that \( x = [a_0, a_1, \ldots, a_{n-1}, m, x_{n+1}]_z \), where \( x_{n+1} = \frac{z}{x_n - m} \), and \( m \) is any integer for which \( x_{n+1} > 1 \). That is, we need \( m \) to satisfy \( 0 < x_n - m < z \). The obvious choice, \( m = \lfloor x_n \rfloor \) gave rise to \( a_n \) in the expansion for \( x \) but we may also use \( m = a_n - 1 \) owing to the fact that \( z \geq 2 \). Hence, \( x = [a_0, a_1, \ldots, a_{n-1}, a_n - 1, y]_z \), with \( y = \frac{z}{x_n - a_n + 1} > 1 \), and we may obtain another expansion for \( x \) by expanding \( y \). Since the choice of \( n \) was arbitrary, this leads to infinitely many \( \operatorname{cf}_z \) expansions. \( \Box \)
The authors do not know what happens with $1 < z < 2$. Perhaps the ergodic approaches employed in [3, 4] can clarify the situation. It appears that most $x$ (in some measure-theoretic sense) have infinitely many expansions. Evidence for this is given in the following result.

**Theorem 13.** Let $1 < z < 2$.

(a) Every $x > 0$ has an infinite $c_f_z$ expansion.

(b) A real number $x = [a_0, a_1, a_2, \ldots]_z$ has a unique $c_f_z$ expansion if and only if the expansion is infinite, $x - \lfloor x \rfloor > z - 1$ and $x_n < \frac{z}{z - 1}$ for all complete quotients $x_n$ with $n \geq 1$.

**Proof.** Suppose that $x$ has a finite maximal expansion $[a_0, a_1, \ldots, a_n]_z$. Since $z > 1$, $a_n \geq 2$ so we may write $x = [a_0, a_1, \ldots, a_n-1, x_n]_z$ and expand $z$ (by the max algorithm) to produce a longer expression. If $z$ has an infinite $c_f_z$ expansion we are done. Otherwise, we iterate.

For the second part, suppose that the $c_f_z$ expansion of $x$ is unique. By part (a), this expansion must be infinite. Moreover, if we write $x = [a_0, a_1, \ldots, a_n, x_n]_z$, then we must select $a_n = \lfloor x_n \rfloor$. Since we are free to let $a_n$ be any positive integer with $x_n - z \leq a_n \leq x_n$, this means that $\lfloor x_n \rfloor - 1 > x_n - z$, or $x_n - a_n > z - 1$. When $n = 0$, this says $x - \lfloor x \rfloor < z - 1$. Given $x_n - a_n > z - 1$, we have $x_{n+1} > \frac{z}{z - 1}$. Conversely, if $x$ has more than just the max expansion then for some $n$, the $n$’th partial quotient need not be $\lfloor x_n \rfloor$. In this case, $x_n - z \leq \lfloor x_n \rfloor - 1$, giving $x_{n+1} \geq \frac{z}{z - 1}$. \qed

**Corollary 14.** If $1 < z < 2$ and $x$ has a unique $c_f_z$ expansion $x = [a_0, a_1, \ldots]_z$, then the expansion is infinite and $a_n \leq \frac{z}{z - 1}$ for all $n \geq 1$.

This gives a necessary, but not a sufficient condition. For example, $3 = [2, 1, 2, 1, \ldots]_z$ when $z = \frac{3}{2}$. In this case, $\frac{z}{z - 1} = 3$, and $a_n < 3$ for all $n$.

We conclude this section by addressing the question of when a real number $x$ has a periodic $c_f_z$ expansion. Using notation similar to that used by Anselm and Weintraub [1], let

$$(a_1, a_2, \ldots, a_n)_k$$

denote a sequence in which $a_1, \ldots, a_n$ are repeated $k$ times. For example, $(1, 2, 3)_2$ represents the sequence 1, 2, 3, 1, 2, 3. If we let $k$ be infinity, then the sequence is periodic. With this notation we have

$$\frac{3}{2} = [1, 3]_z = [(1, 2)_k, 3]_z = [(1, 2)_{\infty}]_z.$$
when $z = \frac{3}{2}$. We also note the following maximal expansions.

\[
\begin{align*}
\frac{7}{4} &= \left[\frac{1}{1}\right]_{1}^{21/16}, \\
\sqrt{2} &= \left[1, \frac{3}{2}\right]_{3/2}, \\
\frac{3}{2} &= \left[1, 2, 1, 2\right]\sqrt{2}, \\
\frac{7}{6} &= \left[1, 8, 2, 1, 2, 2, \frac{3}{3}\right]_{\sqrt{2}}, \\
2\sqrt{2} &= \left[2, 1, 2\right]\sqrt{2}, \\
\sqrt{2} &= \left[1, \frac{3}{3}\right]_{\sqrt{2}}, \\
\sqrt{3} &= \left[1, 1, 2\right]_{\sqrt{2}}, \\
\sqrt{2} &= \left[1, 4, 9, 3, 8, 3, 14, \ldots\right]_{\sqrt{3}}, \\
\sqrt{\pi} + 4 &= \left[2, \frac{4}{4}\right]_{\sqrt{3}}.
\end{align*}
\]

With regard to formulas (3.3) and (3.4), when checking all rational numbers $1 < x < 2$ with denominator less than 500, we found that most of them, about 75%, had finite expansions when $z = \sqrt{2}$, and the rest had periodic expansions with period 1 or 6. Those with period 1 always had some $x_k = 2 + \sqrt{2}$, and associated periodic part $(3)_\infty$. When $z = \sqrt{3}$, most rationals we checked had periodic expansions (about 81%) with periods of length 2, 8, 12 or 72, and the rest had finite expansions. When $z = \sqrt{7}$, only $\frac{45}{43}$ had a finite expansion (of length 4) and there were no periodic expansions of length at most 500.

Formulas (3.5) and (3.6) show that $x$ and $2x$ can have very different expansions, while formulas (3.7) and (3.8) show that there is no relationship when $x$ and $z$ are interchanged.

By formula (2.9) every finite $\text{cf}_z$ expansion represents a number in $\mathbb{Z}(z)$, the ring of rational expressions in $z$ with integer coefficients. Thus, all reals not belonging to $\mathbb{Z}(z)$ must have infinite $\text{cf}_z$ expansions. Reals with periodic expansions must satisfy a quadratic equation.

**Theorem 15.** If $x$ has the $\text{cf}_z$ expansion

\[
[a_0, a_1, \ldots, a_{j-1}, (a_j, \ldots, a_{j+k-1})_{\infty}]_z
\]

then $x$ satisfies the quadratic equation

\[
q_{k-1}x^2 + (zq_{k-2} - p_{k-1})x - zp_{k-2} = 0
\]

when $j = 0$,

\[
q_{k-1}x^2 + (q_k - a_0q_{k-1} - p_{k-1})x - (p_k - a_0p_{k-1}) = 0
\]

when $j = 1$, and

\[
x^2(q_{j+k-1}q_{j-2} - q_{j+k-2}q_{j-1}) - x(p_{j+k-1}q_{j-2} + p_{j-2}q_{j+k-1} - p_{j+k-2}q_{j-1} - p_{j-1}q_{j+k-2}) + p_{j+k-1}p_{j-2} - p_{j+k-2}p_{j-1} = 0
\]
for $j \geq 2$.

If we define $p_{-2} = 0$, $q_{-2} = \frac{1}{z}$ then formulas (3.10) and (3.11) are special cases of (3.12) but it is convenient to have all three forms.

**Proof.** We only show that formula (3.12) holds. If

$$x = [a_0, a_1, \ldots, a_{j-1}, \overline{(a_j, \ldots, a_{j+k-1})}]_z$$

then

$$x = [a_0, a_1, \ldots, a_{j-1}, x_j]_z$$

$$= [a_0, a_1, \ldots, a_{j-1}, a_j, \ldots, a_{j+k-1}, x_j]_z.$$

By Theorem 2,

$$x = \frac{p_{j-1} x_j + z p_{j-2}}{q_{j-1} x_j + z q_{j-2}}, \quad x = \frac{p_{j+k-1} x_j + z p_{j+k-2}}{q_{j+k-1} x_j + z q_{j+k-2}},$$

or

$$x_j = -z \frac{p_{j-2} - x q_{j-2}}{p_{j-1} - x q_{j-1}} = -z \frac{p_{j+k-2} - x q_{j+k-2}}{p_{j+k-1} - x q_{j+k-1}},$$

from which the result follows. $\Box$

The discriminant of the quadratic in (3.10) is $(z q_{k-2} - p_{k-1})^2 + 4 p_{k-2} q_{k-1} = (z q_{k-2} + p_{k-1})^2 + 4 (-1)^n z^n$, by formula (2.6). So if $x$ has a purely periodic expansion $[(a_0, \ldots, a_{n-1})_\infty]_z$, then

$$x = \frac{p_{k-1} - z q_{k-2} + \sqrt{(z q_{k-2} + p_{k-1})^2 + 4 (-1)^n z^n}}{2 q_{k-1}}. \tag{3.13}$$

Theorem 15 has the following converse.

**Theorem 16.** If $a_0, \ldots, a_{j+k-1}$ are positive integers and there is an $x = [a_0, \ldots, a_{j+k-1}, \ldots]_z$ which satisfies formula (3.12) then $x$ has the cf$_z$ expansion

$$[a_0, a_1, \ldots, a_{j-1}, \overline{(a_j, \ldots, a_{j+k-1})}]_z.$$ 

If $a_m \geq z$ for $1 \leq m \leq j + k - 1$ then the expansion is a maximal cf$_z$ expansion.

**Proof.** By Theorem 2,

$$x_j = -z \frac{p_{j-2} - x q_{j-2}}{p_{j-1} - x q_{j-1}} \quad \text{and} \quad x_{j+k} = -z \frac{p_{j+k-2} - x q_{j+k-2}}{p_{j+k-1} - x q_{j+k-1}}.$$ 

Since $x$ satisfies formula (3.12), $x_j - x_{j+k} = 0$, so $x$ is periodic, with the given expansion. If $a_m \geq z$ for all $m \geq 1$ then by Lemma 3.5, the maximal expansion of $x$ has the desired form. $\Box$

We have the following easy consequences of Theorem 15.
Corollary 17. In order for a positive real number $x$ to have a periodic $c_f(z)$ expansion, $x$ must be an element of $\mathbb{Z}(\sqrt{f(z)})$, the set of rational expressions in $\sqrt{f(z)}$, where $f(z)$ is a rational function of $z$ with integer coefficients.

Corollary 18. If $x$ is a positive real number then

(a) $x$ has purely periodic expansion $[[a]_\infty]_z$ if and only if $x = \frac{a + \sqrt{a^2 + 4z}}{2}$. This is the maximal expansion for $x$ provided $z < a + 1$.

(b) $x$ has purely periodic expansion $[[a,b]_\infty]_z$ if and only if $x = \frac{ab + \sqrt{a^2b^2 + 4abz}}{2b}$. This is the maximal expansion for $x$ provided $z < \min(a + \frac{a}{b}, b + \frac{b}{a})$.

Proof. The first part follows from the second. If $x = [[a]_\infty]_z$, then by (3.13), $x = \frac{a + \sqrt{a^2 + 4z}}{2}$. On the other hand, given such an $x$, we have $x_1 = \frac{x}{x-a} = \frac{a + \sqrt{a^2 + 4z}}{2a}$ and $x_2 = \frac{x_1}{x_1-a} = x$ so $x$ has the desired periodic expansion. Moreover, $\lfloor x \rfloor = a$ if and only if $z < a + \frac{a}{b}$, and $\lfloor x_1 \rfloor = b$ if and only if $z < b + \frac{b}{a}$, demonstrating the maximal expansion property.

For example, if $a = 1$ in part (a) and $z = \frac{10}{9}$ then $1 + 4z = \frac{49}{9}$ so $x = \frac{5}{3}$ will have maximal expansion $[[1]_\infty]_z$. If $a = 2, b = 3, z = \frac{3}{2}$, then $x = 1 + \sqrt{2}$ in part (b), essentially giving expansion (3.2). Formulas (3.1), (3.6) and (3.9) are also essentially examples of Corollary 18.

4 Expansions with rational $z$

In this section, we focus on the case were $z$ is rational, say $z = \frac{u}{v}$ where $u$ and $v$ are positive relatively prime integers with $u > v$. With $C_n = \frac{P_n}{q_n}$, as noted in Section 3, $p_n$ and $q_n$ will not, in general, be integers. When $z$ is rational, we can scale $p_n$ and $q_n$ to obtain an integer numerator and denominator for $C_n$, but at the cost of more complicated recurrences. In place of Theorem 2 we have the following.

Theorem 19. Given a sequence of integers $\{a_n\}$ with $a_0 \geq 0$ and $a_k \geq 1$ for $k \geq 1$ define sequences $\{P_n\}$ and $\{Q_n\}$ inductively as follows:

\[
P_{-1} = 1, \quad P_0 = a_0, \quad P_n = \begin{cases} \frac{a_n P_{n-1} + u P_{n-2}}{v a_n P_{n-1} + u P_{n-2}}, & \text{if } n \text{ is even;} \\ \frac{a_n P_{n-1} + u P_{n-2}}{v a_n P_{n-1} + u P_{n-2}}, & \text{if } n \text{ is odd,} \end{cases}
\]

\[
Q_{-1} = 0, \quad Q_0 = 1, \quad Q_n = \begin{cases} \frac{a_n Q_{n-1} + u Q_{n-2}}{v a_n Q_{n-1} + u Q_{n-2}}, & \text{if } n \text{ is even;} \\ \frac{a_n Q_{n-1} + u Q_{n-2}}{v a_n Q_{n-1} + u Q_{n-2}}, & \text{if } n \text{ is odd.} \end{cases}
\]

If $C_n = \frac{P_n}{Q_n}$ then for each $n \geq 0$,

\[
C_n = [a_0, a_1, \ldots, a_n]_z.
\]
Other relevant results from Section 2 translate as follows.

**Theorem 20.** With \( \{a_n\}, \{P_n\}, \{Q_n\} \), defined as in Theorem 19, we have

\[
P_nQ_{n-1} - P_{n-1}Q_n = (-1)^{n-1}u^n,
\]

\[
P_nQ_{n-2} - P_{n-2}Q_n = \begin{cases} (-1)^{n-1}u^{n-1}, & \text{if } n \text{ is even;} \\ v(-1)^{n-1}u^{n-1}, & \text{if } n \text{ is odd,} \end{cases}
\]

\[x = \begin{cases} \frac{P_{n-1}x_n + uP_{n-2}}{Q_{n-1}x_n + uQ_{n-2}}, & \text{if } n \text{ is even;} \\ \frac{vP_{n-1}x_n + uP_{n-2}}{vQ_{n-1}x_n + uQ_{n-2}}, & \text{if } n \text{ is odd,} \end{cases}
\]

\[v \mid Q_{2n-1} \quad \text{for all } n,
\]

\[\gcd(P_n, Q_n) \mid u^n \quad \text{for all } n.
\]

An easy induction gives the following relationship between \( P_n, Q_n \) and \( p_n, q_n \).

**Lemma 21.** For all \( n \geq 0 \),

\[P_n = v^{\lfloor n/2 \rfloor}p_n, \quad Q_n = v^{\lceil n/2 \rceil}q_n.
\]

With regard to periodic expansions, we may replace \( p_n \) and \( q_n \) with \( P_n \) and \( Q_n \) as well.

**Theorem 22.** Let \( z = \frac{u}{v} \) be a positive rational number in lowest terms. If positive real number \( x \) has a purely periodic expansion

\[x = \left[ (a_0, \ldots, a_{n-1}) \right]_z\]

then \( x \) must satisfy the quadratic equation

\[Q_{n-1}x^2 + (uQ_{n-2} - P_{n-1})x - uP_{n-2} = 0\]

if \( n \) is even and

\[vQ_{n-1}x^2 + (uQ_{n-2} - vP_{n-1})x - uP_{n-2} = 0\]

when \( n \) is odd. If \( x \) is rational then \( (uQ_{n-2} + P_{n-1})^2 - 4u^n \) must be a perfect square in \( n \) is even, \( (uQ_{n-2} + vP_{n-1})^2 + 4vu^n \) must be a perfect square if \( n \) is odd.

Odd period lengths are rare in our calculations. Here is one reason for this.

**Theorem 23.** Let \( z = \frac{u}{v} \) be a positive rational number in lowest terms. If \( x \) is a rational number with a periodic cf\(_z\) expansion of odd length, then \( v \) is a square.

**Proof.** If \( x \) is not purely periodic, we may replace \( x \) with a complete quotient \( x_k \), which is purely periodic, so we may assume that \( x \) has a purely periodic expansion. As noted in formula (4.6), all \( Q_n \) of odd index are divisible by \( v \). Since \( n \) is odd we may write \( Q_{n-2} = kv \) for some integer \( k \). By the previous theorem,

\[(uQ_{n-2} + vP_{n-1})^2 + 4vu^n = v^2(ku + P_{n-1})^2 + 4vu^n \quad (4.8)\]
is a square. Every positive integer can be written as a square times a square free part, so suppose \( v = s^2 t \) for some integers \( s \) and \( t \), where \( t \) is square free. Using (4.8) we have \( s^2(A^2t^2 + 4tu^n) = m^2 \) for some integers \( A \) and \( m \). If \( p \) is an odd prime dividing \( t \) then \( p^2 \) will divide \( m^2/s^2 \) forcing \( p^2 \) to also divide \( 4tu^n \). Since \( t \) is prime to \( u \) it must be that \( p^2 \) divides \( t \), a contradiction. Thus, at worst, \( t = 2 \). In this case, \( m \) is even and we may divide by 4 to obtain \( A^2 + 2u^n = \left( \frac{m}{2s} \right)^2 \). Thus, \( 2u^n \) is even and the difference of two squares, forcing it to be divisible by 4. As a consequence, \( u \) is even, a contradiction since \( u \) is prime to \( v \). Since \( t \) is not divisible by any prime, \( v = s^2 \), as desired.

Using Theorem 23 we may classify the \( x \) and \( z \) for which \( x \) has a purely periodic expansion of period 1.

**Theorem 24.** If \( x \) and \( z \) are rational, then \( x = \left[ (n)_{\infty} \right]_z \) if and only if

\[
x = \frac{nw + k}{w}, \quad z = \frac{k(nw + k)}{w^2},
\]

where \( w, k, n \) are positive integers and \( k \) is prime to \( w \). The expansion is maximal when \( 1 \leq k < w \).

**Proof.** If \( x_0 = x = \frac{nw+k}{w} \) and \( z = \frac{k(nw+k)}{w^2} \), then a simple calculation with \( a_0 = n \) shows \( x_1 = x_0 \), allowing for a periodic expansion. In order for the expansion to be maximal, we need \( \lfloor x \rfloor = n \), which requires \( 1 \leq k < w \).

Next, suppose that \( x = \left[ (n)_{\infty} \right]_z \), where \( x \) and \( z \) are rational. By Theorem 23 we may write \( z = \frac{u}{w^2} \) for some positive integers \( u \) and \( w \). By Theorem 22 we have

\[
x = \frac{n + \sqrt{n^2 + 4z}}{2} = \frac{nw + \sqrt{n^2w^2 + 4u}}{2w}.
\]

For \( x \) to be rational, it must be that \( \sqrt{n^2w^2 + 4u} \) is an integer. Since it is larger than \( nw \) we may write \( \sqrt{n^2w^2 + 4u} = nw + m \) for some integer \( m \). Squaring shows \( m \) must be even, so let \( m = 2k \). Again squaring and simplifying gives \( u = k(nw + k) \), giving \( x \) and \( z \) their desired forms.

Formula (3.1), a special case of Theorem 24, shows that a rational number can have an infinite maximal \( cf_z \) expansion even when \( z \) is rational. This contrasts with a result from Anselm and Weintraub [1, Lemma 1.9], that when \( z \) is an integer, the maximal \( cf_z \) expansion of any positive rational number is finite. However, we do have the following conjectures.

**Conjecture 25.** If \( z = \frac{3}{2} \) then every positive rational number has a finite \( cf_z \) expansion.

**Conjecture 26.** If \( z = \frac{5}{3} \) then every positive rational number has either a finite \( cf_z \) expansion or a periodic \( cf_z \) expansion.
We have tested Conjecture 25 on all rational numbers with denominator less than 1000. There are other \( z \) besides \( \frac{3}{2} \) that appear to have this property. In [11] a list of 146 rational \( z \) are given for which it appears that all positive rationals have a finite maximal cf \( z \) expansion.

For Conjecture 26, again, there are other \( z \) besides \( \frac{5}{3} \) that appear to have the given property. In contrast, we have the following.

**Conjecture 27.** For \( z = \frac{11}{8} \) the maximal cf \( z \) expansion of \( \frac{4}{5} \) is neither finite nor periodic.

Using Floyd’s cycle finding algorithm [6, p. 7, Exercise 6], we checked the expansion of \( \frac{4}{5} \) through 1,000,000 partial quotients without it terminating or becoming periodic. Here, \( \frac{11}{8} \) appears to be the smallest rational \( z \)-value having this property. That is, if \( 1 < z < 2 \), \( z = \frac{u}{v} \) and \( u + v < 19 \) then all rational numbers appear to have either finite or periodic maximal cf \( z \) expansions. Also, nearly half of the rational \( x \) we tried appeared to have aperiodic expansions with \( z = \frac{11}{8} \).

Acting against Conjecture 27 is that some rational numbers can have very long finite cf \( z \) expansions. An example is \( x = \frac{2369}{907} \), which has a maximal cf \( z \) expansion of length 37,132 when \( z = \frac{11}{7} \).

## 5 Periodic expansions and reduced quadratic surds

In the theory of periodic continued fractions, both for simple continued fractions and in the work of Anselm and Weintraub [1], the notion of a quadratic irrational being **reduced** is important. The appropriate definition in [1] is the following: A quadratic irrational \( x \) is \( N \)-reduced if \( x > N \) and \(-1 < \overline{x} < 0 \), where \( \overline{x} \) is the Galois conjugate of \( x \). For simple continued fractions, “quadratic irrational” is essentially synonymous with “periodic.” For the more general setting in [1], being a quadratic irrational was a necessary condition for periodicity. This is not the case for general \( z \) so to extend the idea of being reduced to the general \( z \)-setting, one must modify the definition of a conjugate. Intuitively, if \( x \) has a periodic expansion, then \( x \) must satisfy a quadratic equation derived from the periodic expansion and we define its conjugate to be the other solution to that equation.

To be more rigorous, suppose that \( x \) is a positive real number with periodic maximal cf \( z \) expansion of period length \( k \) and tail length \( j \) so

\[
x = [a_0, a_1, \ldots, a_{j-1}, (a_j, \ldots, a_{j+k-1})_{\infty}]_z.
\]

Then \( x \) satisfies the quadratic equation in (3.12). The other solution to this equation is called the **pseudo-conjugate** of \( x \) with respect to \( z \) and is denoted \( \overline{x} \). We note that the this conjugate can be written in the form

\[
\overline{x} = \frac{p_{j+k-1}p_{j-2} - p_{j+k-2}p_{j-1}}{x(q_{j+k-1}q_{j-2} - q_{j+k-2}q_{j-1})}, \quad (5.1)
\]

and this pseudo-conjugate must equal the usual conjugate of \( x \) when \( x \) is a quadratic irrational and \( z \) is rational. The pseudo-conjugate map is not an involution, at least when \( z \) is
not an integer. For example, if \( z = \frac{7}{4} \) and \( x = \frac{79}{30} = [2, 2, (2, 5, 1, 2, 9)_\infty]_z \), then \( \overline{z} = \frac{2287}{1294} = y \), but \( \overline{y} = \frac{11818919047}{6687223982} \neq x \). The following tool is needed.

**Lemma 28.** Suppose that \( x = x_0 = [a_0, a_1, \ldots, a_{j-1}, (a_j, \ldots, a_{j+k-1})_\infty]_z \) and \( x_1 = \frac{z}{x-a_0} \). Then \( x_1 \) is also periodic and \( \overline{x_1} = \frac{z}{x-a_0} \).

**Proof.** The proof is easier if there is no tail so we will assume that \( j \geq 1 \). It is obvious that \( x_1 \) is periodic since (for \( j \geq 1 \))

\[
x_1 = [a_1, a_2, \ldots, a_{j-1}, (a_j, \ldots, a_{j+k-1})_\infty]_z.
\]

We use formula (5.1) to show that \( \overline{x} = a_0 + \frac{z}{\overline{x}_1} \). Using primes to indicate the variables involved are \( a_1, a_2, \ldots \) rather than \( a_0, a_1, \ldots \), we have

\[
a_0 + \frac{z}{x_1} = a_0 + \frac{z x_1 (q'_j q'_{j-3} - q'_j q'_{j-2})}{p'_j q'_{j-3} - p'_j q'_{j-2}} = a_0 + \frac{z^2 (q'_j q'_{j-3} - q'_j q'_{j-2})}{(x-a_0)(p'_j q'_{j-3} - p'_j q'_{j-2})}.
\]

Using parts (a) and (b) of Theorem 3,

\[
\frac{z^2 (q'_j q'_{j-3} - q'_j q'_{j-2})}{(x-a_0)(p'_j q'_{j-3} - p'_j q'_{j-2})} = \frac{(p_j q_j - a_0 q_{j+1})(p_{j-2} - a_0 q_{j-2}) - (p_j q_{j+2} - a_0 q_{j+2})(p_{j-1} - a_0 q_{j-1})}{(x-a_0)(q_{j+1} q_{j-2} - q_{j+2} q_{j-1})} = a_0^2 x - a_0 + \frac{p_j q_{j-2} - p_j q_{j-2} k_{j-2}}{(x-a_0)(q_{j+1} q_{j-2} - q_{j+2} q_{j-1})} - \frac{a_0 p_{j+1} q_{j-2} + p_{j-2} q_{j+1} - p_{j+2} q_{j-1} - p_{j-1} q_{j+2}}{q_{j+1} q_{j-2} - q_{j+2} q_{j-1}}.
\]

Now by formula (3.12),

\[
p_j q_{j-2} + p_{j-2} q_{j+1} - p_{j+2} q_{j-1} - p_{j-1} q_{j+2} = x (q_{j+1} q_{j-2} - q_{j+2} q_{j-1}) + \frac{1}{x} (p_j q_{j-2} - p_{j+2} q_{j-1}).
\]

Consequently,

\[
a_0 + \frac{z}{x_1} = a_0 + \frac{a_0^2}{x - a_0} + \frac{p_j q_{j-2} - p_{j+2} q_{j-1}}{(x-a_0)(q_{j+1} q_{j-2} - q_{j+2} q_{j-1})} - \frac{a_0 x}{x - a_0} - \frac{a_0 x}{x - a_0} = \overline{x},
\]

as desired. \( \square \)
If \( x \) has a maximal cf\(_z\) expansion which is periodic, we define \( x \) to be **reduced** if \( x > z \) and \(-1 < \overline{a} < 0\). Unfortunately, being reduced does not have the power it has in the simple continued fraction case. We say \( x \) is **strongly reduced** if, in addition to being reduced, the maximal expansion of \( x \) satisfies \( a_i \geq z \) for all \( i \geq 1 \). That is, all partial quotients except possibly the first are at least as large as \( z \). We have the following.

**Lemma 29.** If \( x \) is strongly reduced, then so is \( x_1 = \frac{\overline{z}}{x-a_0} \).

**Proof.** To be strongly reduced, \( x \) must be periodic. Since the partial quotients of \( x_1 \) are just shifted partial quotients of \( x \), \( x_1 \) is also periodic. This also shows that the partial quotients of \( x_1 \) are all sufficiently large. Since \( a_0 = \lfloor x \rfloor \), it follows that \( x_1 > z \). By the previous lemma, \( \overline{x_1} = \frac{\overline{z}}{\overline{x} - a_0} \). Since \( \overline{x} \) is negative and \( a_0 \geq z \), \( -1 < \overline{x_1} < 0 \), showing that \( x_1 \) is reduced. \( \square \)

As a consequence of the lemma, if \( x \) is strongly reduced, so is \( x_k \) for every \( k \). The condition that \( x \) be strongly reduced is necessary. For example, \( x = \frac{105}{58} \) has maximal expansion \( \lfloor (1,2,10) \rfloor \) when \( z = \frac{7}{4} \). In this case \( x \) satisfies the quadratic equation \( 348x^2 - 572x + 105 = (6x+1)(58x - 105) \). Thus, \( \overline{x} = -\frac{1}{6} \), so \( x \) is reduced, but not strongly reduced. In this case, \( x_1 = \frac{203}{98} \) which satisfies \( 188x^2 - 124x - 609 = (2x+3)(94x-203) \). Since \( \overline{x_1} = -\frac{3}{2} \), \( x_1 \) is not reduced. It is not important here that \( x \) be rational. For example, \( x = \lfloor (1,2,5) \rfloor \) has the same property: \( x \) is reduced but \( x_1 \) is not reduced. However, if \( x = \overline{\sqrt{3}} + 1 \) and \( z = \frac{11}{5} \) then \( x = \lfloor (2,3,418,3) \rfloor \). Here \( x \) is reduced but not strongly reduced. Nevertheless, all \( x_k \) are reduced. Thus, when \( x \) is reduced but not strongly reduced, \( x_1 \) may or may not be reduced.

**Theorem 30.** If \( x \) is strongly reduced, then \( x \) is purely periodic. Moreover, \(-\frac{\overline{z}}{\overline{x}}\) is also strongly reduced.

**Proof.** We proceed by contradiction to show that \( x \) must be purely periodic. So suppose that \( x \) is periodic with period \( k \), but not purely periodic. Then for some \( j \geq 1 \), \( x = [a_0,a_1,\ldots,a_{j-1},(a_j,\ldots,a_{j+k-1})] \), and \( a_{j-1} \neq a_{j+k-1} \). By periodicity, \( x_j = x_{j+k} \), so

\[
\frac{z}{x_j - a_{j-1}} = \frac{z}{x_{j+k} - a_{j+k-1}}.
\]

Thus,

\[
\frac{z}{x_j - a_{j-1}} = \frac{z}{x_{j+k} - a_{j+k-1}},
\]

or \( x_{j-1} - a_{j-1} = x_{j+k-1} - a_{j+k-1} \). We write this in the form \( a_{j-1} - a_{j+k-1} = x_{j-1} - x_{j+k-1} \). Being reduced, \(-1 < x_{j-1} - x_{j+k-1} < 1 \). Since \( a_{j-1} - a_{j+k-1} \) is an integer, it must be that \( a_{j-1} = a_{j+k-1} \), a contradiction.

For the second part, we prove that if

\[
x = \lfloor (a_0,a_1,\ldots,a_{k-1}) \rfloor, \quad \text{then} \quad -\frac{z}{\overline{x}} = \lfloor (a_{k-1},a_{k-2},\ldots,a_0) \rfloor.
\]
We have \( x_j = \frac{x_j}{x_{j-1} - x_j} \), which we can rewrite \(-\frac{x_j}{x_{j-1}} = a_{j-1} - \frac{x_j}{x_{j-1}}\). Since \( x_j \) is reduced, this shows that for all \( j \), \( -\frac{x_j}{x_{j-1}} = a_{j-1} \), with the interpretation that when \( j = 0 \) the floor is \( a_{k-1} \). Thus, the maximal expansion of \(-\frac{x_j}{x_{j-1}}\) has partial quotients \( a_{k-1}, a_{k-2}, \ldots, a_0, a_{k-1}, \ldots \) \( \Box \)

The converse of Theorem 30 need not be true, as shown in a previous example. That is, if \( x = \frac{203}{94} \) and \( z = \frac{7}{4} \), then the maximal expansion of \( x \) is \([2, 10, 1]_{\infty}z\). Thus, \( x \) has a purely periodic maximal expansion. However, \( x \) is neither strongly reduced nor even reduced since \( x = -\frac{3}{2} \). Also, in this case, \(-\frac{x}{z} = \frac{7}{6} = \left[\begin{array}{ll} 1 & 0 \\ 10 & 2 \end{array}\right]_{\infty}z\), but this is not the maximal expansion of \( \frac{7}{6} \). The maximal expansion is \([1, 10, (3)_{\infty}]z\). Similarly, if \( x = \left[\begin{array}{ll} 2 & 9 \\ 5 & 1 \end{array}\right]_{\infty}z \) then \( x \) is purely periodic, \( x < -1 \), and in this case, \( -\frac{x}{z} = \frac{47}{11} + \sqrt{79} \) does not even appear to be periodic.

If \( x \) is reduced but not strongly reduced, one can ask whether the maximal expansion of \( x \) still has to be purely periodic. This will be the case, by the same proof as in Theorem 30, if all \( x_k \) are reduced. However, if any \( x_k < -1 \) then \( x \) need not be purely periodic. For every \( z > 1 \) which is not an integer, one can construct such \( x \). If we set \( a = \lfloor z \rfloor \) and let \( y = \left[\begin{array}{ll} a+1 & b, a \end{array}\right]_{\infty}z \) then for sufficiently large \( b \) and an appropriate \( k \), \( x = y + k = \lfloor a+1 + k, b, a, a+1 \rfloor_{\infty}z \) will be reduced but not purely periodic. To see this, from the quadratic that \( y \) satisfies, one can easily calculate that \( \bar{y} = -\frac{z}{a} + O(\frac{1}{b}) < -1 \) for large \( b \). Thus, for sufficiently large \( b \), adding \( k = \lfloor \frac{z}{a} \rfloor \) to \( y \) gives a reduced \( x \) which is not purely periodic. One should also show that the maximal algorithm of \( x \) is as stated. Since \( a+1 \) and \( b \) are larger than \( z \), this follows if \( b = \left[\begin{array}{ll} (b, a, a+1)_{\infty} \right]_{z} \), or \([0, a, a+1, b]_{z} < 1 \). Now

\[
[0, a, a+1, b]_{z} = \frac{z}{a + \frac{z}{a+1} + b} = \frac{(a+1)z + \frac{z^2}{b}}{a(a+1) + \frac{az}{b} + z} < \frac{(a+1)z + \frac{z^2}{b}}{(a+1)z + \frac{z^2}{b} + z} < 1,
\]

as desired.

6 Periodic expansions for \( \sqrt{n} \)

As shown by Anselm and Weintraub [1, Theorem 2.2], every quadratic irrational has a periodic \( \text{cf}_z \) expansion when \( z \) is a positive integer. This follows from the well-known fact that \( x \) has a periodic \( \text{cf}_1 \) expansion if and only if \( x \) is a quadratic irrational, coupled with formula (2.3) in the form

\[
[a_0, a_1, a_2, a_3, \ldots]_{1} = [a_0, za_1, za_2, za_3, \ldots]_{z}. \tag{6.1}
\]

However, right hand side of (6.1) is only a maximal expansion when the even terms satisfy \( a_{2k} \geq z \) for all \( k \).

When \( z \) is rational but not an integer, formula (6.1) only produces a proper \( \text{cf}_z \) expansion when \( za_{2k+1} \) is an integer for all \( k \). For example, \( \sqrt{2} = [1, 2, 2, 2, \ldots]_{1} \) so with \( z = \frac{3}{2} \) we have \( \sqrt{2} = [1, 3, 2, 3, \ldots]_{z} = [1, (3, 2)_{\infty}z] \). Formula (6.1) does not apply if, say, \( z = \frac{4}{3} \) instead. In
Theorem 32. If \( n \) is a rational number, \( \sqrt{n} \) has a periodic \( \text{cf}_z \) expansion of tail length \( j \) and period length \( k \) then

\[
\begin{align*}
  n(q_{j+k-1}q_{j-2} - q_{j+k-2}q_{j-1}) + p_{j+k-1}p_{j-2} - p_{j+k-2}p_{j-1} &= 0, \\
  p_{j+k-1}q_{j-2} + p_{j-2}q_{j+k-1} - p_{j+k-2}q_{j-1} - p_{j-1}q_{j+k-2} &= 0.
\end{align*}
\] (6.2) (6.3)

In the case where \( j = 1 \), \( a_0 = a \), \( n = a^2 + b \), these are equivalent to

\[
\begin{align*}
  q_k - aq_{k-1} - p_{k-1} &= 0, \\
  bq_{k-1} + aq_k - p_k &= 0.
\end{align*}
\] (6.4) (6.5)

Proof. These are easy consequences of formulas (3.11) and (3.12). \( \square \)

Burger and his coauthors [2] show that quadratic irrationals have infinitely many positive integers \( z \) for which the maximal \( \text{cf}_z \) expansion has period 1. Nevertheless, as Anselm and Weintraub mention [1], odd period lengths tend to be rare for \( \sqrt{n} \). Several cases of odd period length exist, including infinite families such as \( \sqrt{a^2 + b} = [a, (2a)_\infty]b \). However, these only occur when \( z \) is an integer.

Theorem 32. If \( z \) is rational and \( \sqrt{n} \) has a \( \text{cf}_z \) expansion with odd period length then \( z \) is an integer.

Proof. Suppose that \( \sqrt{n} \) has a periodic \( \text{cf}_z \) expansion with period \( 2k + 1 \),

\[
\sqrt{n} = [a_0, a_1, \ldots, a_{j-1}, (a_j, \ldots, a_{j+2k})_\infty].
\]

For convenience, we assume that \( j \geq 2 \), the proof being slightly easier if \( j = 0 \) or \( j = 1 \). We view equation (6.2) as an equation in \( z \). If \( j \) is even, then by Theorem (4), the terms in (6.2) have degrees \( j + k - 1 \), \( j + k - 2 \), \( j + k - 1 \), and \( j + k \), respectively. That is, the term of highest degree is \( p_{j+k-2}p_{j-1} \). If \( j \) is odd, then the degrees are \( j + k - 3 \), \( j + k - 2 \), \( j + k \), and \( j + k - 1 \), with \( p_{j+k-2}p_{j-1} \) being the term of highest degree. In each case, the term of highest degree is the product of two \( p \)'s of odd index. Again by Theorem (4), this means that the left hand side of the equation in (6.2) is a polynomial with leading coefficient \( \pm 1 \) and integer coefficients. By the rational root theorem, any zero of this polynomial must be an integer. \( \square \)
With the general theory of the previous section, we can describe some of the patterns in periodic expansions for $\sqrt{n}$, at least in the case where $z$ is rational and the periodic part of the expansion is strongly reduced. Again, these results closely parallel those of Anselm and Weintraub [1].

**Theorem 33.** Suppose that $z$ is rational and $\sqrt{n}$ has a strongly reduced periodic maximal expansion, with period length $k$. Let $a = \lfloor \sqrt{n} \rfloor$.

(a) If $z < a + \sqrt{n}$ then $\sqrt{n} = [a, (a_1, \cdots, a_{k-1}, 2a)_{\infty}]_{z}$.

(b) If $z > a + \sqrt{n}$ then $\sqrt{n} = [a, a_1, (a_2, \cdots, a_k, a_1 + h)_{\infty}]_{z}$ where

$$h = \left\lfloor \frac{z}{a + \sqrt{n}} \right\rfloor.$$

Thus, if $\sqrt{n}$ has a strongly reduced periodic expansion, then the tail in the maximal expansion has length 1 or 2.

**Proof.** If $z < a + \sqrt{n}$ and we set $y = a + \sqrt{n}$ then $y > z$, and since $z$ is rational, $\bar{y} = a - \sqrt{n}$ satisfies $-1 < \bar{y} < 0$. Thus, $y$ is strongly reduced and periodic, so it must be purely periodic. Since $\lfloor y \rfloor = 2a$, the result follows.

Next, suppose that $z > a + \sqrt{n}$ and set $h = \left\lfloor \frac{z}{a + \sqrt{n}} \right\rfloor$. We know $\sqrt{n}$ has a maximal expansion $[a, a_1, x_2]_z$, where $a_1$ is the floor of $x_1 = \frac{z}{\sqrt{n} - a}$. Now $x_2 = \frac{z}{x_1 - a_1} > z$ and $\bar{x}_2 = \frac{z}{x_1 - a_1}$. But $\bar{x}_1 = \frac{z}{\sqrt{n} - a} < -1$ so $\frac{-z}{-1 - a_1} < \bar{x}_2 < 0$. Since $a_1 \geq \lfloor z \rfloor$, $-1 < \bar{x}_2 < 0$ so $x_2$ is strongly reduced, and consequently, purely periodic. It remains to show that $a_{k+1} = a_1 + h$. By the proof of Theorem 30 with $j = 2$ we have $a_{k+1} - a_1 = \bar{x}_{k+1} - \bar{x}_1$. Thus, $a_{k+1} - a_1 = \frac{z}{a + \sqrt{n}} + \bar{x}_{k+1} = h$ since $a_{k+1} - a_1$ is an integer and $-1 < \bar{x}_{k+1} < 0$. □

In the first case of Theorem 33, as in the classical case ($z = 1$) and in [1], more structure is present.

**Theorem 34.** Suppose that $z$ is rational and $\sqrt{n}$ has a strongly reduced periodic maximal expansion, with period length $k$. Let $a = \lfloor \sqrt{n} \rfloor$ and assume that $z < a + \sqrt{n}$. Then

$$\sqrt{n} = [a, (a_1, \cdots, a_{k-1}, 2a)_{\infty}]_{z},$$

where for each $j$ with $1 \leq j \leq k - 1$, $a_j = a_{k-j}$. That is, the sequence $a_1, a_2, \ldots, a_{k-1}$ is palindromic.

**Proof.** Setting $x = x_0 = a + \sqrt{n}$ we have $x_1 = \frac{z}{x_0 - 2a} = \frac{z}{\sqrt{n} - a}$. This means that $-\frac{z}{\bar{x}_1} = -(-\sqrt{n} - a) = x$. By Theorem 30 the partial quotients of $x_1$ are the reverse of the partial quotients of $x$. That is,

$$(a_1, a_2, \ldots, a_{k-1}, 2a) = (a_{k-1}, a_{k-2}, \ldots, a_1, 2a),$$

and the result follows. □
If $\sqrt{n}$ does not have a strongly reduced expansion, the results of Theorem 34 may or may not hold. For example,

$$\sqrt{5} = [2, (4, 1, 6, 11180, 6, 1, 4, 4)_{\infty}]_{20/17}$$

has palindromic behavior but as previously noted,

$$\sqrt{2} = [1, (3, 6, 14, 1, 2, 2)_{\infty}]_{4/3},$$

and the palindromic pattern is not present.

Case (a) of Theorem 33 can fail, as well. That is, $\sqrt{n}$ might have a periodic expansion with a tail of size 1 but the period might not end with $2a$. For example, if $z = \frac{21}{8}$ then

$$\sqrt{5} = [2, (11, 21, 2, 3)]_{z}.$$

This example is part of an infinite family:

$$\sqrt{k^2 + 1} = [k, (2k^2 + k + 1, 4k^2 + 2k + 1, k, 2k - 1)_{\infty}]_{z} \quad (6.6)$$

when $z = \frac{4k^2 + 2k + 1}{4k}$. There are also examples with a tail longer than 2 when the tail is not strongly reduced. Among them are

$$\sqrt{34} = [5, 2, 3, (12, 4, 117, 4)_{\infty}]_{z} \quad \text{when } z = \frac{9}{4}, \quad (6.7)$$

$$\sqrt{29} = [5, 8, 3, 4, (12, 5, 688, 5)_{\infty}]_{z} \quad \text{when } z = \frac{24}{7}, \quad (6.8)$$

$$\sqrt{178} = [13, 4, 3, 1, 1, (1, 2, 3, 2, 1, 2, 39, 2)_{\infty}]_{z} \quad \text{when } z = \frac{3}{2}. \quad (6.9)$$

There is a simplification for general periodic expansions with tail length 1, if they have the palindromic behavior of Theorem 34.

**Lemma 35.** As free variables, if $a_j = a_{k-j}$ for $1 \leq j \leq k - 1$ and $a_k = 2a_0$ then $q_k - aq_{k-1} - p_{k-1} = 0$. That is, formula (6.4) of Lemma 31 is a polynomial identity in this situation.

**Proof.** This is a simple consequence of the polynomial identities in Theorem 3. \(\Box\)

Consequently, by Theorem 16, $\sqrt{n}$ will have a $cf_z$ expansion as in Theorem 34 if and only if formula (6.5) is satisfied. If $a_j \geq z$ for all $j \geq 1$ then this will be the maximal expansion for $\sqrt{n}$. From the first several cases of formula (6.5) we have the following expansions.

**Theorem 36.** Let $n = a^2 + b$ where $1 \leq b \leq 2a$ and let $z$ be rational with $1 \leq z \leq 2a$. Strongly reduced expansions for $\sqrt{n}$ satisfying formula (6.5) of period up to 6 have the following forms.

(a) $\sqrt{n} = [a, (2a)_{\infty}]_{z}$, when $z = b$, 

21
(b) $\sqrt{n} = [a, (c, 2a \infty)]_z$, when $z = \frac{bc}{2a}$, for some $c \geq 1$,

(c) $\sqrt{n} = [a, (c, c, 2a \infty)]_z$, when $z^2 + (2ac - b)z - bc^2 = 0$ for some $c \geq z$,

(d) $\sqrt{n} = [a, (c, d, c, 2a \infty)]_z$, when $(2a + d)z^2 + 2c(ad - b)z - bc^2d = 0$ for some $c, d \geq z$,

(e) $\sqrt{n} = [a, (c, d, d, e, c, 2a \infty)]_z$, when

$$z^3 + (2ac + 2ad + d^2 - b)z^2 + c(2ad^2 - bc - 2bd)z - bc^2d^2 = 0$$

for some $c, d \geq z$,

(f) $\sqrt{n} = [a, (c, d, e, d, c, 2a \infty)]_z$, when

$$(2a + 2d)z^3 + (4acd + 2ade + d^2e - 2bc - be)z^2 + 2c(2ad^2e - bcd - 2bde)z - bc^2d^2e = 0$$

for some $c, d, e \geq z$.

By Theorem (3) we may write formula (6.5) in the form

$$bq_{k-1}(a, a_1, \ldots, a_{k-1}) - zq_{k-1}(a_1, \ldots, a_{k-1}, 2a) = 0,$$

from which it follows that for fixed integers $a_1, \ldots, a_{k-1}$ and fixed $z$ we have a linear Diophantine equation of the form $bx - ay = c$. This allows for the construction of families of $n$ for which $\sqrt{n}$ has small period length. For example, in part (d) of Theorem 36 if we let $z = \frac{5}{3}$; $c = 2$, $d = 6$, the resulting equation is $276b - 410a = 150$, with solution $a = 3 + 138t$, $b = 5 + 205t$. As a consequence, for all nonnegative integers $t$ we have

$$\sqrt{(3 + 138t)^2 + 5 + 205t} = [138t + 3, (2, 6, 2, 276t + 6 \infty)]^{1/3}.$$

For a given $z \geq 1$, if there is an $n$ for which $\sqrt{n}$ has a periodic expansion of length $k$ as in Theorem 34, then this construction shows that there are infinitely many $n$ for which $\sqrt{n}$ has period length $k$.

**Conjecture 37.** For every rational $z \geq 1$ and every $k \geq 1$, where $z$ is an integer when $k$ is odd, there are infinitely many integers $n$ for which $\sqrt{n}$ has a maximal $c_{f_z}$ expansion with palindromic behavior as in Theorem 34.

This conjecture is obviously true for periods of length 1 or 2, and not too hard to show for period length 3.

For period length 4, if $n$ is fixed, Pell’s equation comes into play. We have the following theorem.

**Theorem 38.** Let $n = a^2 + b$, where $1 \leq b \leq 2a$. Then $\sqrt{n}$ has maximal expansion of the form $[a, (c, c, d, c, 2a \infty)]_z$ if and only if $(x, d)$ is a positive solution to the Pell equation $x^2 - nd^2 = b^2$ for some integer $x$. When $d$ is such a solution, an expansion will exist for

$$z = \frac{c}{2a + d}(x + b - ad),$$

and $c$ is chosen so that $0 < z \leq \min(2a, d)$.
Proof. The discriminant for \((2a + d)z^2 + 2c(ad - b)z - bc^2d\) is \(nd^2 + b^2\), leading to the Pell equation. When the Pell equation has a solution, solving \((2a + d)z^2 + 2c(ad - b)z - bc^2d = 0\) for \(z\) gives the form of \(z\) above. The condition on \(c\) is needed for the expansion to be strongly reduced.

**Corollary 39.** For each \(n = a^2 + b\) with \(1 \leq b \leq 2a\), there are infinitely many strongly reduced maximal expansions \(\sqrt{n} = [a, (c, d, c, 2a)_{\infty}]_z\).

Proof. There are infinitely many solutions to the Pell equation \(x^2 - nd^2 = b^2\). As \(d\) goes to infinity, \(\frac{x + b - ad}{2a + d}\) approaches \(\sqrt{n} - a < 1\) so there are infinitely many \(d\) with \(\frac{x + b - ad}{2a + d} < 1\). This guarantees that for each such \(d\) there exist \(c\) and \(z\) fulfilling the conditions of Theorem 38. In particular, \(c = 1\) will work. There is also a smallest \(c\) making \(z \geq 1\), and for this \(c\), \(z \leq 2a\) so there are infinitely many expansions with \(z \geq 1\) as well.

The condition on \(c\) in Theorem 38 is not best possible. For example, when \(a = b = 1\), \(n = 2\), the Pell equation is \(x^2 - 2d^2 = 1\). One solution to this equation is \(d = 12, x = 17\), giving \(z = \frac{3}{7}\). We have strongly reduced expansions \(\sqrt{2} = [1, (c, 12, c, 2)_{\infty}]_z\) for \(1 \leq c \leq 4\).

When \(c = 5\), we still have maximal expansion \(\sqrt{2} = [1, (5, 12, 5, 2)_{\infty}]_z\), with \(z = \frac{15}{7} > 2a\). When \(c = 6\), the floor of \(z\) is still \(2\) but the maximal expansion for \(\sqrt{2}\) does not have period \(4\).

When \(z > 2a\), strongly reduced periodic expansions have tail length \(2\). The formulas in these cases are more complicated because Lemma 35 no longer applies. We give a short list below, of the requirements for period lengths up to 3.

**Theorem 40.** Let \(n = a^2 + b\) where \(1 \leq b \leq 2a\) and let \(z\) be rational with \(z > 2a\). Strongly reduced expansions for \(\sqrt{n}\) with tail length \(2\) and period at most \(3\) have the following forms.

- **(a)** \(\sqrt{n} = [a, c, (d)_{\infty}]_z\), when
  \[(2a - 2c + d)z - 2ac^2 + 2acd = 0,\]
  \[z^2 - bz + bc^2 - bcd = 0,\]
  for some \(c, d \geq z,\)

- **(b)** \(\sqrt{n} = [a, c, (d, e)_{\infty}]_z\), when
  \[(2ae - 2cd + de)z - 2ac^2d + 2acde = 0,\]
  \[dz^2 - bez + b^2d - bcde = 0,\]
  for some \(c, d, e \geq z,\)

- **(c)** \(\sqrt{n} = [a, c, (d, e, f)_{\infty}]_z\), when
  \[(2a - 2c + d - e + f)z^2 + (-2ac^2 + 2acd - 2ace + 2acf + 2ae f - 2cde + def)z\]
  \[-2ac^2d + 2acdef = 0,\]
  \[z^3 + (de - b)z^2 + (bc^2 - bcd + bce - bcf - bdf)z + b^2de - bcdef = 0,\]
  for some \(c, d, e, f \geq z,\)

23
Proof. In each case, the top condition is equation (6.3), with the tail length \( j = 2 \). The bottom equation is the result of \( a \) times equation (6.3) subtracted from equation (6.2), after replacing \( n \) by \( a^2 + b \).

For small period, these formulas allow for the construction of infinite families of expansions. We mention the following.

\[
\begin{align*}
\sqrt{9m^2 - 2m} &= [3m - 1, 24m - 6, (24m - 4)_{\infty}]_{16m - 4}, \\
\sqrt{9m^2 - 3m + 1} &= [3m - 1, 8m^2 - 2m, (24m^2 - 6m)_{\infty}]_{12m^2}, \\
\sqrt{9m^2 - m} &= [3m - 1, 9m - 2, (30m - 6, 9m - 1)_{\infty}]_{\frac{12m}{2}}, \\
\sqrt{4m^2 - m} &= [2m - 1, 8m - 3, (12m - 4, 8m - 2)_{\infty}]_{6m - 2}.
\end{align*}
\] (6.10) (6.11) (6.12) (6.13)

The first two of these fit into a doubly infinite family: \( \sqrt{n} = [a, c, (d)_{\infty}]_z \), for all positive integers \( m \) and \( k \) with \( n = a^2 + b \) where \( a = m(4k - 1) - k, \ b = m(4k - 1), \ c = 2m(4m - 1)(4k - 1), \ d = 2m(4m - 1)(4k - 1) + 2m \), \( z = 4m^2(4k - 1) \). There appear to be a large number of formulas similar to (6.12) and (6.13).

In Theorem 40 parts (a) and (b), Pell’s equation again plays a role when \( n \) is fixed.

**Theorem 41.** If \( n = a^2 + b \) then \( \sqrt{n} = [a, c, (d)_{\infty}]_z \) provided that \( nd^2 + b^2 \) is a square and

\[
c = \frac{nd + ab + a\sqrt{nd^2 + b^2}}{2n} \quad \text{and} \quad z = \frac{b^2 + b\sqrt{nd^2 + b^2}}{2n}
\]

are both integers. If \( 2a < z \leq \min(c, d) \), then this is a maximal expansion.

Proof. Adding \( b \) times the top equation to \( 2a \) times the bottom equation in Theorem 6.12 (a) gives the condition \( z = \frac{b}{2a}(2c - d) \). This coupled with \( (2a - 2c + d)z - 2ac^2 + 2acd = 0 \) gives a quadratic equation in \( c \) with positive solution \( c = \frac{nd + ab + a\sqrt{nd^2 + b^2}}{2n} \), implying that \( z = \frac{b^2 + b\sqrt{nd^2 + b^2}}{2n} \). Thus, in order for \( \sqrt{n} \) to be \( [a, c, (d)_{\infty}]_z \), both \( c \) and \( z \) must be integers, which also requires \( nd^2 + b^2 \) to be a square. If \( c \) and \( z \) are integers in the given form, it follows that the equations in Theorem 40 (a) are satisfied.

We conjecture that there are integers \( c \) and \( z \) satisfying the requirements of Theorem 41 for any non square \( n \), though we do not have a proof. However, with part (b) of Theorem 40, we have more freedom.

**Theorem 42.** For each positive integer, \( n \), not a square, there are infinitely many \( c, d, e \) for which \( \sqrt{n} = [a, c, (d, e)_{\infty}]_z \). In particular, with \( n = a^2 + b, \ 1 \leq b \leq 2a \), if \( m^2n + 1 = k^2 \) then \( \sqrt{n} \) has maximal expansion \( [a, c, (d, e)_{\infty}]_z \) with \( c = k - 1 + am, d = bm, e = 2(k - 1), z = bm \), for all solutions with \( bm > 2a \).
Proof. Letting $d = mb$, then $nd^2 + b^2 = b^2(nm^2 + 1)$, and $nm^2 + 1$ is a square infinitely often. Suppose that $nm^2 + 1 = k^2$ for positive integer $k$. If we write

$$z = \frac{be(b + \sqrt{nd^2 + b^2})}{2nd} = \frac{eb(k + 1)}{2mn}$$

and

$$c = \frac{e}{2} + \frac{ae(k + 1)}{2mn}$$

then $a, b, c, d, e, z$ formally satisfy the equations in Theorem 40 (b). Note that $c = \frac{1}{2}(e + \frac{2az}{b})$. Since $b \leq 2a$, $c \geq \frac{1}{2}(e + z)$. Thus, if $e$ can be selected so that $e \geq z$, $2a < z \leq d$ and $c$ is an integer, then the maximal expansion of $\sqrt{n}$ will be $[a, c, (d, e)_{\infty}]_z$. Let $(m, k)$ to be a positive integer solution to the Pell equation $x^2 - ny^2 = 1$ with $k > 1$. If $e = 2(k - 1)$ then

$$z = \frac{2b(k^2 - 1)}{2mn} = \frac{2bm^2n}{2mn} = bm = d$$

and $c = am + k - 1 > am - 1 + m\sqrt{n} > 2am - 1$ so $c \geq bm = z$. □

Cases of expansions with longer tails or longer periodic part become more complicated but presumably they could be investigated with similar techniques.

References


2010 *Mathematics Subject Classification*: 11A55.

*Keywords*: Continued fraction, Linear Diophantine equation, Pell's equation