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Greedy Matching in Young's Lattice

JOHN GREENE
Department of Mathematics and Statistics, University of Minnesota, Duluth, Duluth, MN 55812, U.S.A.

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Abstract. If the level sets of a ranked partially ordered set are totally ordered, the greedy match between adjacent levels is defined by successively matching each vertex on one level to the first available unmatched vertex, if any, on the next level. Aigner showed that the greedy match produces symmetric chains in the Boolean algebra. We extend that result to partially ordered sets which are products of chains.

It is widely thought that for Young's lattices corresponding to rectangles, the greedy match is complete. We show here that the greedy match is, in fact, complete for $n \times 2$, $n \times 3$ and $n \times 4$ rectangles but not for $n \times k$ rectangles if $k \geqslant 5$ and n is sufficiently large.

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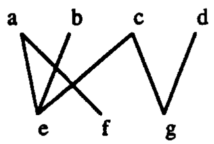
1. Introduction

Let G be a bipartite graph with vertex sets U and V. A matching in G is a bijection between subsets $U' \subseteq U$ and $V' \subseteq V$ such that corresponding pairs of vertices are joined by edges of G. A matching is called complete if either $U' = U$ or $V' = V$.

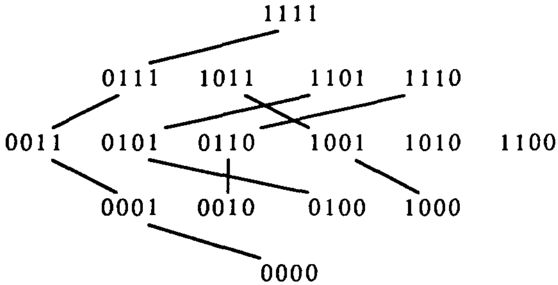
If U and V are each totally ordered, the greedy match on G is defined as follows. Examine the vertices of U one at a time in order. For each vertex $u \in U$, match it to the first compatible vertex in V (if any) which has not been matched. For example, the greedy match for

G: 

(1.1)

(ordered from left to right) is $\{ea, gc\}$.

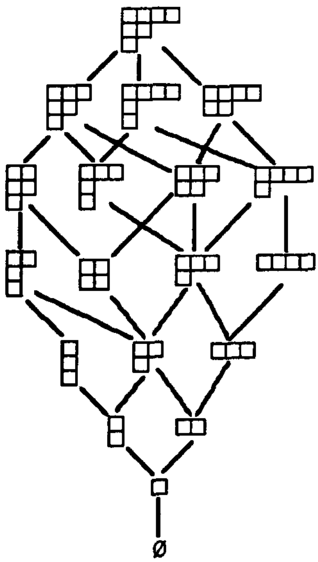
It is well known [1] that for the Boolean algebra B_n, if level sets are ordered lexicographically then the greedy match is, in fact, a complete match between any two adjacent levels. Moreover, the resulting induced chain decomposition is a symmetric chain decomposition. This is illustrated for B_4:



(1.2)

In this paper, we investigate the properties of the greedy match in Young's lattice, $\mathscr{Y}_\lambda$, the set of all partitions whose Ferrer's diagrams fit inside the Ferrer's diagram of a given partition λ. The order relation is defined by containment of Ferrer's diagrams and each level set is ordered lexicographically. For example, the Hasse diagram for $\mathscr{Y}_{421}$ is:

$\mathscr{Y}_{421}$:



(1.3)

The motivation for this paper is that when λ is a rectangle, it is known that there is a complete match between any two successive levels of Young's lattice [5], but there is no known explicit construction of such a match in general. The first attempt at constructing a complete match is usually to investigate the greedy match and it was widely believed that the greedy match worked. Stanley [5] conjectured that in the case of a rectangle, Young's lattice is a symmetric chain order. Again, this has been verified for small cases [2, 4, 8] but not in general. If the Ferrer's diagram of a given partition is not a rectangle, very little is known (see [6]). We show here that the greedy match is complete for $\mathscr{Y}_{n^2}$, $\mathscr{Y}_{n^3}$, $\mathscr{Y}_{n^4}$ and $\mathscr{Y}_{abc}$ but that it is not complete for $\mathscr{Y}_{n^m}$ if $m \geqslant 5$ and n is sufficiently large.

Even though the greedy match fails to be a complete match in general, it is amazing that is works as well as it does. That this is the case suggests that perhaps the greedy match is the place to begin in the search for general complete matches and/or symmetric chain decompositions.

This paper is organized as follows. Section 2 contains general information on the greedy match and basic notation used in the paper. Sections 3 and 4 contain the major results of this paper, stated above. Sections 5 and 6 contain two involved proofs to theorems in Sections 3 and 4, and Section 7 gives remarks and conclusions.

2. Notation and Preliminary Results

The greedy match for $G = (U, V)$ was defined by matching the elements of U into the elements of V. If this is denoted $U \to V$, then one might ask about the greedy match $V \to U$. In Example (1.1), the same match results. In fact, this is always the case.

LEMMA 2.1. *For any bipartite graph $G = (U, V)$, the greedy matches $U \to V$ and $V \to U$ are the same.*

Proof. As with most of the proofs in this paper, we proceed by the method of minimal counterexample. Let $u \in U$ be the first element of U for which the matches $U \to V$ and $V \to U$ differ. That is, either $u \to v$ in $U \to V$ but $v \not\to u$ in $V \to U$ or u is unmatched in $U \to V$ but is matched in $V \to U$. These two cases may be handled simultaneously by appending an element ∞ as the last element of V, if necessary, and matching u to ∞ if u is unmatched in $U \to V$. There are two possibilities to consider:

Case 1. $v \not\to u$ because $v \to u'$ which comes before u in U.

Since there is an edge $u'v$ but u' is not matched to v in $U \to V$, there must be a v' in U appearing before v such that $u' \to v'$ in $U \to V$. By minimality of u, we must have that $v' \to u'$, a contradiction.

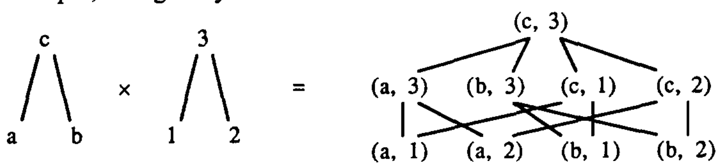
Case 2. $v \not\to u$ because for some v' coming before v', $v' \to u$ in $V \to U$. Since there is an edge uv' but $u \not\to v'$, there must be a u' coming before u in U such that $u' \to v'$. Again, by minimality of u, $v' \to u'$ in $V \to U$. This second contradiction completes the proof.

Let P be a ranked partially ordered set (in this paper, all partially ordered sets are ranked and have linearly ordered level sets). As noted in the introduction, if there is a match between each pair of adjacent levels in P then this set of matches, which by abuse of notation we will refer to as a match, induces a decomposition of P into disjoint chains. An element u will be called maximal (minimal) with respect to a match if u is the element of largest (smallest) rank on a chain. Equivalently, u is maximal (minimal) if u is not matched in the next higher (lower) level.

A match in P is called complete if the associated matches between successive levels are all complete. Obviously, a match is complete if it never happens that the lower of two adjacent levels contains maximal elements while the higher contains minimal elements. If P is rank unimodal (the level sizes of P increase to a maximum and then decrease), then a match is complete if all minimal elements are below the largest level (or all maximal elements are above the largest level).

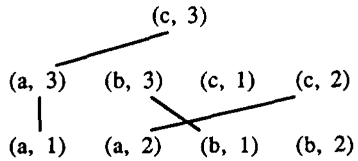
Write $u \to v$ with respect to a match if the match takes u to v. We say v covers u, denoted $v \cdot\!> u$, if $v \geqslant u$ and v and u are on adjacent levels. If $u \to v$, then clearly $v \cdot\!> u$.

Given two partially ordered sets P and Q, a natural order on the level sets of $P \times Q$ is obtained lexicographically: given (u, v) and (u', v') on the same level of $P \times Q$, we say (u, v) precedes (u', v') if the rank of u is less than the rank of u' or if u precedes u' in P or if $u = u'$ but v precedes v' in Q. The greedy match is not well-behaved for product posets in the sense that if P and Q both have complete greedy matches, then $P \times Q$ does not necessarily have a complete greedy match. For example, the greedy match in



(2.2)

is



(2.3)

A ranked partially ordered set P is called a symmetric chain order (SYM) or is said to admit a symmetric chain decomposition if P can be expressed as a union of disjoint saturated chains which are all symmetric about the midlevel of P. A saturated chain is one in which there is an element from each level between the minimal and maximal elements of the chain. If a symmetric chain decomposition is induced by the greedy match, call P a greedy *SYM*.

THEOREM 2.4. *If P is a product of chains, then P is a greedy SYM. We defer the proof of Theorem* 2.4 *to Section* 5.

COROLLARY 2.5. *The Boolean algebra B_n is a greedy SYM.*

Proof. B_n is the product of n 2-element chains.

In this paper, we are primarily interested in Young's lattice. We introduce the following definitions and notation. A partition is a finite nonincreasing sequence of nonnegative integers called parts. We allow parts of size 0. If a partition λ has m parts, denoted $|\lambda| = m$, we will write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. We say λ is a partition of n if $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$, and denote this $\|\lambda\| = n$.

Young's lattice for a partition λ, denoted $\mathscr{Y}_\lambda$, is the set of all partitions μ such that $|\mu| = |\lambda|$ and $\mu_i \leqslant \lambda_i$ for $1 \leqslant i \leqslant |\lambda|$. The order relation on $\mathscr{Y}_\lambda$ where $|\lambda| = m$ is

$$\mu \leqslant v \text{ if and only if } \mu_i \leqslant v_i \text{ for } 1 \leqslant i \leqslant m.$$

The level sets of $\mathscr{Y}_\lambda$ are loaded lexicographically. If λ has m parts all of size n, write $\lambda = n^m$. If $m = 2$ or $m = 3$ we may write $\lambda = ab$ or $\lambda = abc$.

If λ and μ both have m parts, define the difference of λ and μ componentwise:

$$\lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_m - \mu_m).$$

Define the n-complement of λ in a rectangle n^m by

$$\bar{\lambda} = (n - \lambda_m, n - \lambda_{m-1}, \ldots, n - \lambda_1).$$

For $n \geqslant \lambda_1$, $\bar{\lambda}$ is a partition and $\|\bar{\lambda}\| = mn - \|\lambda\|$.

3. The Greedy Match in $\mathscr{Y}_{n^m}$

The greedy match has many nice properties in $\mathscr{Y}_{n^m}$. Several are listed in the following proposition.

PROPOSITION 3.1.

(i) *If $\lambda_m > 0$ and $0 \leqslant k \leqslant \lambda_m$, then $\lambda \to \mu$ if and only if $\lambda - k^m \to \mu - k^m$.*
(ii) *$\lambda \to \mu$ if and only if $\bar{\mu} \to \bar{\lambda}$.*
(iii) *If λ is maximal, then $\lambda_1 = n$.*
(iv) *If $\lambda = (n, n, \ldots, n, n-1, \lambda_j, \ldots, \lambda_m)$ for some $j > 2$, then λ is not maximal.*
(v) *If λ is minimal, then $\lambda_m = 0$.*
(vi) *If λ is not maximal in $\mathscr{Y}_{n^m}$, then if $\lambda \to \lambda'$ in $\mathscr{Y}_{n^m}$, $\lambda \to \lambda'$ in $\mathscr{Y}_{k^m}$ for any $k \geqslant \lambda'_1$.*

Proof. Parts (i) and (ii) follow from minimal counterexample arguments. In (ii), for example, suppose the conclusion is false and let λ be a minimal counterexample. That is, let λ be the partition of smallest rank and lexicographically first on that rank such that $\lambda \to \mu$ but $\bar{\mu} \not\to \bar{\lambda}$. Since $\bar{\lambda} \cdot\!> \bar{\mu}$ if $\mu \cdot\!> \lambda$, either $\bar{\mu} \to \bar{\lambda}'$ which precedes $\bar{\lambda}$ or $\nu \to \bar{\lambda}$ where ν precedes $\bar{\mu}$. In the first case, by minimality of λ, $\lambda' \to \bar{\bar{\mu}} = \mu$ contradicting $\lambda \to \mu$. In the second case, $\nu \to \bar{\lambda}$ implies that $\bar{\nu} \cdot\!> \lambda$. Since $\bar{\lambda} \cdot\!> \nu$, $\bar{\lambda} \cdot\!> \bar{\mu}$ and ν precedes $\bar{\mu}$, for some $j > i$, $\nu_j = \bar{\mu}_j + 1$, $\nu_i = \bar{\mu}_i - 1$ and $\nu_k = \bar{\mu}_k$ if $k \neq i$ or j. Consequently, $\bar{\nu}$ and μ differ in positions $m - i$ and $m - j$, with $m - i > m - j$ and $\bar{\nu}_{m-j} = \mu_{m-j} - 1$. Thus, $\bar{\nu}$ precedes μ. Since $\lambda \not\to \bar{\nu}$, there must be a λ' preceding λ such that $\lambda' \to \nu'$. By minimality of λ, $\nu \to \bar{\lambda}'$ contradiction $\nu \to \bar{\lambda}$. This completes the proof of part (ii).

For (iii), if $\lambda_1 < n$, let $\mu = (\lambda_1 + 1, \lambda_2, \ldots, \lambda_m)$. Since $\mu \cdot\!> \lambda$ and μ covers no partition preceding λ, either $\lambda \to \mu$ or $\lambda \to \mu'$ which precedes μ. A similar idea works for parts (iv) and (v). Part (vi) is an easy consequence of part (i).

If $m = 1$, $\mathscr{Y}_{n^1}$ is a chain of length $n + 1$. With two parts, $\mathscr{Y}_{n^2}$ is also fairly trivial.

THEOREM 3.2. *$\mathscr{Y}_{n^2}$ is a greedy SYM.*

Proof. The explicit characterization of the greedy match in $\mathscr{Y}_{n^2}$ is

$$(a, b) \to \begin{cases} (a+1, b), & \text{if } a - b \text{ is even and } a < n \\ (a, b+1), & \text{if } a - b \text{ is odd.} \end{cases}$$

To verify this characterization, by Proposition 3.1(i), it is enough to show that for $2k < n$, $(2k, 0) \to (2k+1, 0) \to (2k+1, 1)$. Clearly, $(0,0) \to (1,0) \to (1,1)$. Suppose $(2k-2, 0) \to (2k-1, 0) \to (2k-1, 1)$. By Proposition 3.1(i), $(2k-1, 1) \to (2k, 1) \to (2k, 2)$. Thus, $(2k, 0) \not\to (2k, 1)$. Since $(2k, 0)$ is not maximal, $(2k, 0) \to (2k+1, 0)$. Also, $(2k+1, 1)$ covers both $(2k, 1)$ and $(2k+1, 0)$. Since $(2k, 1) \to (2k, 2)$, $(2k+1, 0) \to (2k+1, 1)$.

The chains in $\mathscr{Y}_{n2}$ are now easy to describe. Minimal elements are of the form $(2k, 0)$ and the chain of $(2k, 0)$ is

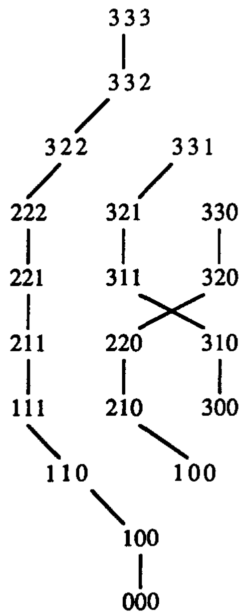
$$(2k, 0) \to (2k+1, 0) \to (2k+1, 1) \to \cdots \to (n, n-2k),$$

which is clearly symmetric. This completes the proof.

An interesting property of the greedy match in $\mathscr{Y}_{n2}$ is that if $\lambda_1 \geqslant \lambda_2 + 2$, then $\lambda \to \mu$ if and only if $\lambda - (2,0) \to \mu - (2,0)$. Given this property and Proposition 3.1(i), the entire match can be built up from the two cases $(0,0) \to (1,0)$ and $(1,0) \to (1,1)$.

The greedy match gets more complicated as the number of parts increases. With three parts, the chains are no longer symmetric as the case $\mathscr{Y}_{33}$ shows.

The greedy match in $\mathscr{Y}_{33}$ is



(3.3)

It is always true, however, by Proposition 3.1(ii) that chains are either symmetric or they come in pairs which are symmetric about the midlevel of $\mathscr{Y}_{nm}$.

The partitions $(0,0)$ and $(1,0)$ have the property that $\lambda_1 - \lambda_2 < 2$. An extension of this property plays an important role for the greedy match in $\mathscr{Y}_{nm}$. Call a partition λ irreducible if for some $i < m$, $\lambda_i - \lambda_{i+1} < 2$. Given the partitions λ and μ with m and n parts respectively, if $\lambda_m \geqslant \mu_1$, let $\lambda\mu$ denote the partition with $m+n$ parts $(\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_n)$. If λ is irreducible, then there is a smallest j such that

$\lambda_j - \lambda_{j+1} < 2$. Write $\lambda = \mu\nu$ where $\nu = (\lambda_j, \ldots, \lambda_m)$ (if $j = 1$, $\mu = \emptyset$). For example, if $\lambda = (9, 7, 3, 2, 0)$, then $\mu = (9, 7)$ and $\nu = (3, 2, 0)$.

THEOREM 3.4. *Suppose λ is not irreducible, $\lambda_1 \leqslant n$, $|\lambda| = m_1$, μ is irreducible, $\lambda_{m_1} \geqslant \mu_1 + 2$, $|\mu| = m_2$ and $\mu \to \mu'$ as an element of $\mathscr{Y}_{nm_2}$. Then $\lambda\mu \to \lambda\mu'$ as an element of $\mathscr{Y}_{nm_1+m_2}$.*

The proof of Theorem 3.4 is given in Section 6.

COROLLARY 3.5. *If $\lambda \in \mathscr{Y}_{nm}$ is irreducible and $\lambda \to \lambda'$, then λ' is irreducible.*

Proof. We use induction on m. By inspection, the corollary is true for $m = 2$; suppose the corollary is true for $m < k$ and $|\lambda| = k$. If $\lambda_1 = \lambda_2$, then $\lambda'_1 - \lambda'_2 \leqslant 1 < 2$ so λ' is irreducible. If $\lambda_1 - \lambda_2 = 1$, then by Lemma 3.1 (iv), (vi), $\lambda_1 = \lambda'_1$ and again $\lambda'_1 - \lambda'_2 \leqslant 1 < 2$. If $\lambda_1 - \lambda_2 \geqslant 2$, then $\lambda = \mu\nu$ where $\mu = (\lambda_1)$ and $\nu = (\lambda_2, \ldots, \lambda_m)$, and ν is irreducible. By Theorem 3.4, $\lambda \to \mu\nu'$ and by induction, ν' is irreducible so λ' is also irreducible.

COROLLARY 3.6. *Suppose $|\lambda| = m_1$, $|\mu| = m_2$, $|\nu| = m_3$, $\lambda_1 \leqslant n$, $\lambda_{m_1} \geqslant \mu_1 + 2$, $\mu_{m_2} \geqslant \nu_1 + 2$, λ and ν are not irreducible but μ is irreducible. Then if $\mu \to \mu'$ as an element of $\mathscr{Y}_{nm_2}$, $\lambda\mu\nu \to \lambda\mu'\nu$ as an element of $\mathscr{Y}_{nm_1+m_2+m_3}$.*

Proof. By Theorem 3.4, we must show that $\mu\nu \to \mu'\nu$ as an element of $\mathscr{Y}_{nm_2+m_3}$. Consider $\overline{\mu'\nu} = \bar{\nu}\bar{\mu}'$ in $\mathscr{Y}_{nm_2+m_3}$. Since $\bar{\nu}$ is not irreducible but $\bar{\mu}'$ is, and since $\bar{\mu}' \to \bar{\mu}$ in $\mathscr{Y}_{nm_2}$, by Theorem 3.4, $\bar{\nu}\bar{\mu}' \to \bar{\nu}\bar{\mu}$ in $\mathscr{Y}_{nm_2+m_3}$ and so $\mu\nu \to \mu'\nu$ as desired.

As an example, consider $\lambda = (9, 7, 3, 2, 0) = (9, 7)(3, 2)(0)$. Since $(3, 2) \to (3, 3)$ in $\mathscr{Y}_{n2}$, by Corollary 3.6, $\lambda \to (9, 7, 3, 3, 0)$.

Now let $\lambda^* = (2m - 2, 2m - 4, \ldots, 2, 0)$ (which depends on m of course). λ^* is the smallest non-irreducible partition with m parts. If λ is not irreducible, then $\lambda - \lambda^*$ is a partition.

COROLLARY 3.7. *If λ is not irreducible, then $\lambda \to \mu$ if and only if $\lambda - \lambda^* \to \mu - \lambda^*$.*

Proof. An easy minimal counterexample argument.

Corollary 3.7 is a very powerful inductive tool which is now used to prove the major results of this paper.

THEOREM 3.8. *The greedy matches in $\mathscr{Y}_{n3}$ and $\mathscr{Y}_{n4}$ are complete.*

Proof. Since chains $\mathscr{Y}_{nm}$ are either symmetric or come in pairs that are symmetric about the midlevel, the theorem will follow if we show that every maximal partition is at or above the midlevel. If $\mathscr{Y}_{nm}$ has an even number of ranks, then there will be two levels of equal size rather than one midlevel. In this case, there can be no maximal elements on the lower of the two levels. Thus, in general, the greedy match will be complete in $\mathscr{Y}_{nm}$ if for any maximal λ, $\|\lambda\| \geqslant \lceil mn/2 \rceil$ where $\lceil x \rceil$ is the nearest integer greater than or equal to x.

By Proposition 3.1 parts (iii) and (iv), if λ is maximal in $\mathscr{Y}_{nm}$, then $\lambda_1 = n$ and $\lambda_2 \neq n - 1$. If $\lambda_2 = n$, then $\|\lambda\| \geqslant 2n$ so λ is at or above the midlevel in $\mathscr{Y}_{n3}$ or $\mathscr{Y}_{n4}$.

If $\lambda_2 < n - 1$ and λ is irreducible, then $\lambda = \mu\nu \to \mu\nu'$ by Theorem 3.4 so λ is not maximal in this case. If λ is not irreducible, then by Corollary 3.7, λ is maximal in $\mathscr{Y}_{n^m}$ if and only if $\lambda - \lambda^*$ is maximal in $\mathscr{Y}_{(n+2-2m)^m}$. We now proceed by induction on n. The theorem can easily be checked for $\mathscr{Y}_{n^m}$ if $n = 1$ and is vacuously true for $n < 1$. So assume that the theorem is true for $\mathscr{Y}_{(n+2-2m)^m}$. Then if λ is maximal in $\mathscr{Y}_{n^m}$,

$$\|\lambda - \lambda^*\| \geqslant \lceil m(n+2-2m)/2 \rceil$$

so

$$\begin{aligned}\|\lambda\| &\geqslant \lceil m(n+2-2m)/2 \rceil + \|\lambda^*\| \\ &\geqslant \lceil m(n+2-2m)/2 \rceil + m(m-1) \\ &\geqslant \lceil mn/2 \rceil ,\end{aligned}$$

as desired.

Unfortunately, this proof breaks down if $m \geqslant 5$. The problem is that if $\lambda_2 = n$, it no longer follows that $\|\lambda\| \geqslant \lceil mn/2 \rceil$. In fact, Corollary 3.6 can be used to know that the match is not complete if $m \geqslant 5$, for it can be shown that in $\mathscr{Y}_{n^4}$, the partition (a, a, b, b) matches to $(a+1, a, b, b)$ if $a - b \geqslant 2$ and $a < n$. This partition is maximal if $a = n \geqslant b + 2$. By Corollary 3.6, $(n, n, b, b, 0)$ is maximal in $\mathscr{Y}_{n^5}$ if $n - b \geqslant 2$ and $b \geqslant 2$. Consequently, $(9, 9, 2, 2, 0)$ is maximal in $\mathscr{Y}_{9^5}$. But $\|(9, 9, 2, 2, 0)\| = 22 < 23 = \lceil 9*5/2 \rceil$. Thus, the greedy match is not complete in $\mathscr{Y}_{9^5}$. In fact, $(9, 9, 2, 2, 0)$ is the only maximal element on level 22 and its 9-complement, $(9, 7, 7, 0, 0)$, is the only minimal element on level 23.

In general, $\lambda = (n, n, 2(m-4), 2(m-4), 2(m-5), 2(m-6), \ldots, 2, 0)$ is maximal in $\mathscr{Y}_{n^m}$ and below the midlevel provided $n \geqslant 2m - 2$ if m is even or $n \geqslant 2m - 1$ if m is odd. Thus, for $m > 4$, $\mathscr{Y}_{n^m}$ is not complete via the greedy match if $n \geqslant \lceil (4m-3)/2 \rceil$. This bound is not the best possible for large m.

4. The Greedy Match in $\mathscr{Y}_{abc}$

If $\lambda \neq n^m$, then the greedy match in $\mathscr{Y}_\lambda$ is more complicated than was the case in Section 3. For example, there is no analog for the complement of a partition in $\mathscr{Y}_\lambda$. We do have a partial generalization of Proposition 3.1, however.

PROPOSITION 4.1. *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, *where* $\lambda_m > 0$.

(i) $\mu \to \nu$ *in* $\mathscr{Y}_\lambda$ *if and only if* $\mu - k^m \to \nu - k^m$ *in* $\mathscr{Y}_\lambda$ *for any* k *with* $0 \leqslant k \leqslant \mu_m$.

(ii) *If* μ *is minimal in* $\mathscr{Y}_\lambda$, *then* $\mu_m = 0$.

(iii) *If* μ *is maximal in* $\mathscr{Y}_\lambda$, *then* $\mu_1 = \lambda_1$.

The proof of Proposition 4.1 is straightforward. We give three more general properties of the greedy match in $\mathscr{Y}_\lambda$.

PROPOSITION 4.2. *If* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ *with* $\lambda_m > 0$, *then the greedy match in* $\mathscr{Y}_\lambda$ *is the same as the greedy match in* $\mathscr{Y}_{n^m}$ *up through level* k, *where* $k = \min(2\lambda_2 + 1, 3\lambda_3 + 2, \ldots, m\lambda_m + m - 1)$.

Proof. So long as the level sets are the same, the matches in $\mathscr{Y}_\lambda$ and $\mathscr{Y}_{n^m}$ must agree. The first element in $\mathscr{Y}_{n^m}$ which is not in $\mathscr{Y}_\lambda$ is of the form $\mu = (a, a, \ldots, a, 0, \ldots, 0)$ with j a's, where $\lambda_{j-1} \geqslant a$ but $\lambda_j = a - 1$. This occurs at level $aj = j\lambda_j + (j-1) + 1$, and the result follows.

PROPOSITION 4.3. *If* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ *with* $\lambda_m > 0$, *then the greedy match in* $\mathscr{Y}_\lambda$ *is complete from level* k *to level* $\|\lambda\|$, *where*

$$k = 2\lambda_2 + \lambda_3 + \cdots + \lambda_m = \|\lambda\| - (\lambda_1 - \lambda_2).$$

Proof. If $\|\mu\| > k$, then $\mu_1 > \mu_2$. It now follows easily by the method of minimal counterexample that the match is given by $\mu \to \mu'$ where $\mu'_1 = \mu_1 + 1$ and that this match is complete.

PROPOSITION 4.4. *Let* $\mu \in \mathscr{Y}_\lambda$ *where* $\lambda = (n, \lambda_2, \ldots, \lambda_m)$ *with* $\lambda_m > 0$. *If* μ *is not minimal in* $\mathscr{Y}_{n^m}$, *then* μ *is not minimal in* $\mathscr{Y}_\lambda$.

Proof. If μ is not minimal in $\mathscr{Y}_{n^m}$, then for some ν, $\nu \to \mu$. If $\mu \cdot\!> \nu$, then $\nu \in \mathscr{Y}_\lambda$. By the method of minimal counterexample, ν cannot map to a predecessor of μ in $\mathscr{Y}_\lambda$ so if $\nu \not\to \mu$ in $\mathscr{Y}_\lambda$, some predecessor of ν must match to μ.

The following is the main theorem of this Section.

THEOREM 4.5. *The greedy match is complete in* $\mathscr{Y}_{ab}$ *and* $\mathscr{Y}_{abc}$.

Proof. We will show that the match between any two adjacent levels is complete by showing that there cannot be a minimal element in the higher level if there is a maximal element in the lower level. The proof is in three parts, the first of which handles the case $\mathscr{Y}_{ab}$.

PART 1. *The greedy match is complete in* $\mathscr{Y}_{ab}$.

Proof. Maximal elements in $\mathscr{Y}_{ab}$ are of the form (a, v) and minimal elements are of the form $(u, 0)$ by Proposition 4.1 parts (ii) and (iii). Thus, the maximal partitions are all of rank greater than or equal to that of the minimal partitions. Note that this shows more than that the match in $\mathscr{Y}_{ab}$ is complete: $\mathscr{Y}_{ab}$ is also rank unimodal.

PART 2. *If* $(a, k, 0)$ *in* $\mathscr{Y}_{abc}$ *is maximal on level* n, *then there are no minimal partitions on level* $n + 1$.

Proof. If $k = b$, then all partitions on level $n + 1$ have three nonzero parts. If $k < b$, since $(a, k, 0) \not\to (a, k + 1, 0)$, it must be that $(a - 1, k + 1, 0) \to (a, k + 1, 0)$. Any partition $(u, v, 0)$ on level n is of the form $(a - i, k + i, 0)$, so by induction on i it follows that if $u < a$, then $(u, v, 0) \to (u + 1, v, 0)$. As a consequence, n must be even since $(u, u - 1, 0)$ will not match to $(u + 1, u - 1, 0)$ if $u \leqslant b$. Now if $(u, v, 0)$ is on level $n + 1$, then $u > v$ (since $n + 1$ is odd) and $(u - 1, v, 0) \to (u, v, 0)$ implies that $(u, v, 0)$ is not minimal.

PART 3. *The match between level k and level k + 1 is complete in* $\mathcal{Y}_{abc}$.

Proof. Assume level k has maximal elements. We will show that level $k+1$ does not have minimal elements. If $(a, v, 0)$ is maximal, we are done by Part 2, so assume every maximal element of level k has three nonzero parts.

Let $L_n(\lambda)$ denote the number of partitions on level n in $\mathcal{Y}_\lambda$. Note that $L_n(a, b, c) = L_n(a, b, 0) + L_{n-3}(a-1, b-1, c-1)$. (A similar result holds for general λ). By way of induction, assume the greedy match is complete in $\mathcal{Y}_{uvw}$ if $u+v+w < a+b+c$. By Proposition 4.1(i), if μ is maximal in $\mathcal{Y}_{abc}$ and $\mu_3 > 0$, then $\mu - 1^3$ is maximal in $\mathcal{Y}_{a-1,b-1,c-1}$. Since this second poset is complete, $L_{k-2}(a-1, b-1, c-1) < L_{k-3}(a-1, b-1, c-1)$.

If $(u, v, 0) \to (u, v, 1)$ in $\mathcal{Y}_{abc}$, then $(u-1, v-1, 0)$ is minimal in $\mathcal{Y}_{a-1,b-1,c-1}$. Since there are no minimal elements on level $k-2$ in $\mathcal{Y}_{a-1,b-1,c-1}$, if follows that in $\mathcal{Y}_{abc}$, partitions with less than three nonzero parts on level k match to partitions with less than three nonzero parts on level $k+1$. To show the match is complete, we must show that $L_{k+1}(a, b) \leqslant L_k(a, b)$ (so that there are no minimal partitions on level k).

It is easy to see that if $L_{n+1}(a, b) > L_n(a, b)$, then n is odd and $n \leqslant \min(a, 2b)$. Also, if $n < \min(a, 2b)$ then $L_{n+1}(a, b) \geqslant L_n(a, b)$. If $n < \min(a, 2b)$, then $n - 3 < \min(a-1, 2(b-1))$ so if $L_{n-1}(a, b) > L_n(a, b)$, then $L_{n+1-3k}(a-k, b-k) \geqslant L_{n-3k}(a-k, b-k)$ for all $k \geqslant b$. But

$$L_{k-3}(a-1, b-1, c-1) = \sum_{i=1}^{c} L_{k-3i}(a-i, b-i)$$

so if $L_{k+1}(a, b) > L_k(a, b)$, then

$$\begin{aligned} L_{k-3}(a-1, b-1, c-1) &\leqslant \sum_{i=1}^{c} L_{k+1-3i}(a-i, b-i) \\ &= L_{k-2}(a-1, b-1, c-1). \end{aligned}$$

This contradicts the condition $L_{k-3}(a-1, b-1, c-1) > L_{k-2}(a-1, b-1, c-1)$, so it must be that $L_{k+1}(a, b) \leqslant L_k(a, b)$. This completes the proof.

5. A Proof of Theorem 2.4

We will prove slightly more: if P satisfies a certain regularity condition and P is a greedy *SYM*, then $C_m \times P$ satisfies that same condition and is a greedy *SYM*. The condition is as follows:

> If there are two chains (induced by the greedy match) $u_1 \to u_2 \to \cdots \to u_n$ and $v_1 \to v_2 \to \cdots \to v_m$ such that u_j and v_k have the same rank, u_j precedes v_k, u_{j+1} precedes v_{k+1} and v_{k+1} covers u_j, then $n - j \geqslant m - k$. (5.1)

Pictorially, if

$$\begin{tikzcd} u_{j+1} & v_{k+1} \\ u_j \arrow[u] \arrow[ur, dotted, dash] & v_k \arrow[u] \end{tikzcd} \tag{5.2}$$

then v_k is at least as close to the top of its chain as u_j is to the top of its chain. Certainly, condition (5.1) is vacuously true if P is itself a chain.

Now suppose P satisfies (5.1) and P is a greedy SYM. We shall prove first that $C_m \times P$ is a greedy SYM and then that $C_m \times P$ satisfies condition (5.1). To show that $C_m \times P$ is a greedy SYM, we construct the match in $C_m \times P$. Given an element (c_i, v) in $C_m \times P$, let $v = v_j$ where $v_1 \to \cdots \to v_j \to \cdots \to v_n$ is the chain of v induced by the greedy match in P. We claim

$$(c_i, v) = (c_i, v_j) \to \begin{cases} (c_i, v_{j+1}), & \text{if } i+j-1 < n \\ (c_{i+1}, v_j), & \text{if } i+j-1 \geqslant n,\ i < m. \end{cases} \tag{5.3}$$

The picture corresponding to condition (5.3) for $m = 3$ and $n = 4$ is

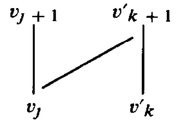
$$\begin{tikzcd}[column sep=tiny] & & (c_3, v_4) \arrow[dl, dash] & & & \\ & (c_2, v_4) \arrow[dl, dash] & & (c_3, v_3) \arrow[dl, dash] & & \\ (c_1, v_4) \arrow[dr, dash] & & (c_2, v_3) \arrow[dr, dash] & & (c_3, v_2) \arrow[dr, dash] & \\ & (c_1, v_3) \arrow[dr, dash] & & (c_2, v_2) \arrow[dr, dash] & & (c_3, v_1) \\ & & (c_1, v_2) \arrow[dr, dash] & & (c_2, v_1) & \\ & & & (c_1, v_1) & & \end{tikzcd} \tag{5.4}$$

Given that property (5.3) holds, the proof of the theorem reduces to the usual proof that the product of two symmetric chain orders is a symmetric chain order (the chains in (5.4) are symmetric in $C_m \times P$).

To show that (5.3) holds, suppose not and let (c_i, v_j) be the minimal counterexample. By minimality of (c_i, v_j), nothing preceding it can match to the element proposed for it by (5.3). So if property (5.3) fails, it must be that (c_i, v_j) matches to a predecessor of the match proposed for it by (5.3). Suppose (c_i, v_j) matches to (c_i, v'). (If $(c_i, v_j) \to (c_{i+1}, v_j)$ then either this is in agreement with (5.3) or an easy contradiction arises.) Now v' is on some chain, so write $v' = v'_{k+1}$, and assume this chain has length n'.

If $i+j-1 < n$, then (c_i, v'_{k+1}) precedes (c_i, v_{j+1}) (the proposed match of (c_i, v_j) from (5.3)), so v'_{k+1} precedes v_{j+1} and v'_{k+1} covers v_j. Consequently, there must be a v'_k preceding v_j such that $v'_k \to v'_{k+1}$ in P. Since both (c_{i-1}, v'_{k+1}) and (c_i, v'_k) precede (c_i, v_j), by minimality, one of them matches to (c_i, v_{k+1}) – a contradiction. (Note that the case $i = 1$ is just as trivial.)

If $i+j-1 \geqslant n$, again cases $i = 1$, $k = 0$, $i = m$, etc. are easy to handle. If (c_{i-1}, v'_{k+1}) and (c_i, v'_k) both precede (c_i, v_j), then again a contradiction arises, so assume v_j precedes v'_k. This forces v_{j+1} to precede v'_{k+1} as well, so we have the figure



(5.5)

in P. By condition (5.1), $n-j \geqslant n'-k$, so if $i+j-1 \geqslant n$, then $i+k-1 \geqslant n'$. But then (c_{i-1}, v'_{k+1}), which precedes (c_i, v_j), must match to (c_i, v'_{k+1}) and this contradiction establishes the result.

To show that $C_m \times P$ satisfies condition (5.1), we may use the fact that $C_m \times P$ satisfies condition (5.3). Now suppose $(c_i, u_j) \to (c_{i'}, u_{j'})$, $(c_k, v_1) \to (c_{k'}, v_{1'})$, (c_i, u_j) precedes (c_k, v_1), $(c_{i'}, v_{1'})$ precedes $(c_{k'}, v_{1'})$ and $(c_{k'}, v_{1'})$ covers (c_i, u_j). Then either $k' = i+1$ or $k' = i$.

CASE 1. $k' = i+1$.

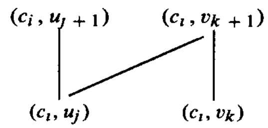
This forces $v_{1'} = u_j$ and by (5.3), $u_{j'} = u_{j+1}$ and $v_1 = u_{j-1}$. It is easy to see that the length of the chain of (c_i, u_j) is $m+n+1-2i$, and the length of the chain of (c_{i+1}, u_{j-1}) is $m+n+1-2(i+1)$ where n is the length of the chain of u. So

$$\begin{aligned} m+n+1-2i-(i+j) &= m+n+1-3i-j \\ &> m+n-1-3i-j \\ &= m+n+1-2(i-1)-(i+1+j-1) \end{aligned}$$

verifies condition (5.1) in this case.

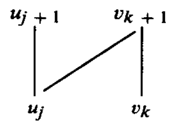
CASE 2. $k' = i$.

Since $i \leqslant i' \leqslant k' = i$ and $i \leqslant k \leqslant k' = i$, $i = i' = k = k'$. The figure is therefore



(5.6)

which implies



(5.7)

so $n_1 - j \geqslant n_2 - k$ by condition (5.1), where n_1 is the length of the chain of u and n_2 is the length of the chain of v.

The length of the chain of (c_i, u_j) is $m+n_1+1-2i$, the length of the chain of (c_i, v_k) is $m+n_2+1-2i$ and again

$$\begin{aligned} m+n_1+1-2i-(i+j) &= m+1-3i+n_1-j \\ &\geqslant m+1-3i+n_2-k \\ &= m+n_2+1-2i-(i+k). \end{aligned}$$

This completes the proof that $C_m \times P$ satisfies condition (5.1).

An easy induction now shows that any product of chains satisfies condition (5.1), from which Theorem 2.4 follows.

6. A Proof of Theorem 3.4

Let $m = m_1 + m_2$. We proceed by induction on m, noting that if Theorem 3.4 is true for all $m_1 + m_2 < m$, then Corollary 3.5 is true in $\mathscr{Y}_{nk}$ if $k < m$ and Corollary 3.6 is true for all $|\lambda| + |\mu| + |\nu| < m$.

If Theorem 3.4 is true for $m_1 + m_2 < m$ but false if $m_1 + m_2 = m$, let $\lambda\mu$ be the minimal counterexample where $|\lambda| = m_1$, λ is not irreducible, $|\mu| = m_2$ and μ is irreducible. Let $\mu \to \mu'$ in $\mathscr{Y}_{nm_2}$. As usual, there are two cases to consider.

CASE 1. *$\lambda\mu \not\to \lambda\mu'$ because $\lambda\mu \to \nu$ which precedes $\lambda\mu'$.*

Since $\nu_i \geqslant (\lambda\mu)_i$ for all i, it must be that $\nu = \lambda\eta'$ for some partition η' in $\mathscr{Y}_{nm_2}$ such that $\eta' \cdot\!> \mu$, for if $\nu_i > \lambda_i$ for any $i \leqslant m_1$, then ν would not precede $\lambda\mu'$. Since η' precedes μ' but $\mu \to \mu'$, there must be an η in $\mathscr{Y}_{nm_2}$ with $\eta \to \eta'$, where η precedes μ. The minimality of $\lambda\mu$ is now contradicted unless η is not irreducible. If η is not irreducible, then by Corollary 3.5, η' is not irreducible. Since $\eta' \cdot\!> \eta$ and $\eta' \cdot\!> \mu$, for $1 \leqslant i \leqslant m_2$, $\eta'_i = \max(\eta_i, \mu_i)$. For some smallest j, $\mu_j - \mu_{j+1} < 2$. It must then be that $\mu_j - \mu_{j+1} = 1$, $\eta'_i = \mu_i$ if $i \neq j$, and $n'_j = \mu_j + 1$. If we write $\mu = \mu^1\mu^2$, where $\mu^1 = (\mu_1, \ldots, \mu_{j-1})$, then by Theorem 3.4 (true for $m = m_2$), $\mu \to \mu^1(\mu^2)'$. By Proposition 3.1(iv), $(\mu^2)'_1 = (\mu^2)_1$ so $\mu'_j = \mu_j$. But then $\mu'_i = \eta'_i$ for $1 \leqslant i < j$ and $\mu'_j < (\mu^1)'_j$ which contradicts the assumption that η' precedes μ'.

CASE 2. *$\lambda\mu \not\to \lambda\mu'$ because $\nu \to \lambda\mu'$ for some ν preceding $\lambda\mu$.*

If $\nu \to \lambda\mu'$, then either $\nu = \lambda\eta$ where μ' covers η or $\nu = \eta\mu'$ where λ covers η.

The first subcase proceeds similarly to Case 1: there is an immediate contradiction if η is irreducible. If η is not irreducible but μ' is, then for some j, $\eta_j - \eta_{j+1} = 2$, $\mu'_i = \eta_i$ if $i \neq j+1$, $\mu'_{j+1} = \eta_{j+1} + 1$. But then by Corollary 3.6, $\mu = (\mu'_1, \ldots, \mu'_j - 1, \mu'_{j+1}, \ldots, \mu'_{m_2})$ (this follows from the fact that $(a, a) \to (a+1, a)$ in $\mathscr{Y}_{n2}$). We now have a contradiction because the forms of μ and η indicate that μ precedes η.

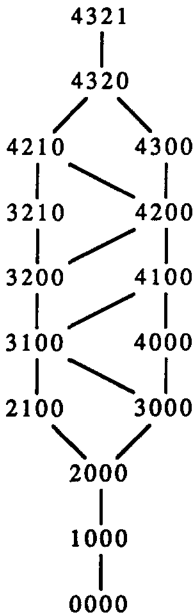
In the second subcase, an immediate contradiction arises if η is not irreducible. If η is irreducible and λ covers η, then for some j, $\eta_j - \eta_{j+1} = 1$. If $j = 1$, then by Proposition 3.1(iv), $\lambda_1 - \lambda_2 < 2$, a contradiction. If $j > 1$, let $\eta = \eta^1\eta^2$ with $\eta^1 = (\eta_1, \ldots, \eta_{j-1})$. Then $\nu = \eta^1\eta^2\mu'$ where η^1 is not irreducible and ν precedes $\lambda\mu$. By minimality of $\lambda\mu$, $\nu \to \eta^1(\eta^2\mu')'$ where $(\eta^2\mu')'$ is the match of $\eta^2 m'$ in $\mathscr{Y}_{n\,m-j+1}$. Since $\eta^2_1 = \eta^2_2 + 1$, Proposition 3.1(iv) again gives that

$(\eta^2\mu')'_1 - (\eta^2\mu')'_2 < 2$ so $\lambda_j - \lambda_{j+1} < 2$. This final contradiction completes the proof.

7. Remarks

Closely related to $\mathcal{Y}_{nm}$ is the partially ordered set $M(n)$ of partitions with nonzero parts all distinct and largest part $\leqslant n$. (An alternative definition for $M(n)$ is that it is the poset of order ideals of $\mathcal{Y}_{(n-1)2}$.) As an example,

M(4):



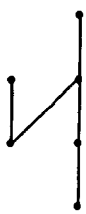
(7.1)

Like $\mathcal{Y}_{nm}$, $M(n)$ is rank unimodal, rank symmetric and strongly Sperner [5]. Also like $\mathcal{Y}_{nm}$, there is no known explicit match between levels, and it is not known if $M(n)$ is a symmetric chain order.

The greedy match in $M(n)$ is not as regular as in $\mathcal{Y}_{nm}$. Chains do not occur in pairs symmetric about the midlevel. The poset $M(n)$ is complete via the greedy match for $1 \leqslant n \leqslant 11$ but not for $n = 12$, and presumably, not for $n > 12$. Surprisingly, the greedy match in $M(12)$ is complete through the midlevel (rank $= 39$) but fails to be complete between levels 40 and 41.

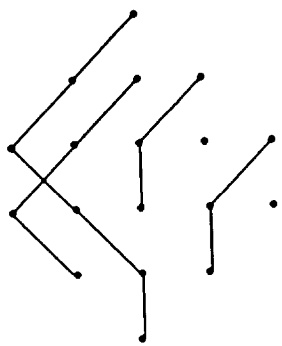
The condition (5.1) in Section 5 cannot be removed. For example if the Hasse diagram for P is

P :



(7.2)

and $Q = C_3 \times P$, then the greedy match in Q is



(7.3)

The proof that $C_m \times P$ is a greedy SYM can be slightly strengthened to show that $Q \times P$ is a greedy SYM if P is a greedy SYM satisfying condition (5.1) and Q is a greedy SYM satisfying a similar condition:

> If there are two chains in Q (induced by the greedy match) $u_1 \to u_2 \to \cdots \to u_n$ and $v_1 \to v_2 \to \cdots \to v_m$ such that u_j and v_k have the same rank, u_j precedes v_k, u_{j+1} precedes v_{k+1} and u_{j+1} covers v_k, then $n - j \geqslant m - k$. (7.4)

This result can not be used inductively, however, because $P \times Q$ need not satisfy property (5.1) or (7.4).

It is proved in [6] that $\mathscr{Y}_{abc}$ is rank unimodal. This result together with Theorem 4.5 implies that $\mathscr{Y}_{abc}$ is Sperner. If $|\lambda| > 3$, $\mathscr{Y}_\lambda$ is not necessarily unimodal [6]. It is not known whether $\mathscr{Y}_\lambda$ is Sperner or if there is a complete match in $\mathscr{Y}_\lambda$ for $|\lambda| > 3$, or even whether the greedy match is complete for $\mathscr{Y}_\lambda$ in the case $|\lambda| = 4$.

The greedy match in $\mathscr{Y}_{n^3}$ can be easily modified to produce symmetric chains, but no such easy modification has been found for $\mathscr{Y}_{n^4}$. In fact, symmetric chains in $\mathscr{Y}_{n^4}$ do not seem to resemble greedy chains much at all. For example, the greedy match in $\mathscr{Y}_{n^m}$ has the property that if $\lambda \to \mu$ in $\mathscr{Y}_{n^m}$, then $\lambda \to \mu$ in $\mathscr{Y}_{k^m}$ for any $k > n$. There does not appear to be a symmetric chain decomposition in $\mathscr{Y}_{n^4}$ with this property.

Whereas it is disappointing that the greedy match fails to be complete in $\mathscr{Y}_{n^5}$, it is intriguing that the greedy match should work as well as it does. For example, the greedy match in $\mathscr{Y}_{8^5}$ is complete. The maximum level size in $\mathscr{Y}_{8^5}$ is 78 and the maximum number of covers of any given partition is 5. When bipartite graphs of this form are constructed at random in computer runs, even though complete matches occur frequently, complete greedy matches do not, even if the vertices at one level are reordered lexicographically with respect to the vertices they cover on the other level.

Even though the greedy match is not complete in $\mathscr{Y}_{n^m}$, the fact that irreducible partitions match to irreducible partitions might lead one to conjecture that the set of all irreducible partitions in $\mathscr{Y}_{n^m}$ form a rank unimodal, rank symmetric subposet. In fact, this conjecture follows easily from the KOH identity in [7]. Since the subposet of nonirreducible partitions in $\mathcal{Y}_{nm}$ is isomorphic to $\mathcal{Y}_{(n-2m+2)m}$, this suggests an inductive approach different from the approaches in [2] and [8]. This idea is very reminiscent of the approach used by Kathy O'Hara [3] in her constructive proof of the unimodality of $\mathcal{Y}_{nm}$.

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