ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 2, Spring 1993

HYPERGEOMETRIC FUNCTIONS OVER FINITE FIELDS AND REPRESENTATIONS OF SL (2, q)

JOHN GREENE

ABSTRACT. It is well known that the matrix elements in the principal representations of SL $(2, R)$ with respect to the appropriate basis are essentially hypergeometric functions. A parallel theory is presented here for the principal representations of SL $(2, F)$ where F is a finite field.

1. Introduction. There has been much work recently on special functions over finite fields. Evans [**4**] derived analogues for various extensions of beta integrals over finite fields. Helversen-Pasotto [**10**] derived Barnes integral analogues. Koblitz [**11**] introduced analogues of hypergeometric functions. An extensive study of many finite field analogues of orthogonal polynomials was conducted by Evans [**6**], and a similar study of hypergeometric functions was conducted by the author [**8, 9**].

That this work might be related to representation theory is indicated by the papers of Helversen-Pasotto [**10**] and Li and Soto-Andrade [**12**] where results follow from properties of representations of GL $(2, q)$ and GL $(3, q)$.

In this paper we demonstrate that, as in the classical case (see, for example, [**16**, Chapter VII]), hypergeometric functions over finite fields arise as matrix elements of certain representations of SL (2).

The organization of this paper is as follows. The construction of the principal series representations for SL $(2, q)$ as representation operators is given in Section 2. In Section 3, matrix elements of these representations are given with respect to a canonical basis. These matrix elements are described in terms of hypergeometric functions in Section 4 which also contains a description of how properties of hypergeometric functions are derived from this framework. Finally, comments, questions and related representations are given in Section 5.

Received by the editors on May 22, 1989.
Partially supported by NSF Grant DMS 8801131.

Throughout this paper we will use the following notation. Capital letters A, B, C and Greek letters $\chi, \psi, \ldots$ will denote multiplicative characters of $\mathrm{GF}(q)$. The quadratic character will be denoted by φ and the trivial character by ε. All multiplicative characters are defined to be 0 at the 0 element of $\mathrm{GF}(q)$.

We define a function δ on elements of $\mathrm{GF}(q)$ by

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0, \end{cases}$$

and on multiplicative characters by

$$\delta(A) = \begin{cases} 1 & \text{if } A = \varepsilon \\ 0 & \text{if } A \neq \varepsilon. \end{cases}$$

Write $\sum_x$ to denote the sum over all $x \in \mathrm{GF}(q)$ and $\sum_\chi$ to denote the sum over all multiplicative characters of $\mathrm{GF}(q)$. Let $\zeta = e^{2\pi i/p}$, where $q = p^n$, and let Tr be the trace map from $\mathrm{GF}(q)$ to $\mathrm{GF}(p)$.

The Gauss sum of a multiplicative character A is defined by

$$G(A) = \sum_x A(x)\zeta^{\mathrm{Tr}(x)},$$

and the Jacobi sum of A and B is defined by

$$J(A, B) = \sum_x A(x)B(1-x).$$

2. The principal representations of $\mathrm{SL}(2, q)$**.** For completeness, we give the construction of the principal representations of $\mathrm{SL}(2, q)$. This material is well known; see, for example, [**3**, Chapter 38] for the construction of the character table of $\mathrm{SL}(2, q)$ and [**14**] for an elementary construction of the irreducible representations of $\mathrm{SL}(2, q)$ from another point of view.

Classically, the principal representations of $G = \mathrm{SL}(2)$ are the irreducible components of the representation given by the action of linear transformations on functions of two variables. The spaces of homogeneous functions are all invariant subspaces, and in most cases (see [**16**, Chapter VII]) the restriction of the representation to these spaces is irreducible. A similar approach works over local fields and finite fields [**7**, Section 3.6].

Specifically, let T be the representation operator on functions of two variables over a finite field defined by

$$\begin{aligned} T(g)f(x,y) &= f((x,y)g) \\ &= f(ax+cy, bx+dy), \end{aligned}$$

where

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Call a function $f(x,y)$ homogeneous of weight A where A is a multiplicative character of GF (q) if

$$f(ax,ay) = A(a)f(x,y) \qquad \text{for all } a,x,y \in \mathrm{GF}\,(q),$$

and let V_A be the space of all homogeneous functions of weight A. It is not hard to show [**14**] that this representation (T acting on V_A) is the representation induced by the one-dimensional representation

$$S\begin{bmatrix} a & b \\ 0 & 1/a \end{bmatrix} = \overline{A}(a)$$

of the subgroup of upper triangular matrices in SL $(2,q)$. An easy calculation (using Frobenius reciprocity) now shows that if λ is the character of T on V_A, then

$$\langle \lambda, \lambda \rangle_G = 1 + \delta(A^2).$$

Thus, T is irreducible so long as $A \neq \varepsilon$ or $A \neq \varphi$. If $A = \varepsilon$, then the function $1-\delta(x)\delta(y)$ is invariant and so, in this case, $T \cong 1 \oplus T_1$, where 1 is the trivial representation and T_1 is irreducible. If $A = \varphi$, it can be shown that V_φ decomposes into two $(q+1)/2$-dimensional subspaces, each invariant under T so, in this case, $T = T_2 \oplus T_3$ where T_2 and T_3 are irreducible of degree $(q+1)/2$. The irreducibility of T will not play a major role in what follows.

3. Matrix elements of T. Let $V = V_A$ be the set of homogeneous functions of weight A. A basis for V is given by

$$\mathcal{B} = \{\sqrt{q-1}A(x)\delta(y), \sqrt{q-1}A(y)\delta(x)\} \cup \{A(y)\chi(y/x)\}_\chi. \tag{3.1}$$

An inner product on V is

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{(q-1)^2}\sum_{x,y} f(x,y)\overline{g(x,y)} \\ &= \frac{1}{q-1}f(1,0)g(1,0) + \frac{1}{q-1}f(0,1)g(0,1) \\ &\quad + \frac{1}{q-1}\sum_{x\neq 0} f(x,1)\overline{g(x,1)}.\end{aligned} \tag{3.2}$$

The basis $\mathcal{B}$ is orthonormal with respect to this inner product. Moreover, T is unitary with respect to this inner product:

$$\langle T(g)f_1, T(g)f_2\rangle = \langle f_1, f_2\rangle \tag{3.3}$$

for any $g \in \mathrm{SL}\,(2,q)$ and $f_1, f_2 \in V$.

If we write $\mathcal{B} = \{v_1, v_2\} \cup \{v_\chi\}_\chi$, then the matrix representation M of T with respect to $\mathcal{B}$ can be written in block diagonal form

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right],$$

where A is 2×2, B is $2\times(q-1)$, C is $(q-1)\times 2$ and D is $(q-1)\times(q-1)$. We are interested primarily in D. With D indexed by the multiplicative characters, $D = (d_{\chi,\psi})$, then since the vectors $\{v_\chi\}$ of $\mathcal{B}$ are orthonormal

$$d_{B,C} = \langle T(g)v_C, v_B\rangle \tag{3.4}$$

or

$$d_{B,C} = \frac{1}{(q-1)^2}\sum_{x,y} AC(bx+dy)\overline{C}(ax+cy)\overline{AB}(y)B(x), \tag{3.5}$$

where $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Define $K_A(g|B,C)$ by

$$K_A(g|B,C) = d_{B,C}. \tag{3.6}$$

If we replace x by xy in (3.5), the y-sum can be evaluated to give

$$K_A(g|B,C) = \frac{1}{q-1}\sum_x B(x)AC(bx+d)\overline{C}(ax+c). \tag{3.7}$$

These matrix elements may also be obtained in a manner more analogous to the classical presentation given in [**16**, Chapter VII] as follows. Define the Mellin transform of a function $f : \mathrm{GF}(q) \to \mathbf{C}$ with respect to the multiplicative character B via

$$M_B f = \frac{1}{q-1}\sum_x B(x)f(x). \tag{3.8}$$

An inversion formula exists for this transform:

$$f(x) = f(0)\delta(x) + \sum_\chi (M_\chi f)\bar{\chi}(x). \tag{3.9}$$

If $f(x,y) \in V_A$, define $M_B f$ by

$$M_B f = \frac{1}{q-1}\sum_x B(x)f(x,1). \tag{3.10}$$

Then
(3.11)

$$\begin{aligned} f(x,y) &= f(0,1)A(y)\delta(x) + f(1,0)A(x)\delta(y) + A(y)\sum_\chi (M_\chi f)\bar{\chi}(x/y) \\ &= \frac{1}{\sqrt{q-1}}\, f(1,0)v_1 + \frac{1}{\sqrt{q-1}}\, f(0,1)v_2 + \sum_\chi (M_\chi f)v_\chi. \end{aligned}$$

Thus, the inversion formula (3.11) expressed $f \in V$ in terms of the basis $\mathcal{B}$. We now factor the Mellin transform through the representation:

$$\begin{aligned} M_B T(g)f &= M_B f(ax+cy, bx+dy) \\ &= \frac{1}{q-1}\sum_x B(x)f(ax+c, bx+d). \end{aligned}$$

Applying (3.11) to this gives

$$M_BT(g)f = \frac{1}{q-1}B(d)\overline{AB}(-b)f(1,0) + \frac{1}{q-1}B(-c)\overline{AB}(a)f(0,1) + \frac{1}{q-1}\sum_{\chi}(M_\chi f)\sum_{x} B(x)A\chi(bx+d)\bar{\chi}(ax+c),$$

or

(3.12)

$$M_BT(g)f = \frac{1}{q-1}f(1,0)B(d)\overline{AB}(-b) + \frac{1}{q-1}f(0,1)B(-c)\overline{AB}(a) + \sum_{\chi}(M_\chi f)K_A(g|B,\chi).$$

This formula may be compared to [**16**, p. 361, **7**, p. 161].

If any of the entries of $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ are 0, these matrix elements can be evaluated in terms of Jacobi sums or δ functions. These evaluations are given in Table 1.

TABLE 1.

	g	$K_A(g \mid B, C)$
1	$\left[\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right]$, $b \neq 0$	$\frac{1}{q-1}\overline{ABC}(a)\overline{B}C(-b)J(AC, B\overline{C})$
2	$\left[\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right]$	$\overline{AC}^2(a)\delta(B\overline{C})$
3	$\left[\begin{smallmatrix} a & 0 \\ c & 1/a \end{smallmatrix}\right]$, $c \neq 0$	$\frac{1}{q-1}\overline{ABC}(a)B\overline{C}(c)B(-1)J(B, \overline{C})$
4	$\left[\begin{smallmatrix} d & b \\ -1/b & 0 \end{smallmatrix}\right]$, $a \neq 0$	$\frac{1}{q-1}\overline{ABC}(a)\overline{B}C(-b)C(-1)J(ABC, \overline{C})$
5	$\left[\begin{smallmatrix} 0 & b \\ -1/b & 0 \end{smallmatrix}\right]$	$AC^2(b)c(-1)\delta(ABC)$
6	$\left[\begin{smallmatrix} 0 & b \\ -1/b & d \end{smallmatrix}\right]$	$\frac{1}{q-1}ABC(d)\overline{B}C(-b)J(B, AC)$

4. Matrix elements and hypergeometric functions. When F is the field of real numbers, the matrix elements in the analogous representations of $\mathrm{SL}(2, F)$ are integrals rather than sums, and these integrals can be written in terms of hypergeometric series [**16**, Chapter 7]. When F is a locally compact topological field, Gelfand et al. [**7**, p. 161] define hypergeometric functions over F as being the matrix elements in the corresponding representation. Hypergeometric functions over finite fields were introduced in [**11, 8**]. In view of the above, it is not surprising that these functions are related to the matrix elements given in (3.7).

The hypergeometric function was defined in [**9**] as

$$(4.1)\quad {}_2F_1\left[\begin{matrix} A, & B \\ & C \end{matrix}\,\middle|\, x\right] = \varepsilon(x)\frac{BC(-1)}{q}\sum_y B(y)\overline{B}C(1-y)\overline{A}(1-xy)$$

which is to be compared with the integral representation for hypergeometric series [**13**, p. 47]. When none of a, b, c or d is 0,

(4.2)

$$d_{B,C} = K_A(g|B,C) = \frac{q}{q-1}ABC(-d)\overline{B}(b)\overline{C}(c){}_2F_1\left[\begin{matrix} C, & B \\ & ABC \end{matrix}\,\middle|\, \frac{ad}{bc}\right],$$

which arises after the change of variable $x \mapsto -dx/b$ in (3.7). Properties of hypergeometric functions can consequently be deduced from properties of the representation T or, more directly, from the matrix elements K. We now derive several such properties from this point of view.

As a first example, let

$$g_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and let $v_B = AB(y)\overline{B}(x) \in \mathcal{B}$. The action of g_0 on v_B is given by $T(g_0)v_B = AB(-1)v_{\overline{AB}}$. As a result,

$$\begin{aligned} K_A(gg_0 \mid B, C) &= \langle T(gg_0)v_C, v_B\rangle \\ &= AC(-1)\langle T(g)v_{\overline{AC}}, v_B\rangle \\ &= AC(-1)K_A(g \mid B, \overline{AC}). \end{aligned}$$

Also, $T(g_0^{-1})v_B = B(-1)v_{\overline{AB}}$. Since T is unitary, we have

$$\begin{aligned} K_A(g_0 g \mid B, C) &= \langle T(g_0 g)v_C, v_B\rangle \\ &= \langle T(g)v_C, T(g_0^{-1})v_B\rangle \\ &= B(-1)\langle T(g)v_C, v_{\overline{AB}}\rangle \\ &= B(-1)K_A(g|\overline{AB}, C). \end{aligned} \tag{4.3}$$

Composing these results gives an additional relation

$$K_A(g|B,C) = BC(-1)K_A(g_0^{-1}gg_0 \mid \overline{AB}, \overline{AC}).$$

In terms of hypergeometric functions, we now have

$$\begin{aligned} {}_2F_1\left[\begin{matrix} B, & C \\ & ABC \end{matrix}\middle|\frac{ad}{bc}\right] &= A(-1)B\left(\frac{bc}{ad}\right){}_2F_1\left[\begin{matrix} B, & \overline{AC} \\ & BC \end{matrix}\middle|\frac{bc}{ad}\right] \\ &= A(-1)C\left(\frac{bc}{ad}\right){}_2F_1\left[\begin{matrix} \overline{AB}, & C \\ & C\overline{B} \end{matrix}\middle|\frac{bc}{ad}\right] \\ &= ABC\left(\frac{bc}{ad}\right){}_2F_1\left[\begin{matrix} \overline{AB}, & \overline{AC} \\ & \overline{ABC} \end{matrix}\middle|\frac{ad}{bc}\right]. \end{aligned}$$

Setting $ad/bc = x$ and relabeling gives

$$\begin{aligned} {}_2F_1\left[\begin{matrix} A, & B \\ & C \end{matrix}\middle| x\right] &= ABC(-1)\overline{A}(x){}_2F_1\left[\begin{matrix} A, & A\overline{C} \\ & A\overline{B} \end{matrix}\middle|\frac{1}{x}\right] \\ &= ABC(-1)\overline{B}(x){}_2F_1\left[\begin{matrix} B\overline{C}, & \overline{B} \\ & \overline{A}B \end{matrix}\middle|\frac{1}{x}\right] \\ &= \overline{C}(x){}_2F_1\left[\begin{matrix} B\overline{C}, & A\overline{C} \\ & \overline{C} \end{matrix}\middle| x\right]. \end{aligned} \tag{4.4}$$

Another set of transformations can be arrived at by the somewhat artificial method of reordering the terms in $T(g)v_C \cdot \bar{v}_B$. In particular, $T(g)v_C \cdot \bar{v}_B = AC(bx+dy)\overline{C}(ax+cy)\overline{AB}(y)B(x)$ is the product of four functions and these functions can be ordered in 24 ways. Upon summing over all x and y, these identities give 24 transformations for the kernels and, hence, for the hypergeometric function. These 24 transformations correspond to Kummer's 24 solutions to the hypergeometric differential equation. The 24 permutations break up naturally into 6 groups of 4 permutations, and, in fact, (4.3) is the resulting list of transformations from the ordering

$$\{AC, \overline{C}, \overline{A}B, B\}, \{AC, \overline{C}, B, \overline{AB}\}, \{\overline{C}, AC, \overline{AB}, B\}, \{\overline{C}, AC, B, \overline{AB}\}.$$

The entire list of 24 transformations can be obtained by composing the transformations in (4.3) with those that arise from the list

$$\begin{array}{ll} \{AC, \overline{C}, \overline{AB}, B\}, & \{\overline{AB}, B, AC, \overline{C}\} \\ \{AC, \overline{AB}, \overline{C}, B\}, & \{\overline{C}, B, AC, \overline{AB}\} \\ \{AC, B, \overline{C}, \overline{AB}\}, & \{\overline{C}, \overline{AB}, AC, B\}. \end{array} \tag{4.5}$$

We now derive one such identity in depth. We have

$$\begin{aligned} K_A(g|B,C) &= \frac{1}{(q-1)^2}\sum_{x,y} AC(bx+dy)\overline{C}(ax+cy)\overline{AB}(y)B(x) \\ &= \frac{1}{(q-1)^2}\sum_{x,y} AC(bx+dy)\overline{AB}(y)\overline{C}(ax+cy)B(x) \\ &= AB(-1)AC(b)\overline{C}(c)\frac{1}{(q-1)^2}\sum_{x,y} AC(x+\frac{d}{b}y) \\ &\qquad \cdot \overline{AB}(-y)\overline{C}\left(\frac{a}{c}x+y\right)B(x). \end{aligned}$$

The functions $AC(y)\overline{AB}(x)$ and $C(y)\overline{B}(x)$ both live in the space of homogeneous functions of weight $C\overline{B}$. In fact, these are the basis functions v_{AB} and v_B in that space. With

$$g_1 = \begin{bmatrix} 0 & 1 \\ -1 & d/b \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} 1 & a/c \\ 0 & 1 \end{bmatrix},$$

we have

$$\begin{aligned} K_A(g|B,C) &= \langle T(g)v_C, v_B\rangle \\ &= \alpha\langle T(g_1)v_{AB}, T(g_2)v_B\rangle \quad \text{in } V_{C\overline{B}} \\ &= \alpha\langle T(g_2^{-1}g_1)v_{AB}, v_B\rangle \quad \text{in } V_{C\overline{B}} \\ &= \alpha K_{C\overline{B}}(g_2^{-1}g_1 \mid B, AB), \end{aligned}$$

where $\alpha = AB(-1)AC(b)\overline{C}(c)$. Table 2 gives the list of transformations obtained from (4.5) by this method.

TABLE 2.

	Order of Terms	Kernel	α	g_1	g_2
1	$AC,\overline{C},\overline{AB},B$	$K_A(g_2^{-1}g_1 \mid B,C)$	1	$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$
2	$\overline{AB},B,AC,\overline{C}$	$K_{\overline{A}}(g_2^{-1}g_1 \mid \overline{C},\overline{B})$	1	$\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$
3	$AC,\overline{AB},\overline{C},B$	$K_{C\overline{B}}(g_2^{-1}g_1 \mid B,AB)$	$AB(-1)AC(b)\overline{C}(c)$	$\left[\begin{smallmatrix} 0 & 1 \\ -1 & a/b \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 1 & a/c \\ 0 & 1 \end{smallmatrix}\right]$
4	$\overline{C},B,AC,\overline{AB}$	$K_{\overline{C}B}(g_2^{-1}g_1 \mid \overline{AB},\overline{B})$	$AB(-1)AC(b)\overline{C}(c)$	$\left[\begin{smallmatrix} 1 & a/c \\ 0 & 1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 0 & 1 \\ -1 & d/b \end{smallmatrix}\right]$
5	$AC,B,\overline{C},\overline{AB}$	$K_{ABC}(g_2^{-1}g_1 \mid \overline{AB},\overline{B})$	$AB(-1)AC(d)\overline{C}(a)$	$\left[\begin{smallmatrix} 1 & b/d \\ 0 & 1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 0 & 1 \\ -1 & c/a \end{smallmatrix}\right]$
6	$\overline{C},\overline{AB},AC,B$	$K_{\overline{ABC}}(g_2^{-1}g_1 \mid B,AB)$	$AB(-1)AC(d)\overline{C}(a)$	$\left[\begin{smallmatrix} 0 & 1 \\ -1 & c/a \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 1 & b/d \\ 0 & 1 \end{smallmatrix}\right]$

Identities: $\alpha_i K_i = \alpha_j K_j$

There is an obvious pairing $1 \to 2$, $3 \to 4$, $5 \to 6$ in Table 2. These correspond to the transformation

$$ {}_2F_1\left[\begin{matrix} A, & B \\ & C \end{matrix}\middle| x\right] = \overline{C}(-x)C\overline{AB}(1-x)\,{}_2F_1\left[\begin{matrix} \overline{B}, & \overline{A} \\ & \overline{C} \end{matrix}\middle| x\right]. \tag{4.6}$$

When composed with the third transformation of (4.4), this transformation becomes

$$ {}_2F_1\left[\begin{matrix} A, & B \\ & C \end{matrix}\middle| x\right] = C(-1)C\overline{AB}(1-x)\,{}_2F_1\left[\begin{matrix} C\overline{A}, & C\overline{B} \\ & C \end{matrix}\middle| x\right], \tag{4.7}$$

a good analogue for Euler's transformation [**13**, p. 60]. The transformations $(1) \to (3) \to (5)$ are

(4.8)

$$ {}_2F_1\left[\begin{matrix} A, & B \\ & C \end{matrix}\middle| x\right] = \overline{B}(1-x)\,{}_2F_1\left[\begin{matrix} C\overline{A}, & B \\ & C \end{matrix}\middle| \frac{x}{x-1}\right]$$
$$ = B(-1)C\overline{A}(1-x)\overline{C}(x)\,{}_2F_1\left[\begin{matrix} \overline{B}, & A\overline{C} \\ & A\overline{B} \end{matrix}\middle| \frac{1}{1-x}\right].$$

In all these formulas, we are assuming that none of a, b, c and d are 0 and $ad - bc = 1$, so these transformation hold provided $x \neq 0$ or 1.

Since T is a representation operator,

$$M(g_1)M(g_2) = M(g_1g_2). \tag{4.9}$$

Calculating the ij'th entry on both sides or, equivalently, applying formula (3.12) to $M_B(T(g_1)T(g_2)f(x,y))$ gives

$$\begin{aligned} K_A(g_1g_2 \mid B, C) = &\sum_\chi K_A(g_1 \mid B, \chi)K_A(g_2 \mid \chi, C) \\ &+ \frac{1}{q-1}\overline{AB}(-b_1)AC(b_2)B(d_1)\overline{C}(a_2) \\ &+ \frac{1}{q-1}\overline{AB}(a_1)AC(d_2)B(-c_1)\overline{C}(c_2), \end{aligned} \tag{4.10}$$

where

$$g_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}.$$

This result, in conjunction with the special cases listed in Table 1, will give additional identities for hypergeometric functions.

For example, suppose $a \neq 0$ and consider the LU factorization

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 1/a \end{bmatrix}.$$

Applying formula (4.10) to the appropriate special cases in Table 1 gives

$$\begin{aligned} K_A(g \mid B, C) = &B(-c)C(-b)\overline{ABC}(a) \\ &\cdot \frac{1}{(q-1)^2}\sum_\chi J(B, \bar\chi)J(AC, \chi\overline{C})\bar\chi(-bc), \end{aligned} \tag{4.11}$$

provided $bc \neq 0$. In terms of the hypergeometric series,

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} C, & B \\ & ABC \end{matrix} \,\middle|\, \frac{ad}{bc}\right] \\ &\quad = BC(-bc)\overline{ABC}(-ad)\frac{1}{q(q-1)}\sum_\chi J(B, \bar\chi)J(AC, \chi\overline{C})\bar\chi(-bc), \end{aligned}$$

or

$$(4.12)\quad {}_2F_1\left[\begin{matrix} C, & B \\ & ABC \end{matrix}\middle| x\right]$$
$$= \overline{ABC}(x)A(1-x)\frac{1}{q(q-1)}\sum_{\chi} J(B,\bar{\chi})J(AC,\chi\overline{C})\chi(1-x).$$

Composing the second identity in (4.8) with the first in (4.4), when applied to the left hand side of (4.12) gives

$$AC(-1)A(1-x)\overline{ABC}(x){}_2F_1\left[\begin{matrix} \overline{B}, & \overline{C} \\ & A \end{matrix}\middle| 1-x\right]$$
$$= A(1-x)\overline{ABC}(x)\frac{1}{q(q-1)}\sum_{\chi} J(B,\bar{\chi})J(AC,\chi\overline{C})\chi(1-x),$$

or

$$(4.13)\qquad {}_2F_1\left[\begin{matrix} A, & B \\ & C \end{matrix}\middle| x\right] = \frac{BC(-1)}{q(q-1)}\sum_{\chi} J(\overline{A},\bar{\chi})J(\overline{B}C,B\bar{\chi})\chi(x),$$

provided $x \neq 0$ or 1. Thus, all kernels and, hence, hypergeometric functions, may be written in terms of Jacobi sums. This formulation of hypergeometric series is most analogous to the series definition [**13**, p. 45].

In [**13**] the identity

$$(4.14)\quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -1/a & 0 \\ 0 & -a \end{bmatrix}\begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1/a \\ 0 & 1 \end{bmatrix}$$

was used to derive the Barnes integral analogue

$$(4.15)\quad \frac{1}{q-1}\sum_{\chi} G(A\chi)G(B\bar{\chi})G(C\chi)G(D\bar{\chi})$$
$$= \frac{G(AB)G(AD)G(BC)G(CD)}{G(ABCD)} + q(q-1)AC(-1)\delta(ABCD).$$

When (4.14) is combined with Table 1 and formula (4.10), the result is

$$\frac{1}{q-1}\sum_{\chi} J(A\chi, B\bar{\chi})J(AC, \overline{AC}\chi)\chi(-1) = A(-1)J(\overline{C}, \overline{B}C). \tag{4.16}$$

This formula, which is equivalent to (4.15) is an analogue of Gauss's evaluation [**13**, p. 49] and can be written

$${}_2F_1\left[\begin{matrix} A, & B \\ & C \end{matrix}\middle|1\right] = \frac{C(-1)}{q}J(B, A\overline{C}). \tag{4.17}$$

We close this section with two examples of orthogonality relations arising from the representations of SL $(2, q)$. First, since T is unitary and $\mathcal{B}$ is an orthonormal basis,

$$M(g)\overline{M(g)}^T = I. \tag{4.18}$$

Equating entries in region D of M gives

$$\begin{aligned} \sum_{\chi} K_A(g|B,\chi)\overline{K_A(g|C,\chi)} \\ = \delta(B\overline{C}) - \frac{1}{q-1}B\overline{C}(-d/b) - \frac{1}{q-1}B\overline{C}(-c/a). \end{aligned} \tag{4.19}$$

In terms of hypergeometric series,

$$\begin{aligned} \sum_{\chi} {}_2F_1\left[\begin{matrix} \chi, & B \\ & AB\chi \end{matrix}\middle|x\right]\overline{{}_2F_1\left[\begin{matrix} \chi, & C \\ & AC\chi \end{matrix}\middle|x\right]} \\ = \left(\frac{q-1}{q}\right)^2\delta(B\overline{C}) - \frac{q-1}{q^2} - \frac{q-1}{q^2}\overline{B}C(x), \end{aligned} \tag{4.20}$$

where $x \neq 0$ or 1.

Another result arises from the following orthogonality property [**16**, p. 45]. If S and T are nonequivalent irreducible unitary representations of a compact group G of dimensions d_s and d_t with matrix elements $s_{ij}(g)$ and $t_{ij}(g)$, then

$$\int s_{ij}(g)\overline{t_{mn}(g)}\,dg = 0,$$
$$\int s_{ij}(g)\overline{s_{mn}(g)}\,dg = \frac{1}{d_s}\delta_{i,m}\delta_{j,n}.$$

In our case, T acting on V_A is irreducible provided $A^2 \neq \varepsilon$ and T on V_A is inequivalent to T on V_B provided $A \neq \overline{B}$. Consequently,

$$\frac{1}{|\mathrm{SL}\,(2,q)|} \sum_{g\in \mathrm{SL}\,(2,q)} K_A(g|B,C)K_D(g|E,F) = \frac{1}{q-1}\delta(A\overline{D})\delta(B\overline{E})\delta(C\overline{F}) \tag{4.21}$$

provided $A^2 \neq \varepsilon$, $D^2 \neq \varepsilon$ and $AD \neq \varepsilon$. A hypergeometric series result is obtained from (4.21) by separating the summation into cases depending on whether an entry in g is 0. After a lengthy calculation, we have

(4.22)

$$\begin{aligned}\frac{1}{q-1}\sum_x C\overline{F}(1-x)\,{}_2F_1\left[\begin{matrix}C, & B\\ & ABC\end{matrix}\middle| x\right]&\overline{{}_2F_1\left[\begin{matrix}F, & B\overline{C}F\\ & ABC\end{matrix}\middle| x\right]}\\ = \frac{1}{q}\delta(C\overline{F}) - \frac{1}{q^2}\delta(ABC) &- \frac{1}{q^2}CF(-1)\delta(B\overline{C})\\ - \frac{1}{q^2(q-1)}CF(-1)&\\ \cdot [J(B,AC)J(\overline{B}C\overline{F},\overline{AC}^2F) &+ J(AC,B\overline{C})J(\overline{AC}^2F,\overline{B}C)\\ + J(B,\overline{C})J(\overline{B}C\overline{F},F) &+ J(ABC,\overline{C})J(\overline{ABC},F)].\end{aligned}$$

In the case where $C = F$, this becomes, after relabeling parameters,

(4.23)

$$\begin{aligned}\frac{1}{q-1}\sum_x \left|{}_2F_1\left[\begin{matrix}A, & B\\ & C\end{matrix}\middle| x\right]\right|^2 = \frac{q-4}{q(q-1)} + \frac{1}{q^2}[&2\delta(A)+\delta(C)+\delta(B)\\ &+\delta(C\overline{A})+2\delta(C\overline{B})+\delta(A\overline{B})\\ &-(q-1)\delta(A)\delta(B)-(q-1)\delta(A)\delta(C)\\ &-(q-1)\delta(B)\delta(C)-(q-1)\delta(A\overline{B})\delta(C\overline{B})],\end{aligned}$$

provided $C \neq AB$ and $C \neq \varphi AB$.

A hypergeometric series can be simplified if any of $A, B, A\overline{C}$ or $B\overline{C}$ is trivial. If we ignore these cases,

$$\frac{1}{q-1}\sum_x \left|{}_2F_1\left[\begin{matrix}A, & B\\ & C\end{matrix}\middle| x\right]\right|^2 = \frac{q-4}{q(q-1)} + \frac{1}{q^2}(\delta(C)+\delta(A\overline{B})), \tag{4.24}$$

provided $A, B, A\overline{C}, B\overline{C}, C\overline{AB}$ and $\psi C\overline{AB}$ are all nontrivial.

5. Remarks. As indicated in Table 1, $K_A(g|B,C)$ reduces to a Jacobi sum or a delta function if any of a, b, c or d is 0. If none of a, b, c or d is 0, the kernel will still reduce for the right values of the parameters A, B and C. In particular, from

$$K_A(g|B,C) = \frac{1}{(q-1)^2} \sum_{x,y} AC(bx+dy)\overline{C}(ax+cy)\overline{AB}(y)B(x) \tag{5.1}$$

it follows easily that K reduces to a Jacobi sum if any of AC, C, AB or B is the trivial character. This implies that

$$ {}_2F_1\left[\begin{matrix} A, & B \\ & C \end{matrix}\Big| x\right]$$

reduces in case $A = \varepsilon$, $B = \varepsilon$, $A = C$ or $B = C$. These formulas are given in [**9, 3, 16**].

Quadratic transformations for hypergeometric functions are given in [**8, 9**]. The conditions for the existence of a quadratic transformation translate into the conditions $AC = \overline{C}$, $AC = \overline{AB}$, $AC = B$, $\overline{C} = \overline{AB}$ or $\overline{C} = B$ for $K_A(g|B,C)$. In each of these cases, two terms in the sum in (5.1) can be combined to obtain a character evaluated at a quadratic form. Completing the square now gives rise to the quadratic transformation. It would be interesting to see a more representation-theoretic treatment of these transformations.

The orthogonality relations given in (4.20) and (4.22) are analogous to orthogonalities for certain orthogonal polynomials. In the case of (4.20), the analogy is with Krawtchouk polynomials. The Krawtchouk polynomials may be defined by [**2**, p. 161]

$$K_n(x) = (-p)^n \begin{bmatrix} N \\ n \end{bmatrix} {}_2F_1\left[\begin{matrix} -x, & -n \\ & -N \end{matrix}\Big| \frac{1}{p}\right] \tag{5.2}$$

and satisfy the orthogonality relation [**2**, p. 161]

$$\sum_{x=0}^{N} K_n(x)K_m(x) \begin{bmatrix} N \\ x \end{bmatrix} p^x(1-p)^{N-x} = \begin{bmatrix} N \\ n \end{bmatrix} p^n(1-p)^n \delta_{mn}. \tag{5.3}$$

In our case, the normalization of hypergeometric series differs from the classical case. An appropriate analogue for Krawtchouk polynomials in our normalization is

$$K_N(\chi) = N(-p)\,{}_2F_1\left[\begin{matrix}\bar{\chi}, & \overline{N} \\ & \overline{L}\end{matrix}\,\Big|\,\frac{1}{p}\right]. \tag{5.4}$$

It can be shown [**9**, Corollary 3.21] that

$$K_N(\chi) \approx N(-p)\frac{J(L,\overline{N})}{J(L,\bar{\chi})}\,{}_2F_1\left[\begin{matrix}\overline{N}, & \bar{\chi} \\ & \overline{L}\end{matrix}\,\Big|\,\frac{1}{p}\right]. \tag{5.5}$$

The two sides of (5.5) are equal provided χ, L and N do not take on certain values; we will not worry about such details here but concentrate on the general form of the resulting calculations. Analogous to the right hand side of (5.3) is

$$\begin{aligned}
&\sum_{\chi} M(-p)\,{}_2F_1\left[\begin{matrix}\bar{\chi}, & \overline{M} \\ & \overline{L}\end{matrix}\,\Big|\,\frac{1}{p}\right] N(-p)\frac{J(L,\overline{N})}{J(L,\bar{\chi})} \\
&\qquad \cdot\ {}_2F_1\left[\begin{matrix}\overline{N}, & \bar{\chi} \\ & \overline{L}\end{matrix}\,\Big|\,\frac{1}{p}\right] \chi(-1)J(L,\bar{\chi})\chi(p)L\bar{\chi}(1-p) \\
&\quad = MN(-p)L(1-p)J(L,\overline{N})\sum_{\chi}{}_2F_1\left[\begin{matrix}\bar{\chi}, & \overline{M} \\ & \overline{L}\end{matrix}\,\Big|\,\frac{1}{p}\right] \\
&\qquad \cdot\ {}_2F_1\left[\begin{matrix}\overline{N}, & \bar{\chi} \\ & \overline{L}\end{matrix}\,\Big|\,\frac{1}{p}\right] \chi(-p)\bar{\chi}(1-p).
\end{aligned}$$

By transformations (4.6), (4.8) and (4.4), this sum can be written

$$\begin{aligned}
MN(-p)N\left(\frac{p-1}{p}\right)J(L,\overline{N})\sum_{\chi}{}_2F_1\left[\begin{matrix}\chi, & N \\ & \overline{L}N\chi\end{matrix}\,\Big|\,\frac{p-1}{p}\right] \\
\cdot\ \overline{{}_2F_1\left[\begin{matrix}\chi, & \overline{M} \\ & \overline{L}M\chi\end{matrix}\,\Big|\,\frac{p-1}{p}\right]}.
\end{aligned}$$

So by (4.20), we have

$$\begin{aligned}
&\sum_{\chi} K_M(\chi)K_N(\chi)J(L,\bar{\chi})\chi(-p)L\bar{\chi}(1-p) \\
&\quad = \left(\frac{q-1}{q}\right)^2 J(L,\overline{N})N(-p)N(1-p)\delta(M\overline{N}) + \text{ extra terms}
\end{aligned} \tag{5.6}$$

which we contrast with (5.3).

For (4.22), the analogy is with Jacobi polynomials. Analogous to [**13**, p. 254],

$$P_n^{(\alpha,\beta)}(x) = \begin{bmatrix} n+\alpha \\ n \end{bmatrix} \left(\frac{1+x}{2}\right)^n {}_2F_1\left[\begin{matrix} -B-n, & -n \\ & 1+\alpha \end{matrix}\,\middle|\, \frac{x-1}{x+1}\right] \tag{5.7}$$

is

$$P_N^{(A,B)}(x) = N\left(\frac{1+x}{2}\right) {}_2F_1\left[\begin{matrix} \overline{BN}, & \overline{N} \\ & A \end{matrix}\,\middle|\, \frac{x-1}{x+1}\right]. \tag{5.8}$$

The orthogonality relation for Jacobi polynomials is [**13**, p. 260]

$$\begin{aligned}\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\, dx \\ = \frac{2^{\alpha+\beta+1}}{(1+\alpha+\beta+2n)} \frac{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{n!\Gamma(1+\alpha+\beta+n)} \delta_{mn}.\end{aligned} \tag{5.9}$$

On the other hand,

$$\begin{aligned}&\sum_x A(1-x)B(1+x)P_N^{(A,B)}(x)P_M^{(A,B)}(x) \\ &\quad = MN(2)\sum_x A(1-x)BMN(1+x) {}_2F_1\left[\begin{matrix} \overline{BN}, & \overline{N} \\ & A \end{matrix}\,\middle|\, \frac{x-1}{x+1}\right] \\ &\quad\quad \cdot {}_2F_1\left[\begin{matrix} \overline{BM}, & \overline{M} \\ & A \end{matrix}\,\middle|\, \frac{x-1}{x+1}\right] \\ &\quad = A(-1)AB(2)\sum_y A(y)\overline{ABMN}(1-y) {}_2F_1\left[\begin{matrix} \overline{BN}, & \overline{N} \\ & A \end{matrix}\,\middle|\, y\right] \\ &\quad\quad \cdot {}_2F_1\left[\begin{matrix} \overline{BM}, & \overline{M} \\ & A \end{matrix}\,\middle|\, y\right].\end{aligned}$$

Using

$${}_2F_1\left[\begin{matrix} \overline{BM}, & \overline{N} \\ & A \end{matrix}\,\middle|\, y\right] = \frac{G(AN)G(BN)}{G(N)G(AB)} {}_2F_1\left[\begin{matrix} \overline{N}, & \overline{BN} \\ & A \end{matrix}\,\middle|\, y\right]$$

(which again holds provided A, B and N do not take on certain values) and (4.6), we have

$$(5.10)\quad \sum_x A(1-x)B(1+x)P_N^{(A,B)}(x)P_M^{(A,B)}(x)$$
$$= A(-1)AB(2)\frac{G(AN)G(BN)}{G(N)G(ABN)}\sum_y {}_2F_1\left[\begin{matrix}\overline{N}, & \overline{BN} \\ & A\end{matrix}\Big| y\right]$$
$$\cdot\ \overline{{}_2F_1\left[\begin{matrix}\overline{M}, & \overline{BM} \\ & A\end{matrix}\Big| y\right]}$$
$$= \frac{q-1}{q}A(-1)AB(2)\frac{G(AN)G(BN)}{G(N)G(ABN)}\delta(M\overline{N}) + \text{extra terms},$$

which we contrast with (5.9).

It is not surprising that the classical orthogonalities (5.3) and (5.9) can also be derived from a representation-theoretic approach [**15, 16**, p. 162].

Another natural basis for V_A is given by

$$(5.11)\qquad \mathcal{B}' = \{A(y)\delta(x)\} \cup \{A(x)\zeta^{\mathrm{Tr}\,(ay/x)}\}_a.$$

If we change the inner product in (3.2) to

$$(5.12)\qquad \langle f, g\rangle = \frac{1}{q(q-1)}\sum_{x,y} f(x,y)\overline{g(x,y)},$$

then the basis $\mathcal{B}'$ is orthogonal and orthonormal except for the vector $A(y)\delta(x)$. With respect to this basis, we have

$$d_{u,v} = \frac{1}{q(q-1)}\sum_{x,y} A(ax+cy)\zeta^{\mathrm{Tr}\,(u(bx+dy)/(ax+cy))}\overline{A}(x)\zeta^{-\mathrm{Tr}\,(vy/x)}$$

taking y to xy and replacing $a+cy$ by x if $c \neq 0$ we have

$$(5.13)\quad d_{u,v} = \begin{cases} \frac{1}{q}\zeta^{\mathrm{Tr}\,((ud+va)/c)}\sum_x A(x)\zeta^{-\mathrm{Tr}\,((1/c)(u/x+vx))} & \text{if } c \neq 0 \\ A(a)\zeta^{\mathrm{Tr}\,(ub/a)}\delta(v-ud/a) & \text{if } c = 0. \end{cases}$$

Thus, in this basis the matrix elements of the principal series representations are essentially generalized Kloosterman sums and represent analogues of Bessel functions (see [**6**] and compare with [**7**, p. 160]).

The irreducible representations of $\mathrm{SL}\,(2, q)$ divide up into two categories: the principal representations discussed in this paper and the discrete representations. Gelfand et al. [**7**, p. 185] give an operator formulation for the discrete representations:

$$T_A(g)f(x) = \sum_{x \neq 0} K_A(g|u, v)f(x), \tag{5.14}$$

where
(5.15)

$$K_A(g|u, v) = \begin{cases} -\zeta^{\mathrm{Tr}\,((du+av)/c)} \sum_{z\bar{z}=v/u} \zeta^{-\mathrm{Tr}\,((1/b)(uz+v/z))} A(z) \\ \qquad \text{if } b \neq 0 \\ A(d)\zeta^{\mathrm{Tr}\,(cdu)}\delta(v - d^2 u) \\ \qquad \text{if } b = 0. \end{cases}$$

In this expression A is a multiplicative character of $\mathrm{GF}\,(q^2)$, $z \in \mathrm{GF}\,(q^2)$, $\bar{z} = z^q$ and the sum is over all $z \in \mathrm{GF}\,(q^2)$ for which $z\bar{z} = v/u$. To show that (5.14) and (5.15) define a representation is itself a rather involved calculation. A study of this representation would be interesting. Possibly, it would tie in with quadratic transformations.

There are two obvious representations defined on the subgroup of upper triangular matrices in $\mathrm{SL}\,(2, q)$:

$$T_1(g)f(x) = f(x + b), \qquad T_2(g)f(x) = \zeta^{\mathrm{Tr}\,(bx)} f(x).$$

These may be combined to give a representation for the Heisenberg group

$$G = \left\{ g = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle|\ a, b, c \in \mathrm{GF}\,(q) \right\} \tag{5.16}$$

via

$$T(g)f(x) = \zeta^{\mathrm{Tr}\,(cx+b)} f(x + a). \tag{5.17}$$

A simple calculation shows that T is an irreducible representation of G.

The matrix elements for the classical analogue of (5.17) are related to confluent hypergeometric functions (see [**16**, Chapter VIII]). If we define a basis $\mathcal{B}$ for the set of all functions from GF (q) to $\mathbf{C}$ by

$$\mathcal{B} = \{\delta(x)\} \cup \{\chi(x)\}_{\chi}, \tag{5.18}$$

then the matrix elements of $d_{A,B}$ of T with respect to $\mathcal{B}$ are

$$K(g|A,B) = \frac{1}{q-1}\sum_{x}\overline{A}(x)B(x+a)\zeta^{\mathrm{Tr}\,(cx+b)}. \tag{5.19}$$

Confluent hypergeometric functions over finite fields were introduced in [**8**, 7.35]. They may be defined by

$$ {}_1F_1\left[\begin{matrix}A\\B\end{matrix}\middle|x\right] = \varepsilon(x)AB(-1)\sum_{y}A(y)\overline{A}B(1-y)\zeta^{-\mathrm{Tr}\,(xy)}. \tag{5.20}$$

Thus, the change of variables $x \mapsto -ax$ in (5.19) and gives

$$K(g|A,B) = \frac{\overline{A}B(-a)}{q-1}\zeta^{\mathrm{Tr}\,(b)}{}_1F_1\left[\begin{matrix}\overline{A}\\\overline{A}B\end{matrix}\middle|ac\right], \tag{5.21}$$

provided $ac \neq 0$.

From $\overline{A}(x)B(x+a) = B(x+a)\overline{A}(x)$, we obtain

$$K(g|A,B) = K(g'|\overline{B},\overline{A}), \tag{5.22}$$

where

$$g' = \begin{bmatrix}1 & -a & b-ac\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}.$$

In terms of hypergeometric series,

$$ {}_1F_1\left[\begin{matrix}A\\B\end{matrix}\middle|x\right] = B(-1)\zeta^{-\mathrm{Tr}\,(x)}{}_1F_1\left[\begin{matrix}\overline{A}B\\B\end{matrix}\middle|-x\right], \tag{5.23}$$

an analogue for Kummer's ${}_1F_1$-transformation [**13**, p. 125].

From

$$K(g_1 g_2|A,B) = \frac{1}{(q-1)^2}\zeta^{\mathrm{Tr}\,(b_1+b_2-a_1c_1)}A(-a_1)B(a_2) + \sum_{\chi} K(g_1|A,\chi)K(g_2|\chi,B) \tag{5.24}$$

and

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = g$$

it follows that

$$K(g|A,B) = \frac{1}{(q-1)^2}\overline{A}(-a)\overline{B}(c)\sum_{\chi} J(\overline{A},\chi)G(B\bar{\chi})\chi(ac). \tag{5.25}$$

Changing χ to $B\chi$ and converting to hypergeometric series, gives

$$ {}_1F_1\left[\begin{matrix} A \\ B \end{matrix}\bigg| x\right] = B(-1)\zeta^{-\mathrm{Tr}\,(x)}\frac{1}{q-1}\sum_{\chi} J(A,\overline{A}B\chi)G(\bar{\chi})\chi(x). \tag{5.26}$$

The idea of using group theoretic or representation theoretic methods in relation to special functions is highly developed (see [**16, 1**] for extensive bibliographies in these areas). Given the strength of the analogy between the calculations in this section and the previous one to the classical case ($F = R$ or $\mathbf{C}$), it is to be expected that much of this theory will carry over to special functions over finite fields.

REFERENCES

1. R.A. Askey, T.H. Koornwinder and W. Schempp (eds.), *Special functions: Group theoretical aspects and applications*, D. Reidel Publishing Company, Boston, 1984.

2. T.S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach Science Publishers, New York, 1978.

3. L. Dornhoff, *Group representation theory* Part A, *Ordinary representation theory*, Marcel Dekker, New York, 1971.

4. R. Evans, *Identities for products of Gauss sums over finite fields*, Enseign. Math. **27** (1981), 197–209.

5. ———, *Character sum analogues of constant term identities for root systems*, Israel J. Math. **46** (1983), 189–196.

6. ———, *Hermite character sums*, Pacific J. Math. **122** (1986), 357–390.

7. I.M. Gelfand, M.I. Graev and I.I. Pzatetskii-Shapiro, *Representation theory and automorphic functions*, W.B. Saunders Company, Philadelphia, 1969.

8. J. Greene, *Character sum analogues for hypergeometric and generalized hypergeometric functions over finite fields*, Ph.D. Thesis, Univ. of Minnesota, Minneapolis, 1984.

9. ———, *Hypergeometric functions over finite fields*, Trans. Amer. Math. Soc. **301** (1987), 77–101.

10. A. Helversen-Pasotto, *L'identité de Barnes pour les corps finis*, C.R. Acad. Sci. Paris Sér. A. **286** (1978), 297–300.

11. N. Koblitz, *The number of points on certain families of hypersurfaces over finite fields*, Compositio Math. **48** (1983), 3–23.

12. W. Li and J. Soto-Andrade, *Barnes' identities and representations of* GL (2), *Part* I: *Finite field case*, J. Reine Angew Math. **344** (1983), 171–197.

13. E. Rainville, *Special functions*, The Macmillan Company, New York, 1960.

14. A.J. Silberger, *On elementary construction of the representations of* SL $(2, q)$, Osaka J. Math. **6** (1969), 329–338.

15. D. Stanton, private communication.

16. N.J. Vilenkin, *Special functions and the theory of group representations*, Amer. Math. Soc., Providence, 1968.

Department of Mathematics and Statistics, University of Minnesota, Duluth, Duluth, MN 55812