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Note

On a Conjecture of Krammer

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In a letter to David Bressoud passed on to George Andrews [1], Daan Krammer made the following conjecture:

$$1+2\sum_{k=1}^{\lfloor n/2\rfloor}\begin{bmatrix}2k-1\\ k\end{bmatrix}_q(-1)^k q^{-k(k-1)/2}=\begin{cases}\left(\dfrac{n}{5}\right), & \text{if } 5\nmid n\\ 1+4\cos\dfrac{2\pi}{5}, & \text{if } 5\mid n,\end{cases}\tag{1}$$

where $q=e^{2\pi i/n}$. Here

$$\left(\frac{n}{5}\right)=\begin{cases}1, & \text{if } n\equiv 1 \text{ or } 4 \pmod 5\\ -1, & \text{if } n\equiv 2 \text{ or } 3 \pmod 5,\end{cases}$$

$\lfloor x\rfloor$ denotes the greatest integer function and $[{n \atop k}]_q$ is the Gaussian q-binomial coefficient defined by

$$\begin{bmatrix}n\\ 0\end{bmatrix}_q=\begin{bmatrix}n\\ n\end{bmatrix}_q=1,$$

$$\begin{bmatrix}n\\ k\end{bmatrix}_q=\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}.\tag{2}$$

By convention, $[{n \atop k}]=0$ if $k<0$ or $k>n$.

Krammer was led to his conjecture through manipulations of series related to the Rogers–Ramanujan identities. These manipulations suggested that the left-hand side of (1) should have a nice evaluation and after com-

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puting this sum for several values of n, he arrived at (1). In this note we show that, in fact, (1) follows from the finite form of one of the Rogers–Ramanujan identities [2, p. 50, Example 10]

$$\sum_{k=0}^{\infty} \begin{bmatrix} n-k \\ k \end{bmatrix}_q q^{k^2} = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(5k+1)/2} \begin{bmatrix} n \\ \lfloor \frac{1}{2}(n-5k) \rfloor \end{bmatrix}_q. \tag{3}$$

First, note that

$$\begin{aligned} \begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q &= \frac{(1-q^{2k-1}) \cdots (1-q^k)}{(1-q) \cdots (1-q^k)} \\ &= (-1)^k q^{k(3k-1)/2} \frac{(1-q^{-k}) \cdots (1-q^{1-2k})}{(1-q) \cdots (1-q^k)}. \end{aligned}$$

Since $q^n = 1$, we may write

$$\begin{bmatrix} 2k-1 \\ k \end{bmatrix}_q = (-1)^k q^{k(3k-1)/2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q. \tag{4}$$

Let S be the left-hand side of (1). By (4),

$$S = 1 + 2 \sum_{k=1}^{\lfloor n/2 \rfloor} \begin{bmatrix} n-k \\ k \end{bmatrix}_q q^{k^2} = -1 + 2 \sum_{k=0}^{\infty} \begin{bmatrix} n-k \\ k \end{bmatrix}_q q^{k^2}.$$

By (3),

$$S = -1 + 2 \sum_{k=0}^{\infty} (-1)^k q^{k(5k+1)/2} \begin{bmatrix} n \\ \lfloor \frac{1}{2}(n-5k) \rfloor \end{bmatrix}_q. \tag{5}$$

But since $q^n = 1$, $[{n \atop k}]_q = 0$ unless $k = 0$ or $k = n$. Setting $n = 5m + r$ with $r = 0$, ± 1, or ± 2, we now proceed by cases.

If $r = \pm 2$, $\lfloor \frac{1}{2}(n-5k) \rfloor$ can never be 0 or n, so $S = -1$ in this case.

If $r = 1$, $\lfloor \frac{1}{2}(n-5k) \rfloor = 0$ if $k = m$. If $r = -1$, $\lfloor \frac{1}{2}(n-5k) \rfloor = n$ if $k = -m$. Thus, in these cases

$$S = -1 + 2(-1)^{\pm m} q^{\pm mn/2}.$$

Since $q^{\pm mn/2} = (-1)^m$, $S = 1$ in these cases.

Finally, if $r = 0$, then $\lfloor \frac{1}{2}(n-5k) \rfloor = 0$ if $k = m$ and $\lfloor \frac{1}{2}(n-5k) \rfloor = n$ if $k = -m$, so

$$\begin{aligned} S &= -1 + 2(-1)^m q^{m(n+1)/2} + 2(-1)^{-m} q^{-m(-n+1)/2} \\ &= -1 + 2q^{m/2} + 2q^{-m/2} \end{aligned}$$

$$= -1 + 4\cos\frac{\pi}{5}$$

$$= 1 + 4\cos\frac{2\pi}{5},$$

as desired.

References

1. G. Andrews, Private communication.
2. G. Andrews, The theory of partitions, *in* "Encyclopedia of Mathematics and its Applications," Vol. 2, Addison-Wesley, Reading, MA 1976.