LUCAS SEQUENCES AND TRACES OF MATRIX PRODUCTS

JOHN GREENE

Abstract. Given two noncommuting matrices, A and B, it is well known that AB and BA have the same trace. This extends to cyclic permutations of products of A's and B's. Thus if A and B are fixed matrices, then products of two A's and four B's can have 3 possible traces. For 2×2 matrices A and B we show that there are restrictions on the relative sizes of these traces. For example, if $M_1 = AB^2AB^2$, $M_2 = ABAB^3$ and $M_3 = A^2B^4$ then it is never the case that $\mathrm{Tr}(M_2) > \mathrm{Tr}(M_3) > \mathrm{Tr}(M_1)$, but the other five orderings of the traces can occur. By utilizing the connection between Lucas sequences and powers a 2×2 matrix, a formula is given for the number of orderings of the traces that can occur in products of two A's and n B's.

1. Introduction and main results

Given two square matrices A and B, it well known [7, 8] that

$$Tr(AB) = Tr(BA), \tag{1.1}$$

where $Tr(A)$ is the trace of the matrix. Consequently, for cyclic permutations [8, p. 110]:

$$Tr(A_1A_2 \cdots A_n) = Tr(A_nA_1A_2 \cdots A_{n-1}). \tag{1.2}$$

Given a matrix written as the product of a collection of matrices, define the necklace of that matrix to be the set of all products of cyclic permutations of the collection. Thus the necklace of ABC is $\{ABC, CAB, BCA\}$, the necklace of $ABAB$ is $\{ABAB, BABA\}$, and the necklace of A^2B^2 is $\{A^2B^2, BA^2B, B^2A^2, AB^2A\}$. By (1.2), all products in a necklace have the same trace.

One might ask how traces of different necklaces compare. The author finds it somewhat surprising that in general, the trace of $ABAB$ tends to be larger than the trace of A^2B^2. To be more rigorous, if A and B are square matrices with independent random variables as entries, then the following table from [4] gives results on how often $\mathrm{Tr}(ABAB) > \mathrm{Tr}(A^2B^2)$ in a simulation with 1,000,000 trials.

The first row in this table suggests that for 2×2 matrices with independent random normal variables, $\mathrm{Tr}(ABAB) > \mathrm{Tr}(A^2B^2)$ with probability $\frac{1}{\sqrt{2}}$. This was proved in [4]. The exact probability for larger matrices is unknown.

Some of the results in [4] apply to other necklaces. If A and B are 2×2 matrices, then $\mathrm{Tr}(AB^2AB^2) > \mathrm{Tr}(A^2B^4)$ with probability $\frac{1}{\sqrt{2}}$ as well. However, with two A's and four B's, there are three necklaces to consider, denoted by AB^2AB^2, $ABAB^3$ and A^2B^4. In simulations, whereas $\mathrm{Tr}(AB^2AB^2) > \mathrm{Tr}(A^2B^4)$

1991 *Mathematics Subject Classification.* 15A15, 15A42.
Key words and phrases. random matrix, trace.

n	normal variables	uniform variables
2	707,456	720,660
3	703,004	703,320
4	701,885	700,959
5	702,375	700,259
10	706,124	704,561
20	709,715	710,189
50	714,473	714,627
100	716,805	717,009

Table 1.1

in 707,607 of 1,000,000 trials (as expected if the probability is $\frac{1}{\sqrt{2}}$), $\mathrm{Tr}(AB^2AB^2) > \mathrm{Tr}(ABAB^3)$ in 641,846 trials and $\mathrm{Tr}(ABAB^3) > \mathrm{Tr}(A^2B^4)$ in 583,781 trials. Presumably the exact probabilities could be calculated as in [4] provided the proper 8-fold integrals could be evaluated.

One could also ask about the six possible total orderings of the traces of these necklaces. Again using independent random normal variables as entries for A and B, in 1,000,000 trials, the following table emerged. Letting $M_1 = AB^2AB^2$, $M_2 = ABAB^3$ and $M_3 = A^2B^4$,

Trace combination	Number of cases
$Tr(M_1) > Tr(M_2) > Tr(M_3)$	300,092
$Tr(M_1) > Tr(M_3) > Tr(M_2)$	123,546
$Tr(M_2) > Tr(M_1) > Tr(M_3)$	282,568
$Tr(M_2) > Tr(M_3) > Tr(M_1)$	0
$Tr(M_3) > Tr(M_1) > Tr(M_2)$	218,484
$Tr(M_3) > Tr(M_2) > Tr(M_1)$	75,310

Table 1.2

Of interest to us here is that the order $Tr(M_2) > Tr(M_3) > Tr(M_1)$ did not occur in the 1,000,000 trials. Exploring further, it was discovered that this is common. In fact, as the number of B's grows, a smaller and smaller portion of orders occurred in simulations. The following is a table from [12].

Our main theorem is the following.

Theorem 1.1. *Consider products of two A's and n B's, where A and B are 2×2 matrices and $n \geq 2$. If ϕ is Euler's totient function, then among those matrices for which no two necklaces have the same trace there are*

$$4 + \frac{1}{2}\sum_{k=3}^{n-1} \phi(k) \tag{1.3}$$

possible arrangements for the orders of the traces when n is even and

$$2 + \sum_{k=3}^{n-1} \phi(k) \tag{1.4}$$

# of B's	necklaces	possible orders	orders occurring
2	2	2	2
3	2	2	2
4	3	6	5
5	3	6	6
6	4	24	8
7	4	24	12
8	5	120	12
9	5	120	22
10	6	720	17
11	6	720	32

TABLE 1.3

possible arrangements when n is odd.

For example, when $n = 10$ the number of allowable orders is $4 + \frac{1}{2}(2 + 2 + 4 + 2 + 6 + 4 + 6) = 17$. Now

$$\sum_{k=1}^{n} \phi(k) = \frac{3n^2}{\pi^2} + O(n \ln n), \tag{1.5}$$

an estimate from [6, Theorem 330], and the number of necklace orderings is the factorial of $\lceil \frac{n+1}{2} \rceil$. Thus, the frequency of possible orders rapidly goes to 0 as n increases. Two distinct necklaces have the same trace with probability 0 if their entries are selected independently at random from a normal distribution. One can easily construct A and B for which different necklaces have the same trace, even when A and B do not commute. For example, if

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

then $\mathrm{Tr}(AB^2AB^2) > \mathrm{Tr}(ABAB^3) = \mathrm{Tr}(A^2B^4)$. In this paper, we are only interested in strict inequalities so in what follows, we restrict ourselves to distinct traces.

As usual [10, pp. 41-61], [11, pp. 107-108], Lucas sequences $U_n = U_n(P, Q)$ may be defined by the recurrence $U_0 = 0,\ U_1 = 1,\ U_n = PU_{n-1} - QU_{n-2}$, for $n \geq 2$. Lucas sequences naturally enter into this study as follows. Let B be a 2×2 matrix with trace P and determinant Q. Then by the 2×2 version of the Cayley-Hamilton theorem,

$$B^2 = PB - QI.$$

An easy induction gives

$$B^n = U_nB - QU_{n-1}I. \tag{1.6}$$

In the next section, we relate traces of necklaces to Lucas sequences. In Section 3, we prove the theorem. We give some concluding remarks in section 4.

2. Lucas sequences and necklace traces

In this section, A and B will always denote 2×2 matrices. Moreover, we let $P = \mathrm{Tr}(B)$ and $Q = \det(B)$, and define the Lucas sequence $\{U_n(P,Q)\}$ as in the previous section. The main result of this section is the following.

Theorem 2.1. *Let A and B are 2×2 matrices and let $T = Tr\ (ABAB - A^2B^2)$. If $n \geq m \geq k$ then*

$$Tr\ (AB^mAB^n) - Tr\ (AB^{m-k}AB^{n+k}) = Q^{m-k}U_k\,U_{n-m+k}\,T. \tag{2.1}$$

This theorem allows us to convert a question about trace orders to the positivity of a collection of products on the right hand side of equation (2.1).

Proof. We prove (2.1) via a number of applications of formula (1.6). We have

$$\begin{aligned} AB^mAB^n - AB^{m-k}AB^{n+k} &= (AB^mAB^{m-k} - AB^{m-k}AB^m)B^{n-m+k} \\ &= U_{n-m+k}(AB^mAB^{m+1-k} - AB^{m-k}AB^{m+1}) \\ &\quad - QU_{n-m+k-1}(AB^mAB^{m-k} - AB^{m-k}AB^m). \end{aligned}$$

Since $\mathrm{Tr}(AB^mAB^{m-k}) = \mathrm{Tr}(AB^{m-k}AB^m)$ we have

$$\mathrm{Tr}(AB^mAB^n - AB^{m-k}AB^{n+k}) = U_{n-m+k}\mathrm{Tr}(AB^mAB^{m+1-k} - AB^{m-k}AB^{m+1}).$$

We now use $B^k = U_kB - QU_{k-1}I$ to obtain

$$\begin{aligned} AB^mAB^{m+1-k} - AB^{m-k}AB^{m+1} &= AB^{m-k}B^kAB^{m+1-k} - AB^{m-k}AB^{m+1-k}B^k \\ &= U_k(AB^{m+1-k}AB^{m+1-k} - AB^{m-k}AB^{m+2-k}). \end{aligned}$$

Letting $m - k = j$, we are left to evaluate

$$AB^{j+1}AB^{j+1} - AB^jAB^{j+2}.$$

We have

$$\begin{aligned} AB^{j+1}AB^{j+1} - AB^jAB^{j+2} &= PAB^jAB^{j+1} - QAB^{j-1}AB^{j+1} \\ &\quad - PAB^jAB^{j+1} + QAB^jAB^j \\ &= Q(AB^jAB^j - AB^{j-1}AB^{j+1}). \end{aligned}$$

A simple induction now gives

$$AB^{j+1}AB^{j+1} - AB^jAB^{j+2} = Q^j(ABAB - A^2B^2),$$

and the proof follows. □

If we put the products in a necklace in lexicographic order, then the first element of the necklace will have the form AB^mAB^n with $m \leq n$. We will use such matrices to represent their necklace in what follows. A natural way to order necklaces is by how far the A's are apart in the product (viewed cyclically). With this ordering, if there are a total of $2n$ B's then there are $n+1$ necklaces, denoted by $AB^nAB^n,\ AB^{n-1}AB^{n+1}, \ldots, A^2B^{2n}$. We introduce a numbering scheme where we associate k with the necklace $AB^{n-k+1}AB^{n+k-1}$. When the number of B's is odd, say $2n+1$ we use a similar scheme, but with $1 \leftrightarrow AB^nAB^{n+1}$, and more generally, $k \leftrightarrow AB^{n-k+1}AB^{n+k}$. We write jk to denote that necklace j has a larger trace than necklace k and we let a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ refer to the property that necklaces $\pi_1, \ldots, \pi_n$ have their traces in decreasing order. For example, if there are six B's then the permutation 3412 would be used to represent the statement that

$$\mathrm{Tr}(ABAB^5) > \mathrm{Tr}(A^2B^6) > \mathrm{Tr}(AB^3AB^3) > \mathrm{Tr}(AB^2AB^4).$$

Given a collection of two A's and n B's, let $l = \lfloor \frac{n}{2} \rfloor$. If $i < j$ then $i\,j$ corresponds to the statement

$$\mathrm{Tr}(AB^{l-i+1}AB^{l+i-1}) > \mathrm{Tr}(AB^{l-j+1}AB^{l+j-1})$$

when n is even and

$$\mathrm{Tr}(AB^{l-i+1}AB^{l+i}) > \mathrm{Tr}(AB^{l-j+1}AB^{l+j})$$

when n is odd. Consequently, we have the following corollaries to Theorem 2.1.

Corollary 2.2.

$$i\,j \longleftrightarrow \begin{cases} Q^{l-j+1}U_{j-i}\,U_{i+j-2}\,T > 0 & \textit{when } n \textit{ is even,} \\ Q^{l-j+1}U_{j-i}\,U_{i+j-1}\,T > 0 & \textit{when } n \textit{ is odd.} \end{cases} \tag{2.2}$$

Proof. This is a direct application of Theorem 2.1. □

Note that $i\,j$ when $i > j$ corresponds to the appropriate expression being negative rather than positive.

Corollary 2.3. *If n is even and Q and T are both positive, then the necklace containing AB^lAB^l has the largest trace.*

Proof. Since AB^lAB^l corresponds to 1 in our notation, if this necklace is not largest, then there must be a pair of the form $k\,1$. This translates to

$$k\,1 \longleftrightarrow Q^{l-k+1}U_{k-1}^2\,T < 0,$$

implying either $T < 0$ or $l - k + 1$ is odd and $Q < 0$. □

We introduce one final piece of terminology. For a permutation π, let $(i\,j)$ be 1 if i is to the left of j in π and -1 if it is to the right. For $i < j$,

$$(i\,j) = \begin{cases} \mathrm{sgn}(Q^{l-j+1}U_{j-i}\,U_{i+j-2}\,T) & \text{when } n \text{ is even,} \\ \mathrm{sgn}(Q^{l-j+1}U_{j-i}\,U_{i+j-1}\,T) & \text{when } n \text{ is odd,} \end{cases} \tag{2.3}$$

where $\mathrm{sgn}(x)$ denotes the sign of x. We note that $(i\,j) = -(j\,i)$.

We make use of the following properties of Lucas sequences.

Lemma 2.4. *Viewing $U_n(P, Q)$ as a polynomial in P and Q we have the following.*

(a) *As a polynomial in P, U_n has degree $n - 1$. If n is even, then U_n is an odd polynomial in P, if n is odd, then it is an even polynomial in P.*
(b) *As a polynomial in Q, U_n has degree $\left\lfloor \dfrac{n-1}{2} \right\rfloor$. Also, U_n has exactly $\left\lfloor \dfrac{n+1}{2} \right\rfloor$ terms, one for each allowable power of Q and the coefficient of Q^k has the form $P^{n-1-2k}(-1)^k c_k$ for some integer $c_k > 0$.*
(c) *If $P^2 \geq 4Q$ then $U_n > 0$ if n is odd and $\frac{1}{P}U_n > 0$ if n is even.*

Proof. The proofs of (a) and (b) are easy inductions. Part (c) follows from the representation [10, p. 44]

$$U_n(P,Q) = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} P^{n-2k-1}(P^2-4Q)^k.$$

□

Finally for this section, we mention the following result.

Lemma 2.5. *If A and B are 2×2 matrices and Tr($ABAB$) $<$ Tr(A^2B^2) then $P^2 \geq 4Q$, where P is the trace of B and Q is the determinant of B.*

Proof. In Lemma 3.6 of [4] it is shown that $\mathrm{Tr}(ABAB) > \mathrm{Tr}(A^2B^2)$ when either A or B has complex eigenvalues. Thus, in order for $\mathrm{Tr}(ABAB) < \mathrm{Tr}(A^2B^2)$, B must have real eigenvalues, call them λ_1 and λ_2. Now $P = \lambda_1 + \lambda_2$ and $Q = \lambda_1\lambda_2$, so $P^2 - 4Q = (\lambda_1 - \lambda_2)^2 \geq 0$. □

3. A proof of theorem 1.1

The Q-parameter in $U_n(P,Q)$ may be scaled away by multiplying B by $\frac{1}{\sqrt{|Q|}}$. This will have no effect on the orders of the traces of the necklaces. Thus, we need only consider two types of Lucas sequences: $U_n(x,1)$ and $U_n(x,-1)$. The second of these are usually referred to as Fibonacci polynomials, the first are a scaled version of Chebyshev polynomials of the second kind, with the actual Chebyshev polynomials being $U_{n+1}(2x,1)$. We require the following facts about $U_n(x,1)$.

Lemma 3.1. *The zeros of $U_n(x,1)$ have the form $x = 2\cos\frac{k\pi}{n}$ where $1 \leq k \leq n-1$. In particular, for all $n \geq 3$, $U_n(x,1)$ has exactly $\lfloor \frac{n-1}{2} \rfloor$ simple positive zeros and the zeros of $U_n(x,1)$ and $U_{n+1}(x,1)$ separate each other. That is, between each pair of successive positive zeros of one polynomial there is exactly one zero of the other.*

Proof. That the zeros are simple and separate each other follows from the fact that $\{U_n(2x,1)\}$ is a set of orthogonal polynomials. See [1, Theorem 5.4.1, Theorem 5.4.2], for example. In fact, a standard representation for Chebyshev polynomials [1, p 101] is $U_n(2\cos\theta, 1) = \frac{\sin n\theta}{\sin\theta}$, giving the formula for the zeros. Since U_n has degree $n-1$ and is even or odd depending on whether n is odd or even, the count for the number of positive zeros follows. □

In proving Theorem 1.1 we show that the expressions in (1.3) and (1.4) give upper bounds for the numbers of possible trace orders, and that these bounds are achieved. For the upper bound we use Corollary 2.2 and Lemma 2.4 to give information on permutations of necklace trace orders. We need information on the sign of $Q^{m-k}U_kU_{n-m+k}T$. We break up the investigation into three cases: $Q < 0$, $Q > 0$ but $T < 0$, and both $Q > 0$, $T > 0$. We investigate these cases in order.

Lemma 3.2. *If $Q < 0$ then there are exactly two possible permutations of trace orders.*

Proof. If $Q < 0$ then $P^2 > 4Q$ so by Lemma 2.4, $U_n > 0$ for odd n and $\frac{1}{P}U_n$ is positive for even n. By Corollary 2.2, if $i < j$ then $(i\,j) = \mathrm{sgn}((-1)^{n-j+i}TU_{j-i}U_{j+i-k})$ where $k = 1$ if there are an odd number of B's and $k = 2$ otherwise. This means that

$$(i\,j) = \begin{cases} \operatorname{sgn}((-1)^{l-j+1}\,T) & \text{when } n \text{ is even,} \\ \operatorname{sgn}((-1)^{l-j+1}\,TP) & \text{when } n \text{ is odd.} \end{cases}$$

For any given B, the sign of T and the sign of PT are fixed. That is, the relevant sign is the same for every permutation. Consequently, there are two possible permutations corresponding to whether T (or PT) is positive or negative. □

We can state explicitly what these permutations are. Since $l-j+1$ does not depend on i, but only on the parity of j, once we know $(1\,2)$, we know the permutation. If $(1\,2) = 1$ then all even numbers follow all odd numbers. Also, $(i\;i+2j)$ will be 1 if i is even, -1 if i is odd. This means that π starts with the largest odd number and descends through the odds to 1, followed by the even numbers in increasing order. If $n = 9$, for example, then there are five necklaces and this permutation would be $5\,3\,1\,2\,4$. On the other hand, if $(1\,2) = -1$, we have the reverse of this permutation, $4\,2\,1\,3\,5$. These two permutations must occur since they will be produced by the matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

To see this, we note that

$$\operatorname{Tr}(A_1BA_1B - A_1^2B^2) = -9,$$
$$\operatorname{Tr}(A_2BA_2B - A_2^2B^2) = 9.$$

In these examples, $P = 1,\ Q = -2 < 0$. Since $T = \operatorname{Tr}(ABAB - A^2B^2)$, by the proof of the theorem, $(1\ 2) = sgn((-1)^{n-1}T)$, and both sign patterns will occur. One thing is left to establish: that all necklaces have different traces. For this, by Theorem 2.1, two traces can only be the same when $Q^{m-k}U_kU_{n-m+k}T = 0$. In this case, $U_n = U_{n-1} + 2U_{n-2}$ implies that no U_k is zero if $k > 0$.

Lemma 3.3. *If $Q > 0$ but $T < 0$ then there is one possible permutation when there are an even number of B's, and two permutations if the number of B's is odd.*

Proof. By Lemma 2.5, we again have $P^2 > 4Q$, but now we know the sign of T. Thus, for $i < j$ we have

$$(i\,j) = \begin{cases} -\operatorname{sgn}(Q^{l-j+1}) = -1 & \text{when } n \text{ is even,} \\ -\operatorname{sgn}(Q^{l-j+1}\,P) = -\operatorname{sgn}(P) & \text{when } n \text{ is odd.} \end{cases}$$

If there are m necklaces, then when n is even, the permutation must be $\pi = m\,(m-1)\,\cdots\,2\,1$. When n is odd, there are two possible permutations, one for each sign of P. The first is the same as the above, the other is $\pi = 1\,2\,\cdots\,(m-1)\,m$. □

Again, these cases are realized in the examples

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

This leaves us with the following case.

Theorem 3.4. *Consider the set*

$$S = \{a > 0 \mid U_k(a, 1) = 0 \text{ for some } 3 \le k \le 2n-1\},$$

the set of distinct positive zeros of $U_3, \dots, U_{n-1}$, and suppose S has size m. The number of permutations of trace orders in the case where $Q > 0$ and $T > 0$ is

$$\begin{cases} 1+m, & \textit{when } N \textit{ is even,} \\ 2(1+m), & \textit{when } N \textit{ is odd.} \end{cases}$$

Proof. Since $Q > 0$ and $T > 0$, when $i < j$,

$$(i\,j) = \begin{cases} \operatorname{sgn}(U_{j-i}U_{i+j-2}), & \text{when } n \text{ is even,} \\ \operatorname{sgn}(U_{j-i}U_{i+j-1}), & \text{when } n \text{ is odd.} \end{cases} \tag{3.1}$$

Since $Q > 0$ we may scale away Q and only consider the polynomials $U_k(x, 1)$. We focus on $(i \;\; i+1)$ and $(i \;\; i+2)$. When N is even, these have the form $\operatorname{sgn}(U_1(x,1)U_{2i-1}(x,1)) = \operatorname{sgn}(U_{2i-1})$, and $\operatorname{sgn}(U_2(x,1)U_{2i}(x,1)) = \operatorname{sgn}(xU_{2i})$, respectively. When N is odd, the important quantities are $\operatorname{sgn}(U_{2i})$ and $\operatorname{sgn}(xU_{2i+1})$. Given an $x \notin S$, the conditions of (3.1) will determine a permutation, call it $\pi(x)$.

Suppose we order the set of positive zeros $0 < x_1 < x_2 < \cdots < x_m$. These zeros partition the half line $(0, \infty)$ into $m+1$ regions. If x and y belong to the same region, say $x_i < x, y < x_{i+1}$, then $\pi(x) = \pi(y)$ since signs of $U_k(x, 1)$ and $U_k(y, 1)$ will match for all k. Thus, there can be no more than $m+1$ permutations associated with the regions between the elements of S. When N is even, all the products of the U's will be even polynomials, leaving us with at most these $m+1$ permutations. When n is odd, the products of the U's will be odd polynomials. Thus $\pi(-x)$ will be the reverse of $\pi(x)$, doubling the possible number of permutations. If

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix},$$

then $P = x,\ Q = 1 > 0,\ T = x^2 > 0$ so for every region between the zeros of the U_k there is a matrix B with an x-value in that region, along with its associated permutation.

Finally, we must show that all resulting permutations are distinct. To that end, suppose that a and b are real numbers with $x_i < a < x_{i+1} < \ldots < x_{i+j} < b < x_{i+j+1}$, with the obvious interpretation if $x_{i+1} = x_1$ or $x_{i+j+1} = x_m$. That is, suppose a and b are separated by a positive number, j, of zeros of the U_k. Then there is a smallest k for which a and b are separated by a single zero of some U_k. To see this, let k be minimal with the property that there is a zero of U_k separating a and b. If there were more than two zeros between a and b, then by the interlacing property, U_{k-1} would also have a zero between a and b, a contradiction.

Now given a and b from different regions of the half line, let k be an index for which a and b are separated by a single zero of U_k. Then $\operatorname{sgn}(U_k(a,1)) = -\operatorname{sgn}(U_k(b,1))$. This means that depending on the parity of k, there will be an i with either $(i\;i+1)$ or $(i\;i+2)$ differing from $\pi(a)$ to $\pi(b)$. Consequently, $\pi(a) \neq \pi(b)$. In the case where N is odd, we must also show that for positive a and b, $\pi(a) \neq \pi(-b)$. Since $(1\;2) = \operatorname{sgn}(x)$, when x is positive, 1 will be to the left of 2 but for negative x, 1 is to the right of 2 so these permutations are all distinct. □

A consequence of Theorem 3.4 is that when n is even, there are $m+4$ orders for the permutations of necklace traces. This is because the three permutations arising from the cases where $Q < 0$ and $Q > 0$, $T < 0$ do not begin with 1, making them distinct from the $m+1$ permutations of Theorem 3.4. When n is odd, however, the four permutations associated with $Q < 0$ and $Q > 0$, $T < 0$ also occur among the permutations of Theorem 3.4. In fact, if x_1 is the smallest element of S, and $0 < x < x_1$ then $\pi(x)$ and $\pi(-x)$ are the permutations that arise in Lemma 3.2. This is because an easy calculation shows $(i\ j) = (-1)^j$, which does not depend on i, and the discussion following Lemma 3.2 applies to this case.

Similarly, when n is odd, the two permutations from Lemma 3.3 are $\pi(x)$ and $\pi(-x)$ where $x > x_m$, the largest of the zeros in S. In this case, $U_k(x,1) > 0$ for all k, so $\pi(x)$ is the identity permutation and $\pi(-x)$ is its reverse, as in Lemma 3.3. Consequently, when n is odd, the only permutations we have are those arising from Theorem 3.4. Consequently, we have a count on the number of permutations. It is

$$\begin{cases} 4+m, & \text{when } n \text{ is even,} \\ 2+2m, & \text{when } n \text{ is odd.} \end{cases}$$

The proof of Theorem 1.1 follows from the observation that for S as in Theorem 3.4, $|S| = \dfrac{1}{2}\displaystyle\sum_{k=3}^{n-1}\phi(k)$. This, in turn, follows from the fact that

$$S = \left\{ 2\cos\frac{k\pi}{m} \mid 1 \le k \le \frac{1}{2}(m-1),\ 3 \le m \le n-1 \right\}.$$

That is, given that S contains the zeros for $U_k(x,1)$, with $3 \le k < m$, the contribution of the zeros of U_m to S will consist of those numbers $2\cos\frac{j\pi}{m}$, with j prime to m. There are $\frac{1}{2}\phi(m)$ of these, by periodicity and the fact that we seek only positive zeros. This concludes the proof of Theorem 1.1.

4. COMMENTS

Products of A's and B's with at least three A's and three B's are more complicated. There are two issues. First, for products of 2×2 matrices A and B, there is another trace symmetry in addition to cyclic permutations: If a product is written in reverse order, it has the same trace, as proved in [?] or [4]. That is,

$$\mathrm{Tr}(AABBAB) = \mathrm{Tr}(BABBAA)$$

for all 2×2 matrices A and B. If a product consists of just two A's, then the reverse of a product is in the same necklace, but for larger numbers of A's, as in the example, this need not be the case. This makes ordering the necklaces more challenging.

A second issue is that with at least three A's and three B's, the matrices interact more than just through the trace of $ABAB - A^2B^2$. For example,

$$\mathrm{Tr}(ABABAB - A^2BAB^2) = \mathrm{Tr}(AB)\mathrm{Tr}(ABAB - A^2B^2).$$

We do not have an analog for Theorem 2.1 when there are more than two A's. However, we at least have the following weak version.

Theorem 4.1. *Suppose that M_1 and M_2 are each products of m A's and n B's where A and B are 2×2 matrices. Then*

$$Tr(M_1 - M_2) = c\,Tr(ABAB - A^2B^2),$$

where c is a polynomial in the entries of A and B.

Proof. We may induct on $m+n$, the number of matrices in the two products. If $m+n \leq 4$, the trace is zero when m is 0, 1, 3 or 4, and for $m = 2$, the result is true by Theorem 2.1. For larger $m+n$, we first note that by cyclic permutation we may write

$$M_1 = A^{a_1}B^{b_1}A^{a_2}B^{b_2}\cdots A^{a_j}B^{b_j},$$
$$M_2 = A^{c_1}B^{d_1}A^{c_2}B^{d_2}\cdots A^{c_k}B^{d_k},$$

where each of the exponents is a positive integer, $a_1 + \cdots + a_j = m = c_1 + \cdots + c_k$, and $b_1 + \cdots + b_j = n = d_1 + \cdots + d_k$. Moreover, we may take a_1 to be the largest of the a's and c_1 to be the largest of the c's. If $a_1 \geq 2$ and $c_1 \geq 2$ we may use $A^2 = \mathrm{Tr}(A)A - \det(A)I$ and induct. Similarly, if one of the b's and one of the d's is at least 2 we may induct. If no A has an exponent larger than 1 then $j = k$, and the only way to prevent some exponent of B to be at least 2 is to have $M_1 = M_2$. Thus we may assume that, say c_1, is at least 2, and $a_1 = a_2 = \cdots = a_j = 1$. In this case, $j = m \leq n$ and $k < m$. Consequently, the largest b and largest d will both be at least 2 unless $m = n$ and $M_1 = (AB)^m$. Since $m+n > 4$, $m \geq 3$. Let $M_3 = A(AB)^{m-1}B = A^2(BA)^{m-2}B^2$ and consider

$$\mathrm{Tr}(M_1 - M_2) = \mathrm{Tr}(M_1 - M_3) + \mathrm{Tr}(M_3 - M_2).$$

By the previous discussion, $\mathrm{Tr}(M_3 - M_2) = c_1\mathrm{Tr}(ABAB - A^2B^2)$ since each matrix contains A^2. Since $m - 1 \geq 2$ we may use $(AB)^2 = \mathrm{Tr}(AB)AB - \det(AB)I$ to write

$$\begin{aligned}\mathrm{Tr}(M_1 - M_3) = \mathrm{Tr}(AB)((AB)^{m-1} - A(AB)^{m-2}B)\\ - \det(AB)((AB)^{m-2} - A(AB)^{m-3}B),\end{aligned}$$

and the inductive hypothesis gives $\mathrm{Tr}(M_1 - M_3) = c_2\mathrm{Tr}(ABAB - A^2B^2)$, from which the proof follows. □

We note that the polynomial c in this proof can be thought of as a polynomial in the five variables $\mathrm{Tr}(A)$, $\mathrm{Tr}(B)$, $\mathrm{Tr}(AB)$, $\det(A)$, $\det(B)$, rather than the eight entries of A and B.

We only briefly investigated cases with a higher number of A's. When there are three A's or four A's we obtained the following table.

# of A's	# of B's	necklaces	orders occurring
3	3	3	6
3	4	4	24
3	5	5	52
3	6	7	175
3	8	8	246
4	4	8	616
4	5	10	?

TABLE 4.1

We have not verified that the entries in the fourth column are the true numbers of possible orders. However, it is not too difficult to show that certain orders do not occur. For example, when there are three A's and five B's, if one labels the necklaces via $1 \leftrightarrow ABAB^2AB^2,\ 2 \leftrightarrow ABABAB^3,\ 3 \leftrightarrow A^2B^2AB^3,\ 4 \leftrightarrow A^2BAB^4,\ 5 \leftrightarrow A^3B^5$, then we may construct a table of polynomials as in the previous section and use this to show that the trace order 1 2 4 3 5 does not occur. If we let $a = \mathrm{Tr}(A),\ b = \mathrm{Tr}(AB),\ c = \mathrm{Tr}(B),\ d = \det(B),\ e = \mathrm{Tr}(ABAB - A^2B^2)$, then

References

[1] G. Andrews, R. Askey and R. Roy, *Special Functions,* Encyclopedia of Mathematics and its Applications 71, Cambridge University Press, McGrawHill, Cambridge, 1999.
[2] A. Edelman, *The probability that a random real gaussian matrix has real eigenvalues, related distributions, and the circular law,* J. Multivariate Anal. 60, 1997 203-232.
[3] A. Edelman, E. Kostlan and M.Shub, *How many eigenvalues of a random matrix are real?* J. Amer. Math. Soc. 7, 1994 247-267.
[4] J. Greene, *Traces of matrix products*
[5] J. Greene and K. Niedzielski, *Periodic solutions to some difference equations over the integers,* Difference Equations and Their Applications, 16 (2010), 321-346.
[6] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* Fifth ed, Oxford University Press, Oxford, 1979.
[7] K. Hoffman and R. Kunze, *Linear Algebra, Prentice Hall,* Englewood Cliffs, 1971.
[8] C. Meyer, *Matrix Analysis and Applied Linear Algebra,* SIAM, Philadelphia, 2000.
[9] B. Noble and J. Daniel, *Applied Linear Algebra,* Prentice Hall, Englewood, 1988.
[10] P. Ribenboim, *The Book of Prime Number Records,* Springer-Verlag, New York, 1988.
[11] H. Riesel, *Prime Numbers and Computer Methods for Factorization,* Progress in Mathematics 126 (2nd ed.) Birkhäuser, Boston, 1994.
[12] A. Schneider, *Investigating traces of matrix products,* Master's project, 2015.

Department of Mathematics and Statistics, University of Minnesota Duluth