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John Greene
University of MInnesota, jgreene@d.umn.edu

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TRACES OF MATRIX PRODUCTS*

JOHN GREENE†

Abstract. Given two noncommuting matrices, A and B, it is well known that AB and BA have the same trace. This extends to cyclic permutations of products of A's and B's. It is shown here that for 2×2 matrices A and B, whose elements are independent random variables with standard normal distributions, the probability that $Tr(ABAB) > Tr(A^2B^2)$ is exactly $\frac{1}{\sqrt{2}}$.

Key words. Random matrix, Trace.

AMS subject classifications. 15A15, 15A42.

1. Introduction and main results. Given two square matrices A and B, it follows from [8, 10] that

$$\det(AB) = \det(A)\det(B), \qquad Tr(AB) = Tr(BA), \tag{1.1}$$

where $Tr(A)$ is the trace of the matrix. Consequently, for any product $A_1A_2 \cdots A_n$ and any permutation σ,

$$\det(A_1A_2 \cdots A_n) = \det(A_{\sigma(1)}A_{\sigma(2)} \cdots A_{\sigma(n)}).$$

By the second formula in (1.1), a similar formula holds for the trace, but only for cyclic permutations [10, p. 110]:

$$Tr(A_1A_2 \cdots A_n) = Tr(A_nA_1A_2 \cdots A_{n-1}). \tag{1.2}$$

Given a matrix written as the product of a collection of matrices, define the necklace of that matrix to be the set of all products of cyclic permutations of the collection. Thus, the necklace of ABC is $\{ABC, CAB, BCA\}$, the necklace of $ABAB$ is $\{ABAB, BABA\}$, and the necklace of A^2B^2 is $\{A^2B^2, BA^2B, B^2A^2, AB^2A\}$. By (1.2), all products in a necklace have the same trace.

Given a product $A_1A_2 \cdots A_n$, define its reversal to be $A_nA_{n-1} \cdots A_1$ and denote this

$$(A_1A_2 \cdots A_n)^R = A_nA_{n-1} \cdots A_1. \tag{1.3}$$

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†Department of Mathematics and Statistics, University of Minnesota Duluth (jgreene@d.umn.edu).

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Under the hypotheses in Theorem 1.1 below, a product and its reversal have the same trace. Since $ABAB$ and A^2B^2 belong to different necklaces, and neither is the reversal of the other, one may ask about the relative sizes of their traces. The relative sizes depend on A and B, but surprisingly, there is a sense in which $ABAB$ usually has the larger trace. We make this rigorous in Theorem 1.3 and Theorem 1.4, the main results in this paper. These theorems make use of the following two results.

THEOREM 1.1. *Fix two 2×2 matrices A and B. If*

$$M = M_1 M_2 \cdots M_n,$$

where each M_i is A or B, then M has the same trace as its reversal:

$$Tr(M) = Tr(M^R).$$

For example,

$$Tr(AABBAB) = Tr(BABBAA),$$

even though the two matrices are not in the same necklace.

THEOREM 1.2. *If A and B are 2×2 matrices and M is a product of A's and B's, then*

$$M - M^R = c(AB - BA),$$

where c is a scalar.

The hypotheses in Theorem 1.1 are necessary: If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then $Tr(ABC) = 2$, $Tr((ABC)^R) = 1$. If

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

then

$$Tr(AABBAB) = 149, \qquad Tr((AABBAB)^R) = 148.$$

Thus, Theorem 1.1 only applies to 2×2 matrices, and only to products with two types of matrices.

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The scalar, c, in Theorem 1.2 depends on the order of the matrices, and on A and B. For example,

$$A^2B^3 - B^3A^2 = c_1(AB - BA), \qquad AB^2AB - BAB^2A = c_2(AB - BA),$$

where

$$c_1 = Tr(A)(Tr(B)^2 - \det(B)), \qquad c_2 = Tr(B)Tr(AB) - Tr(A)\det(B).$$

If the matrix M is a product of A's and B's, call M a symbolic palindrome if viewing the A's and B's as characters yields a palindrome. For example, $AB^4A^2B^4A$ is a symbolic palindrome. Obviously, if M is a symbolic palindrome, then $M - M^R = 0$ regardless of the values of A and B. If M is not a symbolic palindrome then $M - M^R$ might still be zero for certain choices of A and B. For example, the scalar c_1 above is 0 whenever $\mathrm{Tr}(A) = 0$.

With these preliminaries, our main results are the following.

Theorem 1.3. *If A and B are 2×2 matrices with independent normally distributed elements of mean 0 and variance 1, M is a product of A's and B's, and M is not a symbolic palindrome, then*

$$Tr(M^2) > Tr(MM^R)$$

with probability $\frac{1}{\sqrt{2}}$. In particular, $Tr(ABAB) > Tr(AABB)$ with probability $\frac{1}{\sqrt{2}}$.

Theorem 1.4. *If A and B are 2×2 matrices with entries selected uniformly and independently at random on $[-1, 1]$, M is a product of A's and B's, and M is not a symbolic palindrome then there is a $p > \frac{1}{2}$, independent of M, for which*

$$Tr(M^2) > Tr(MM^R)$$

with probability p. In particular, $Tr(ABAB) > Tr(AABB)$ with probability p.

Numerical evidence suggests that the probability, p, in Theorem 1.4 is about .72 but it is only proved here that $p \geq \frac{2783}{5184} > .536$. Sketches for the proofs of Theorem 1.1 and Theorem 1.2 are given in the next section. In Section 3, a proof is given of Theorem 1.4, and Theorem 1.3 is proven in Section 4. We close in Section 5 with a result on the determinant of a matrix of trace 0 and give several tables of simulations.

2. Proofs of Theorem 1.1 and Theorem 1.2. The proof of Theorem 1.1 is sketched in [7]. It also appears in the Masters papers [9] and [11]. We give a very brief sketch here.

Proof of Theorem 1.1. The major tool is the Cayley-Hamilton theorem for 2×2 matrices: If I is the 2×2 identity matrix then for any 2×2 matrix C,

$$C^2 = Tr(C)C - \det(C)I. \tag{2.1}$$

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ELA
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Given $M = M_1M_2\cdots M_n$ where each M_i is A or B, we may use (2.1) in any product with $M_i = M_{i+1}$ for some i, resulting in two products of shorter length. One may invoke induction to handle this case since the reversal will be well behaved with respect to (2.1). If $M_i \neq M_{i+1}$ for all i, then the A's and B's must alternate in M. If n is odd, then $M = M^R$, so M and M^R have the same trace. If n is even, then M^R is in the same necklace as M, and again they have the same trace. This completes the proof. ∎

Proof of Theorem 1.2. This proof is also in [9]. As in Theorem 1.1, we let $M = M_1M_2\cdots M_n$ where each M_i is A or B, and proceed by induction on n. When $n = 1$, $M - M^R = 0$. When $n = 2$, there are four cases, with the important one being $M = AB$, for which $M - M^R = AB - BA$. Assuming the result for products of fewer than n matrices, we again consider the case where $M_i = M_{i+1}$ for some i. Letting $a = Tr(M_i)$ and $b = -\det(M_i)$,

$$\begin{aligned} M - M^R = a&[(M_1\cdots M_{i-1}M_iM_{i+2}\cdots M_n) - (M_1\cdots M_{i-1}M_iM_{i+2}\cdots M_n)^R] \\ &+ b[(M_1\cdots M_{i-1}M_{i+2}\cdots M_n) - (M_1\cdots M_{i-1}M_{i+2}\cdots M_n)^R]. \end{aligned}$$

Invoking the inductive hypothesis,

$$M - M^R = ac_1(AB - BA) + bc_2(AB - BA) = (ac_1 + bc_2)(AB - BA)$$

for some constants c_1 and c_2. If there is no i for which $M_i = M_{i+1}$, then the A's and B's alternate in M. As before, if n is odd, $M = M^R$, so $M - M^R = 0(AB - BA)$. Finally, if n is even, then without loss of generality, let $M = (AB)^k$ for some integer k. Writing $(AB)^2 = x(AB) + yI$ in (2.1), we have

$$M = x(AB)^{k-1} + y(AB)^{k-2},$$

so

$$\begin{aligned} M - M^R &= x[(AB)^{k-1} - (BA)^{k-1}] + y[(AB)^{k-2} - (BA)^{k-2}] \\ &= c(AB - BA) \end{aligned}$$

for suitable c. This completes the proof of Theorem 1.2. ∎

3. A proof of Theorem 1.4. The scalar c in Theorem 1.2 is a polynomial in the entries of A and B. We begin by mentioning that for the specific pair of matrices A and B in the lemma below, this polynomial is always nonzero unless M is a symbolic palindrome.

LEMMA 3.1. *If* $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ *and M is a product of A's and B's, then $M \neq M^R$ unless M is a symbolic palindrome.*

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
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Proof. Certainly, if M is a symbolic palindrome then $M = M^R$ for any choice of A and B, so we may assume that M is not a symbolic palindrome. We make the following reduction: If $M = M_1M_2 \cdots M_n$ is not a symbolic palindrome, then there is a smallest index k with $M_k \neq M_{n-k+1}$. Letting $C = M_1 \cdots M_{k-1}$ (or $C = I$ if $k = 1$), it follows that $M = CNC^R$ and $M - M^R = C(N - N^R)C^R$, where N is a product of A's and B's with first matrix different from the last. Since A and B are invertible, $M \neq M^R$ if $N \neq N^R$. Thus, without loss of generality, we may assume that $M_1 \neq M_n$. In fact, we may assume $M_1 = A$ and $M_n = B$. Thus, we may let $M = A^{x_1}B^{y_1} \cdots A^{x_m}B^{y_m}$ where each exponent is a positive integer. We will show that for products of this type, c is a positive integer, with a proof by induction on m.

We need the following fact: With A and B as in the hypotheses, if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $M^R = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$, a fact easily proved by induction. Thus, we have $M - M^R = (a-d)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For M of the form $A^{x_1}B^{y_1} \cdots A^{x_m}B^{y_m}$ we must show that $a > d$. We show that $a > \max(b, c)$ and $d \leq \min(b, c)$. In the case where $m = 1$, $M = A^{x_1}B^{y_1} = \begin{pmatrix} x_1y_1 + 1 & x_1 \\ y_1 & 1 \end{pmatrix}$. If true for $m-1$ then for some p, q, r, s, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} x_my_m + 1 & x_m \\ y_m & 1 \end{pmatrix} = \begin{pmatrix} px_my_m + qy_m + p & px_m + q \\ rx_my_m + sy_m + r & rx_m + s \end{pmatrix}$. By inductive hypothesis, $p > r$, $q \geq s$. Since x_m and y_m are positive integers, it follows that $a > \max(b, c)$ and $d \leq \min(b, c)$, completing the induction. ∎

As a corollary we have the following.

LEMMA 3.2. *If M is not a symbolic palindrome, then the scalar c in Theorem 1.2 is nonzero with probability* 1.

Proof. By Lemma 3.1, c is not identically 0 so suppose that $\deg(c) = n$, as a polynomial in the entries of A and B. By Theorem 1.2 of [2], given any finite sets $S_1, S_2, \ldots, S_8$, each of size $n+1$ there is a point in the 8-fold product $S_1 \times S_2 \times \cdots \times S_8$ for which c is nonzero. This precludes the possibility that c could be zero on any set of positive measure. ∎

There is a link between traces and determinants for products of 2×2 matrices.

LEMMA 3.3. *Let A and B be 2×2 matrices and let $M = M_1M_2 \cdots M_n$, where each M_i is A or B. Then,*

$$Tr(M^2) - Tr(MM^R) = -\det(M - M^R).$$

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Proof. Since $Tr(M - M^R) = 0$, by (2.1),

$$(M - M^R)^2 = -\det(M - M^R)I. \tag{3.1}$$

Also,

$$(M - M^R)^2 = M^2 - MM^R - M^RM + (M^R)^2. \tag{3.2}$$

Since the reversal of $(M^R)^2$ is M^2 and the reversal of M^RM is MM^R, it follows from Theorem 1.1 that

$$Tr((M - M^R)^2) = 2Tr(M^2) - 2Tr(MM^R). \tag{3.3}$$

By (3.1),

$$Tr((M - M^R)^2) = -2\det(M - M^R). \tag{3.4}$$

The result now follows by combining (3.3) and (3.4). □

COROLLARY 3.4. *For 2×2 matrices A and B,*

$$Tr(ABAB) - Tr(A^2B^2) = -\det(AB - BA). \tag{3.5}$$

Combining Theorem 1.2, $M - M^R = c(AB - BA)$, with Lemma 3.3 and formula (3.5), we have

$$Tr(M^2) - Tr(MM^R) = -c^2 \det(AB - BA) = c^2(Tr(ABAB) - Tr(A^2B^2)),$$

where c is the constant in Theorem 1.2. Thus, we have the following corollary.

COROLLARY 3.5. *Let $M = M_1M_2 \cdots M_n$, where each M_i is A or B. If c in Theorem 1.2 is nonzero, then*

$$Tr(M^2) - Tr(MM^R) \quad \textit{and} \quad Tr(ABAB) - Tr(A^2B^2)$$

have the same sign.

The proof of Theorem 1.4 relies on the impact of complex eigenvalues on traces of matrix products.

LEMMA 3.6. *If either A or B has complex eigenvalues, then*

$$Tr(ABAB) \geq Tr(A^2B^2).$$

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ELA
http://math.technion.ac.il/iic/ela

Proof. We need only prove the result for A since interchanging A and B gives $\det(BA - AB) = \det(AB - BA)$. If A has complex eigenvalues $a + bi$, $a - bi$, then there is a real matrix P so that

$$P^{-1}AP = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

This is a special case of an exercise in [3, p. 106, Exercise 40]. It follows from the fact that if u is a nonzero eigenvector for $a + bi$, then $\overline{u}$ is an eigenvector for $a - bi$, and the matrix whose columns are $Re(u)$ and $Im(u)$ will work for P.

Letting $A' = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $B' = P^{-1}BP = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, we have

$$\det(AB - BA) = \det(A'B' - B'A').$$

By direct calculation,

$$\det(A'B' - B'A') = \det \begin{bmatrix} b(x+y) & b(z-w) \\ b(z-w) & -b(x+y) \end{bmatrix} = -b^2((x+y)^2 + (z-w)^2) \leq 0.$$

By Corollary 3.4, the result follows. $\square$

Next, we calculate how often a real matrix has real or complex eigenvalues.

LEMMA 3.7. *If* $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *has its entries selected independently from the uniform distribution on* $[-1, 1]$, *then the probability that* A *has real eigenvalues is* $\frac{49}{72}$.

Proof. The characteristic polynomial of A is: $\lambda^2 - (w + z)\lambda + wz - xy$. This polynomial has real zeros if and only if its discriminant, $(w - z)^2 - 4xy$, is nonnegative. Thus, the probability we seek is

$$\frac{1}{16}\left(\text{the volume of that portion of the hypercube with } (w - z)^2 + 4xy \geq 0\right).$$

By symmetry, we may replace z with $-z$ and y with $-y$. We seek the volume of the region $(w + z)^2 \geq 4xy$, $-1 \leq w, x, y, z \leq 1$. If $xy \leq 0$, the inequality is trivially satisfied. This contributes 8 to the volume. Again, by symmetry, we seek twice the

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Volume 27, pp. 716-734, October 2014

ELA
http://math.technion.ac.il/iic/ela

volume of the region where $x > 0$ and $y > 0$. Thus, our probability is

$$
\begin{aligned}
&\frac{1}{16}\left(8 + 2\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1}\int_{0}^{min(1,(w+z)^2/(4y))} dx\,dy\,dw\,dz\right)\\
&= \frac{1}{16}\left(8 + 2\int_{-1}^{1}\int_{-1}^{1}\int_{(w+z)^2/4}^{1} \frac{(w+z)^2}{4y}\,dy\,dw\,dz + 2\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{(w+z)^2/4} dy\,dw\,dz\right)\\
&= \frac{1}{2} - \frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\frac{(w+z)^2}{4}\ln\frac{(w+z)^2}{4}\,dw\,dz + \frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\frac{(w+z)^2}{4}\,dw\,dz\\
&= \frac{1}{2} + \frac{7}{72} + \frac{1}{12} = \frac{49}{72},
\end{aligned}
$$

as desired. □

The corresponding result where the entries of A are normal rather than uniform was less helpful. We state the result for completeness

LEMMA 3.8. *If* $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ *has independent normally distributed elements of mean 0 and variance 1, then the probability that A has real eigenvalues is* $\frac{1}{\sqrt{2}}$.

Proof. This is the simplest case in [4, 5], where the authors calculate the probability that a random $n \times n$ matrix has real eigenvalues. We give the following sketch as well. In what follows, and in Section 4, let

$$
\chi(a) = \begin{cases} 1, & \text{if } a \geq 0, \\ 0, & \text{if } a < 0. \end{cases} \tag{3.6}
$$

The probability we seek is given by the integral

$$
\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+w^2+y^2+z^2)}\chi((w-z)^2 + 4xy)\,dx\,dy\,dx\,dz.
$$

Replacing (x, y, w, z) with $\frac{1}{\sqrt{2}}(x + y, x - y, w + z, w - z)$, this becomes

$$
\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+w^2+y^2+z^2)}\chi(w^2 + x^2 - y^2)\,dx\,dy\,dx\,dz\\
&= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+w^2+y^2)}\chi(w^2 + x^2 - y^2)\,dx\,dy\,dx.
\end{aligned}
$$

Changing to spherical coordinates, our integral transforms to

$$
\frac{1}{(2\pi)^{3/2}}\int_{0}^{2\pi}\int_{\pi/4}^{3\pi/4}\int_{0}^{\infty} e^{-\frac{1}{2}\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{(2\pi)^{3/2}}\frac{\sqrt{2\pi}}{2}(2\pi)\sqrt{2} = \frac{1}{\sqrt{2}},
$$

as desired. □

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ELA
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To prove Theorem 1.4, let $M = M_1M_2 \cdots M_n$, where each M_i is A or B. By Corollary 3.3 and Theorem 1.2,

$$Tr(M^2) - Tr(MM^R) = -c^2 \det(AB - BA).$$

Since M is not a symbolic palindrome, $c \neq 0$ with probability 1 so the probability that $Tr(M^2) - Tr(MM^R) > 0$ is the same as for $Tr(ABAB) - Tr(A^2B^2)$. This difference is a polynomial in the entries of A and B (the negative of the polynomial given in (4.1)). As such, by an argument similar to that in Lemma 3.2, $Tr(ABAB) - Tr(A^2B^2) = 0$ with probability 0 and can only be negative if A and B both have real eigenvalues, which has probability $(\frac{49}{72})^2$. Thus, the probability that $Tr(M^2) - Tr(MM^R) > 0$ is at least $1 - (\frac{49}{72})^2 = \frac{2783}{5384} > \frac{1}{2}$.

4. A proof of Theorems 1.3. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $\det(AB - BA) < 0$ if and only if

$$S := bc(e-h)^2 - (bg+cf)(a-d)(e-h) + fg(a-d)^2 - (bg-df)^2 < 0. \tag{4.1}$$

Following [4], A is similar to a matrix of the form $A' = \begin{bmatrix} x & y \\ z & x \end{bmatrix}$, via an orthogonal matrix $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, with $0 \le \theta < \frac{\pi}{2}$. If A does not have the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, the matrix Q is unique. Using the change of variables $A' = Q^{-1}AQ$ and $B' = Q^{-1}BQ$, we have

$$\begin{aligned}
a &= x + \frac{y+z}{2}\sin 2\theta, \\
b &= \frac{y-z}{2} + \frac{y+z}{2}\cos 2\theta, \\
c &= -\frac{y-z}{2} + \frac{y+z}{2}\cos 2\theta, \\
d &= x - \frac{y+z}{2}\sin 2\theta, \\
e &= \frac{e'+h'}{2} + \frac{e'-h'}{2}\cos 2\theta + \frac{f'+g'}{2}\sin 2\theta, \\
f &= \frac{f'-g'}{2} + \frac{f'+g'}{2}\cos 2\theta - \frac{e'-h'}{2}\sin 2\theta, \\
g &= -\frac{f'-g'}{2} + \frac{f'+g'}{2}\cos 2\theta - \frac{e'-h'}{2}\sin 2\theta, \\
h &= \frac{e'+h'}{2} - \frac{e'-h'}{2}\cos 2\theta - \frac{f'+g'}{2}\sin 2\theta.
\end{aligned}$$

The 8×8 Jacobian matrix of this change of variables is block lower triangular, with

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 27, pp. 716-734, October 2014

ELA
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the upper left 4×4 block having determinant $2(y+z)$ and lower right 4×4 block having determinant 1. Thus, the change of variables factor is $2|y+z|$. With this change of coordinates, we have

$$a^2 + b^2 + c^2 + d^2 = 2x^2 + y^2 + z^2$$

and

$$e^2 + f^2 + g^2 + h^2 = e'^2 + f'^2 + g'^2 + h'^2.$$

If, by abuse of notation, we let $B' = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then S simplifies to $(yg - fz) - 2yz(e-h)^2$. Using the function χ defined in formula (3.6), our probability, p, is

$$\begin{aligned} p &= \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2)} \\ &\quad \times \chi((bg - df)^2 + (bg + cf)(a-d)(e-h) - bc(e-h)^2 - fg(a-d)^2)\, dA\, dB \\ &= \frac{1}{(2\pi)^4} \int_{0}^{\pi/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(2x^2+y^2+z^2+e^2+f^2+g^2+h^2)} \\ &\quad \times \chi((yg - fz)^2 - yz(e-h)^2)\, 2|y+z|\, dA'\, dB'. \end{aligned}$$

Replacing (e,h) by $\frac{1}{\sqrt{2}}(e+h, e-h)$, we may integrate out (x, e, θ) to obtain

$$\begin{aligned} &\frac{\sqrt{2}}{16\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2+z^2+f^2+g^2+h^2)} \\ &\qquad \times \chi((yg - fz)^2 - 2yzh^2)\, |y+z|\, dy\, dz\, df\, dg\, dh. \end{aligned}$$

Using two polar coordinate changes: $y = r\cos\alpha$, $z = r\sin\alpha$, $f = \rho\cos\beta$, $g = \rho\sin\beta$, the expression S in (4.1) simplifies:

$$\begin{aligned} S &= (r\rho\cos\alpha\sin\beta - r\rho\sin\alpha\cos\beta)^2 - 2r^2h^2\cos\alpha\sin\alpha \\ &= r^2(\rho^2\sin^2(\beta-\alpha) - h^2 sin2\alpha). \end{aligned}$$

The factor r^2 will not affect the sign of S so

$$\begin{aligned} p &= \frac{\sqrt{2}}{16\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(r^2+\rho^2+h^2)} \\ &\quad \times \chi(\rho^2\sin^2(\beta-\alpha) - h^2 sin2\alpha)\, r^2\rho|\sin\alpha + \cos\alpha|\, dh\, d\rho\, dr\, d\beta\, d\alpha \\ &= \frac{1}{16\pi^{3/2}} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\rho^2+h^2)} \\ &\quad \times \chi(\rho^2\sin^2(\beta-\alpha) - h^2 sin2\alpha)\, \rho|\sin\alpha + \cos\alpha|\, dh\, d\rho\, d\beta\, d\alpha. \end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 716-734, October 2014
ELA
http://math.technion.ac.il/iic/ela

If $\sin 2\alpha \leq 0$, then $\chi(S) = 1$. This occurs for $\frac{\pi}{2} < \alpha < \pi$, and $\frac{3\pi}{2} < \alpha < 2\pi$. These two intervals each contribute the same amount:

$$\begin{aligned}
&\frac{1}{16\pi^{3/2}} \int_{\pi/2}^{\pi} \int_0^{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\rho^2+h^2)} \rho |\sin\alpha + \cos\alpha| \, dh\, d\rho\, d\beta\, d\alpha \\
&= \frac{1}{16\pi^{3/2}} \sqrt{2\pi}(1)(2\pi) \int_{\pi/2}^{\pi} |\sin\alpha + \cos\alpha| \, d\alpha \\
&= \frac{\sqrt{2}}{8} 2(\sqrt{2}-1) = \frac{2-\sqrt{2}}{4}.
\end{aligned}$$

The total contribution from the region with $\sin 2\alpha < 0$ is twice this, giving

$$\begin{aligned}
p &= 2\left(\frac{2-\sqrt{2}}{4}\right) + \frac{1}{8\pi^{3/2}} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\rho^2+h^2)} \\
&\quad \times \chi(\rho^2 \sin^2(\beta-\alpha) - h^2 \sin 2\alpha)\, \rho(\sin\alpha + \cos\alpha)\, dh\, d\rho\, d\beta\, d\alpha \\
&= 1 - \frac{1}{\sqrt{2}} + \frac{1}{4\pi^{3/2}} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{\infty} \int_0^{\infty} e^{-\frac{1}{2}(\rho^2+h^2)} \\
&\quad \times \chi(\rho^2 \sin^2(\beta-\alpha) - h^2 \sin 2\alpha)\, \rho(\sin\alpha + \cos\alpha)\, dh\, d\rho\, d\beta\, d\alpha.
\end{aligned}$$

Now by periodicity, $\sin^2(\beta - \alpha)$ may be replaced by $\sin^2\beta$. If we replace h by ρh, this integral transforms:

$$\begin{aligned}
p &= 1 - \frac{1}{\sqrt{2}} + \frac{1}{4\pi^{3/2}} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{\infty} \int_0^{\infty} e^{-\frac{1}{2}\rho^2(1+h^2)} \\
&\quad \times\ \chi(\sin^2\beta - h^2 \sin 2\alpha)\, \rho^2(\sin\alpha + \cos\alpha)\, dh\, d\rho\, d\beta\, d\alpha.
\end{aligned}$$

Changing ρ to $\frac{\rho}{(1+h^2)^{1/2}}$, we have

$$\begin{aligned}
p &= 1 - \frac{1}{\sqrt{2}} + \frac{1}{4\pi^{3/2}} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{\infty} \int_0^{\infty} e^{-\frac{1}{2}\rho^2} \\
&\quad \times\ \chi(\sin^2\beta - h^2 \sin 2\alpha) \frac{1}{(1+h^2)^{3/2}} \rho^2(\sin\alpha + \cos\alpha)\, dh\, d\rho\, d\beta\, d\alpha \\
&= 1 - \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{8\pi} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{\infty} \chi(\sin^2\beta - h^2 \sin 2\alpha) \\
&\quad \times\ \frac{1}{(1+h^2)^{3/2}} (\sin\alpha + \cos\alpha)\, dh\, d\beta\, d\alpha.
\end{aligned}$$

The condition $\sin^2\beta - h^2 \sin 2\alpha > 0$ is equivalent to $h < u$, where $u = \frac{|\sin\beta|}{\sqrt{\sin 2\alpha}}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 716-734, October 2014

Thus,

$$\begin{aligned}
p &= 1 - \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{8\pi} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{u} \frac{1}{(1+h^2)^{3/2}} (\sin\alpha + \cos\alpha)\, dh\, d\beta\, d\alpha \\
&= 1 - \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{8\pi} \int_0^{\pi/2} \int_0^{2\pi} \frac{u}{(1+u^2)^{3/2}} (\sin\alpha + \cos\alpha)\, d\beta\, d\alpha \\
&= 1 - \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{8\pi} \int_0^{\pi/2} \int_0^{2\pi} \frac{|\sin\beta|}{\sqrt{\sin 2\alpha + \sin^2\beta}} (\sin\alpha + \cos\alpha)\, d\beta\, d\alpha \\
&= 1 - \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{2\pi} \int_0^{\pi/2} \arcsin\left(\frac{1}{\sqrt{1+\sin 2\alpha}}\right) (\sin\alpha + \cos\alpha)\, d\alpha.
\end{aligned}$$

The contributions to the integral from $\sin\alpha$ and $\cos\alpha$ are the same, so we must calculate

$$p = 1 - \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{\pi} \int_0^{\pi/2} \cos\alpha \arcsin\left(\frac{1}{\sqrt{1+\sin 2\alpha}}\right) d\alpha.$$

Integrating by parts gives

$$p = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{\pi} \int_0^{\pi/2} \frac{\sin\alpha \cos 2\alpha}{(1+\sin 2\alpha)\sqrt{\sin 2\alpha}}\, d\alpha.$$

Finally, substitution $\alpha = \arctan x^2$ rationalizes the expression giving

$$\begin{aligned}
p &= 1 + \frac{2}{\pi} \int_0^{\infty} \frac{x^2(1-x^2)}{(1+x^2)(1+x^4)}\, dx \\
&= 1 + \frac{2}{\pi} \left(\frac{\pi\sqrt{2}}{4} - \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}},
\end{aligned}$$

as desired.

5. Comments. In the case where the elements of A and B are selected from the uniform distribution, we have proven that when M is not a symbolic palindrome,

$$Tr(M^2) > Tr(MM^R)$$

with probability at least

$$1 - \left(\frac{49}{72}\right)^2 \approx .537,$$

but numerical evidence suggests that the real probability is about .72. For example, see Table 2 in [9, p. 20], reproduced below. This table used Mathematica$^{\text{TM}}$ to

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 27, pp. 716-734, October 2014

ELA
http://math.technion.ac.il/iic/ela

construct 1,000,000 matrices A and B, with entries selected uniformly at random from the interval $[-1, 1]$. Matrices were categorized by whether they had real or complex eigenvalues, and whether the trace of $ABAB$ was larger than the trace of A^2B^2. In Table 5.1, an r in the first column means that a matrix has real eigenvalues, a c signifies complex eigenvalues.

A	B	Tr(ABAB) is larger	frequency
r	r	no	279,340
r	r	yes	183,701
r	c	no	0
r	c	yes	217,715
c	r	no	0
c	r	yes	217,542
c	c	no	0
c	c	yes	101,702

TABLE 5.1

It was this table that inspired Lemma 3.6. As one can see from the table, $ABAB$ had the larger trace in 720,660 cases. The cases where either A or B had complex eigenvalues accounted for 536,959 of these cases, in good agreement with the estimate .537. What the author can not account for is the likelihood that $ABAB$ has the larger trace when both A and B have real eigenvalues.

One may ask what changes in Table 5.1 when entries are selected from a normal distribution rather than a uniform distribution. The results from this calculation are presented in Table 5.2. Calculations for this and subsequent tables were performed in Maple$^{\text{TM}}$.

A	B	Tr(ABAB) is larger	frequency
r	r	no	292,544
r	r	yes	207,597
r	c	no	0
r	c	yes	207,510
c	r	no	0
c	r	yes	206,602
c	c	no	0
c	c	yes	85,747

TABLE 5.2

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 716-734, October 2014

ELA
http://math.technion.ac.il/iic/ela

Again, the results in this table are consistent with theory: The sum of the yes's is 707,456 in good agreement with $\frac{1}{\sqrt{2}}$. It was this calculation that led to Theorem 1.3. Also, A and B should both have real eigenvalues with probability $\frac{1}{2}$; the probability that both A and B have complex eigenvalues should be $\left(1-\frac{1}{\sqrt{2}}\right)^2 \approx .0858$, and the probability that A and B both have real eigenvalues with $Tr(ABAB) > Tr(A^2B^2)$ should be $\frac{1}{\sqrt{2}} - \frac{1}{2} \approx .2071$. These compare with 500,141, 85,747, and 207,597, respectively.

Theorems 1.3 and 1.4 apply to matrices with an even number of A's and an even number of B's, and even here, to a subset of possible pairings of necklaces. With regard to the first comment, given two necklaces represented by M_1 and M_2 where the number of A's or the number of B's is odd, $Tr(M_1) > Tr(M_2)$ with probability $\frac{1}{2}$. The reason for this is the involution $A \mapsto -A$, which reverses the inequality when there are on odd number of A's. Thus, if we have three A's and two B's, there are two necklaces represented by A^3B^2 and A^2BAB, and each is equally likely to be larger than the other.

As the number of A's and B's grows, so does the number of necklaces. If, for example, there are two A's and four B's, then there are three necklaces represented by A^2B^4, $ABAB^3$ and AB^2AB^2. Here, Theorems 1.3 and 1.4 only apply to the comparison of the first and last. That is, $Tr(AB^2AB^2) > Tr(A^2B^4)$ with probability $\frac{1}{\sqrt{2}}$ when entries are selected from the normal distribution via Theorem 1.3 with $M = AB^2$.

Trace combination	Number of cases
$Tr(AB^2AB^2) > Tr(A^2B^4)$	642,122
$Tr(AB^2AB^2) > Tr(ABAB^3)$	706,206
$Tr(ABAB^3) > Tr(A^2B^4)$	582,660

TABLE 5.3

Estimates for the relative probabilities of the other necklace comparisons can be made by again picking $1,000,000$ pairs A and B at random. Using the normal distribution the results are given in Table 5.3. The author does not know the probabilities for the other trace pairings, though their values are 8-dimensional integrals similar to the one used in the proof of Theorem 1.3.

If one wished for the probabilities for the six orderings of the traces, then with $M_1 = AB^2AB^2$, $M_2 = ABAB^3$ and $M_3 = A^2B^4$, and $1,000,000$ simulations the author obtained Table 5.4.

In this table, the author has no explanation for the frequencies except for the 0

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 716-734, October 2014

ELA
http://math.technion.ac.il/iic/ela

Trace combination	Number of cases
$Tr(M_1) > Tr(M_2) > Tr(M_3)$	300,092
$Tr(M_1) > Tr(M_3) > Tr(M_2)$	123,546
$Tr(M_2) > Tr(M_1) > Tr(M_3)$	282,568
$Tr(M_2) > Tr(M_3) > Tr(M_1)$	0
$Tr(M_3) > Tr(M_1) > Tr(M_2)$	218,484
$Tr(M_3) > Tr(M_2) > Tr(M_1)$	75,310

Table 5.4

in the forth row. In this case, two applications of formula (2.1) and the result that A^2B, ABA and BA^2 all have the same trace gives $Tr(ABAB^3) > Tr(A^2B^4)$ when $(Tr(B)^2 - \det(B))(Tr(ABAB) - Tr(A^2B^2)) > 0$ and $Tr(A^2B^4) > Tr(AB^2AB^2)$ when $Tr(B)^2(Tr(A^2B^2) - Tr(ABAB)) > 0$. For this second result to be true, Lemma 3.6 requires that A and B have real eigenvalues. If these are λ_1 and λ_2 then $Tr(B)^2 - \det(B) = (\lambda_1 + \lambda_2)^2 - \lambda_1\lambda_2 \geq 0$ meaning that we need $Tr(ABAB)$ to be both larger and smaller than $Tr(A^2B^2)$, giving the 0 count.

The probabilities of the various orderings of the traces is more complicated than the set of all comparisons of two traces as evidenced by the case of three A's and three B's. In this case, there are three necklaces, representable by $M_1 = ABABAB$, $M_2 = A^2BAB^2$ and $M_3 = A^3B^3$. Since there is an odd number of A's, in any pairing, one necklace has probability $\frac{1}{2}$ of having a larger trace than another necklace. However, when we order all three necklaces we have the following table.

Trace combination	Number of cases
$Tr(M_1) > Tr(M_2) > Tr(M_3)$	324,418
$Tr(M_1) > Tr(M_3) > Tr(M_2)$	115,235
$Tr(M_2) > Tr(M_1) > Tr(M_3)$	60,980
$Tr(M_2) > Tr(M_3) > Tr(M_1)$	114,729
$Tr(M_3) > Tr(M_1) > Tr(M_2)$	61,158
$Tr(M_3) > Tr(M_2) > Tr(M_1)$	323,480

Table 5.5

In this table, the involution $A \mapsto -A$, again reverses all inequalities. This means an ordering and its reverse have the same probability. Thus, the probability that, say, $Tr(M_2) > Tr(M_3) > Tr(M_1)$ is the same as the probability that $Tr(M_1) > Tr(M_3) > Tr(M_2)$. Thus, these two frequencies in the table above, $114,729$ and $115,235$ are nearly equal.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 716-734, October 2014

ELA
http://math.technion.ac.il/iic/ela

One might also ask what happens with larger matrices. The following table contains $1,000,000$ trials each, asking when $ABAB$ had a larger trace than A^2B^2, where A and B are $n \times n$ matrices for various values of n. As one might guess, for larger matrices, the type of distribution appears to matter less. The results show that in general, $ABAB$ appears to have the larger trace about 70% of the time, with the probability appearing to slowly rise for $n \geq 5$.

n	normal variables	uniform variables
2	707,456	720,660
3	703,004	703,320
4	701,885	700,959
5	702,375	700,259
10	706,124	704,561
20	709,715	710,189
50	714,473	714,627
100	716,805	717,009

Table 5.6

Since, for 2×2 matrices A and B, $Tr(ABAB) - Tr(A^2B^2) = -\det(AB - BA)$, and $Tr(AB - BA) = 0$, one might ask if there is a relationship to matrices of trace 0. For 2×2 matrices, $Tr(N) = 0$ if and only if $N = AB - BA$ for some matrices A and B [8, p. 21]. However, the connection between N and A and B is not one-to-one. If $N = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, then $\det(N) = -a^2 - bc$ is negative whenever bc is positive, which is to say, over half the time. In fact, one may calculate the probability that $\det(N) < 0$.

Theorem 5.1. *Let* $N = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$.

(a) *If the entries of* N *are selected independently from the uniform distribution on* $[-1, 1]$*, then* $\det(N) < 0$ *with probability* $\frac{7}{9}$.

(b) *If the entries of* N *are independent normally distributed elements of mean* 0 *and variance* 1*, then* $\det(N) < 0$ *with probability* $\frac{1}{2} + \frac{1}{2\pi}K\left(\frac{1}{2}\right)$*, where* $K(k)$ *is the complete elliptic integral of the first kind.*

Proof. For (a), the probability we seek is

$$\frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \chi(a^2 + bc)\, da\, db\, dc.$$

If we replace b by $-b$, then we seek the region with $a^2 > bc$. Certainly this is the

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 27, pp. 716-734, October 2014

ELA
http://math.technion.ac.il/iic/ela

case if $bc < 0$, which contributes $\frac{1}{2}$ to the probability. Thus, the probability we seek is

$$\frac{1}{2} + \frac{1}{2}\int_0^1 \int_0^1 \int_0^1 \chi(a^2 - bc)\, da\, db\, dc.$$

Now $a^2 > bc$ if $a > \sqrt{bc}$, so our probability is

$$\begin{aligned}\frac{1}{2} + \frac{1}{2}\int_0^1 \int_0^1 \int_{\sqrt{bc}}^1 da\, db\, dc \\ &= \frac{1}{2} + \frac{1}{2}\int_0^1 \int_0^1 (1 - b^{1/2}c^{1/2})\, db\, dc \\ &= \frac{1}{2} + \frac{1}{2}\int_0^1 \left(1 - \frac{2}{3}c^{1/2}\right) dc \\ &= \frac{7}{9},\end{aligned}$$

as desired.

For part (b), a general reference for the complete elliptic function is [1]. The important value is

$$\begin{aligned}K\left(\frac{1}{2}\right) &= \int_0^{\pi/2} \frac{1}{\sqrt{1 - \frac{1}{4}\sin^2\theta}}\, d\theta \\ &= 2\int_0^1 \frac{1}{\sqrt{1 + x^2 + x^4}}\, dx,\end{aligned}$$

by the change of variables $\theta = 2\arctan x$. Replacing x by $\frac{1}{x}$ converts the interval of integration from $[0, 1]$ to $[1, \infty)$, so

$$K\left(\frac{1}{2}\right) = \int_0^{\infty} \frac{1}{\sqrt{1 + x^2 + x^4}}\, dx. \tag{5.1}$$

We mention that there is a misprint in [6, sec. 3.165 integral 2], which evaluates the given integral as $\frac{1}{2}K(\frac{1}{2})$ instead of $K(\frac{1}{2})$. The probability we seek is

$$\frac{1}{2} + \frac{4}{(2\pi)^{3/2}}\int_0^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-\frac{1}{2}(a^2+b^2+c^2)}\chi(a^2 - bc)\, db\, dc\, da.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 716-734, October 2014

ELA
http://math.technion.ac.il/iic/ela

Replacing b by ab and c by ac, we have

$$\begin{aligned}
\text{probability} &= \frac{1}{2} + \frac{4}{(2\pi)^{3/2}} \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{1}{2}a^2(1+b^2+c^2)} \chi(1-bc) a^2 \, db \, dc \, da \\
&= \frac{1}{2} + \frac{4}{(2\pi)^{3/2}} \int_0^\infty \int_0^\infty \int_0^\infty a^2 e^{-\frac{1}{2}a^2} \chi(1-bc) \frac{1}{(1+b^2+c^2)^{3/2}} \, db \, dc \, da \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \int_0^\infty \chi(1-bc) \frac{1}{(1+b^2+c^2)^{3/2}} \, db \, dc \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \int_0^{1/c} \frac{1}{(1+b^2+c^2)^{3/2}} \, db \, dc \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{1}{(1+c^2)\sqrt{1+c^2+c^4}} \, dc \\
&= \frac{1}{2} + \frac{1}{2\pi} K\left(\frac{1}{2}\right).
\end{aligned}$$

This last step follows from formula (5.1) since

$$\int_0^\infty \frac{1}{(1+x^2)\sqrt{1+x^2+x^4}} \, dx = \frac{1}{2} \int_0^\infty \frac{1}{\sqrt{1+x^2+x^4}} \, dx,$$

via the change of variables $x \to \frac{1}{x}$. $\square$

These results again compare well to simulated results. In 1,000,000 trials, where A is a random 2×2 matrix with trace 0, the determinant was negative in 777,787 trials when the elements were selected from the uniform distribution, and in 767,770 when the elements were selected from a normal distribution. Here, $\frac{1}{2} + \frac{1}{2\pi}K(\frac{1}{2}) \approx .7683$.

For 2×2 matrices, $ABAB$ has a larger trace than A^2B^2 when $\det(AB-BA) < 0$. This connection disappears for $n \times n$ matrices when $n \geq 3$. Out of curiosity, we ran $1,000,000$ trials on when $\det(AB - BA) < 0$ for larger matrices. The following table gives the results.

size of A	normal variables	uniform variables
2×2	707,456	720,660
3×3	500,485	499,576
4×4	453,292	453,022
6×6	511,588	510,130
8×8	496,174	497,505
10×10	501,096	500,229

Table 5.7

This table suggests many possible conjectures. Of course when n is odd, the probability that $\det(AB - BA) < 0$ is $\frac{1}{2}$, since interchanging A and B changes the

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela

sign of the determinant in this case. Based on Table 5.7, the following are at least plausible, however. Let p_n be the probability that $\det(AB - BA) < 0$, where A and B are $n \times n$ matrices with independent standard normal entries. Let q_n be the probability that $\det(AB - BA) < 0$, where A and B are $n \times n$ matrices with entries independent entries selected uniformly on $[-1, 1]$.

CONJECTURE 1. $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \dfrac{1}{2}$.

CONJECTURE 2. *If n is even, then $p_n \neq \frac{1}{2}$ and $q_n \neq \frac{1}{2}$.*

CONJECTURE 3. *If $n \equiv 2 \pmod 4$, then $p_n > \frac{1}{2}$ and $q_n > \frac{1}{2}$.*

CONJECTURE 4. *If $n \equiv 0 \pmod 4$, then $p_n < \frac{1}{2}$ and $q_n < \frac{1}{2}$.*

It is unclear from Table 5.4 how p_n and q_n compare to each other in magnitude.

REFERENCES

[1] M. Abramowitz and I.A. Stegun (editors). Elliptic Integrals, Ch. 17 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover Publications, New York, 587–607, 1972.

[2] N. Alon. Combinatorial nullstellensatz. *Combinatorics, Probability and Computing*, 8:7–29, 1999.

[3] E. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. McGraw Hill, New York, 1955.

[4] A. Edelman. The probability that a random real gaussian matrix has real eigenvalues, related distributions, and the circular law. *J. Multivariate Anal.* 60:203–232, 1997.

[5] A. Edelman, E. Kostlan, and M.Shub. How many eigenvalues of a random matrix are real? *J. Amer. Math. Soc.*, 7:247–267, 1994.

[6] I.S. Gradshteyn and I.M. Ryzhik. *Tables of Integrals, Series, and Products*, sixth edition. Academic Press, San Diego, 2000.

[7] J. Greene and K. Niedzielski. Periodic solutions to some difference equations over the integers. *Difference Equations and Their Applications*, 16:321–346, 2010.

[8] K. Hoffman and R. Kunze. *Linear Algebra*. Prentice Hall, Englewood Cliffs, 1971.

[9] Z. Huang. *Traces of Matrix Products*. Master Thesis, University of Minnesota Duluth, 2009.

[10] C. Meyer. *Matrix Analysis and Applied Linear Algebra*. SIAM, Philadelphia, 2000.

[11] K. Niedzielski. *Periodic Solutions to Difference Equations*. Master Project, University of Minnesota Duluth, 2008.

[12] B. Noble and J. Daniel. *Applied Linear Algebra*. Prentice Hall, Englewood, 1988.