Elementary proofs of various facts about 3–cores

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Abstract

Using elementary means, we derive an explicit formula for $a_3(n)$, the number of 3–core partitions of n, in terms of the prime factorization of $3n+1$. Based on this result, we are able to prove several infinite families of arithmetic results involving $a_3(n)$, one of which specializes to the recent result of Baruah and Berndt which states that, for all $n \geq 0$, $a_3(4n+1) = a_3(n)$.

Given the integer partition π of an integer n, we say that π is a t**–core** if it has no hook numbers divisible by t. See James and Kerber [7] for a fuller discussion of the definition of t–cores. We denote the number of t–core partitions of n by the function $a_t(n)$.

The goal of this brief note is to prove several arithmetic properties for the function $a_3(n)$, the number of 3–cores of n, via elementary means. (Indeed, everything we prove below follows from nothing more than Jacobi's Triple Product Identity and some elementary generating function manipulations.)

It should be noted that such results have appeared in recent publications, but the techniques used therein have been much deeper than necessary. For example, in the work below, we will prove the following: For all $n \geq 0$,

$$a_3(n) = d_{1,3}(3n+1) - d_{2,3}(3n+1)$$

where $d_{r,3}(n)$ is the number of divisors of n congruent to $r \pmod 3$. This result appeared in Granville and Ono [4]. However, there the authors use the theory of modular forms to prove the result, while we do not.

A more recent example of a 3–cores identity appears as Theorem 4.1 of Baruah and Berndt [1] where the authors note that, for all $n \geq 0$, $a_3(4n+1) = a_3(n)$. We prove this result below as a special case of a much larger theorem.

With the above said, we now proceed to prove our results. We begin by proving various representations of the generating function of $a_3(n)$, some of which stem from work in [3, 5, 6].

Theorem 1.

$$\begin{aligned}
\sum_{n\geq 0} a_3(n)q^n &= \prod_{n\geq 1} \frac{(1-q^{3n})^3}{(1-q^n)} \\
&= \frac{1}{3} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} \\
&= \sum_{n\geq 1} \left(\frac{q^{n-1}}{1-q^{3n-2}} - \frac{q^{2n-1}}{1-q^{3n-1}} \right)
\end{aligned}$$

The proof of this theorem follows from three lemmas below which involve the functions $a(q)$ and $c(q)$ which were introduced in [2]. Our proofs of these three lemmas are drawn from [5] and [6]. We begin by defining the functions $a(q)$ and $c(q)$ by

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \quad \text{and}$$

$$\begin{aligned}
c(q) &= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} \\
&= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2-m-n}.
\end{aligned}$$

We then have the following:

Lemma 2.

$$a(q) = 1 + 6\sum_{n\geq 1} \left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}} \right) \tag{1}$$

and

$$c(q) = 3\prod_{n\geq 1}\frac{(1-q^{3n})^3}{(1-q^n)}. \tag{2}$$

Proof. We have

$$\begin{aligned} a(q^2) &= \sum_{m,n=-\infty}^{\infty} q^{2m^2+2mn+2n^2} \\ &= \sum_{m+n+p=0} q^{m^2+n^2+p^2} \\ &= 1+6\sum_{n\geq 1}\left(\frac{q^{6n-4}}{1-q^{6n-4}} - \frac{q^{6n-2}}{1-q^{6n-2}}\right) \quad \text{as proved in [6]} \end{aligned}$$

and

$$\begin{aligned} qc(q^2) &= \sum_{m,n=-\infty}^{\infty} q^{2m^2+2mn+2n^2-2m-2n+1} \\ &= \sum_{m+n+p=1} q^{m^2+n^2+p^2} \\ &= 3q\prod_{n\geq 1}\frac{(1-q^{6n})^3}{(1-q^{2n})} \quad \text{again as proved in [6].} \end{aligned}$$

The results follow. □

Lemma 3.

$$a(q) - a(q^3) = 2qc(q^3). \tag{3}$$

Proof. We have

$$\begin{aligned} a(q) &= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \\ &= \sum_{m-n\equiv 0 \pmod 3} q^{m^2+mn+n^2} + \sum_{m-n\equiv 1 \pmod 3} q^{m^2+mn+n^2} + \sum_{m-n\equiv -1 \pmod 3} q^{m^2+mn+n^2}. \end{aligned}$$

In the first sum above, set $m+2n = 3r$ and $m-n = 3s$. In the second sum, set $m+2n = 3r+1$ and $m-n = 3s+1$. In the third sum, set $m+2n = -3s-1$ and $m-n = -3r-1$. Then we have

$$\begin{aligned}
a(q) &= \sum_{r,s=-\infty}^{\infty} q^{(r+2s)^2+(r+2s)(r-s)+(r-s)^2} \\
&\qquad + \sum_{r,s=-\infty}^{\infty} q^{(r+2s+1)^2+(r+2s+1)(r-s)+(r-s)^2} \\
&\qquad + \sum_{r,s=-\infty}^{\infty} q^{(2r+s+1)^2-(2r+s+1)(r-s)+(r-s)^2} \\
&= \sum_{r,s=-\infty}^{\infty} q^{3r^2+3rs+3s^2} + 2q \sum_{r,s=-\infty}^{\infty} q^{3r^2+3rs+3s^2+3r+3s} \\
&= a(q^3) + 2qc(q^3).
\end{aligned}$$

□

Lemma 4.

$$c(q) = 3\sum_{n\geq 1}\left(\frac{q^{n-1}}{1-q^{3n-2}} - \frac{q^{2n-1}}{1-q^{3n-1}}\right). \tag{4}$$

Proof. From Lemmas 2 and 3, we know

$$\begin{aligned}
a(q^3) + 2qc(q^3) &= a(q) \\
&= 1 + 6\sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}}\right) \\
&= 1 + 6\sum_{n\geq 1}\left(\frac{q^{3n-2}(1+q^{3n-2}+q^{6n-4})}{1-q^{9n-6}} - \frac{q^{3n-1}(1+q^{3n-1}+q^{6n-2})}{1-q^{9n-3}}\right) \\
&= 1 + 6\sum_{n\geq 1}\left(\frac{q^{9n-6}}{1-q^{9n-6}} - \frac{q^{9n-3}}{1-q^{9n-3}}\right) \\
&\qquad +6\sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{9n-6}} - \frac{q^{6n-2}}{1-q^{9n-3}}\right) \\
&\qquad +6\sum_{n\geq 1}\left(\frac{q^{6n-4}}{1-q^{9n-6}} - \frac{q^{3n-1}}{1-q^{9n-3}}\right) \\
&= a(q^3) + 6\sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{9n-6}} - \frac{q^{6n-2}}{1-q^{9n-3}}\right)
\end{aligned}$$

since

$$\begin{aligned}
6\sum_{n\geq 1}\left(\frac{q^{6n-4}}{1-q^{9n-6}} - \frac{q^{3n-1}}{1-q^{9n-3}}\right) &= 6\sum_{n\geq 0}\left(\frac{q^{6n+2}}{1-q^{9n+3}} - \frac{q^{3n+2}}{1-q^{9n+6}}\right) \\
&= 6\sum_{m,n\geq 0}\left(q^{9mn+3m+6n+2} - q^{9mn+6m+3n+2}\right) \\
&= 0.
\end{aligned}$$

It follows that

$$2qc(q^3) = 6\sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{9n-6}} - \frac{q^{6n-2}}{1-q^{9n-3}}\right).$$

Dividing by q and replacing q^3 by q yields the result. □

With these elementary lemmas in hand, we can prove the following pivotal theorem:

Theorem 5. *For all* $n \geq 0$,

$$a_3(n) = d_{1,3}(3n+1) - d_{2,3}(3n+1)$$

where $d_{r,3}(n)$ *is the number of divisors of* n *congruent to* $r \pmod 3$.

Proof. We have

$$\begin{aligned}\sum_{n\geq 0} a_3(n)q^{3n+1} &= \sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{9n-6}} - \frac{q^{6n-2}}{1-q^{9n-3}}\right) \quad \text{from Lemma 2}\\ &= \sum_{n\geq 0}\left(\frac{q^{3n+1}}{1-q^{9n+3}} - \frac{q^{6n+4}}{1-q^{9n+6}}\right)\\ &= \sum_{m,n\geq 0}\left(q^{(3m+1)(3n+1)} - q^{(3m+2)(3n+2)}\right).\end{aligned}$$

It follows that

$$a_3(n) = \sum_{\substack{d\,|\,3n+1\\ d\equiv 1 \pmod 3}} 1 - \sum_{\substack{d\,|\,3n+1\\ d\equiv 2 \pmod 3}} 1 \tag{5}$$

This is the desired result. □

Theorem 5 appears in Granville and Ono [4], but is proved there using heavy mathematical machinery.

Using a standard counting argument and (5), we can give an explicit formula for $a_3(n)$ in terms of the prime factorization of $3n+1$.

Theorem 6. *Let* $3n+1 = \prod_{p_i\equiv 1 \pmod 3} p_i^{\alpha_i} \prod_{q_j\equiv 2 \pmod 3} q_j^{\beta_j}$ *with each* $\alpha_i, \beta_j \geq 0$ *be the prime factorization of* $3n+1$. *Then*

$$a_3(n) = \begin{cases}\prod(\alpha_i+1) & \text{if all } \beta_j \text{ are even,}\\ 0 & \text{otherwise.}\end{cases}$$

Thanks to Theorem 6, we have the following corollaries.

Corollary 7. *Let $p \equiv 1 \pmod 3$ be prime and let k be a positive integer. Then, for all $n \geq 0$ with $p^\alpha \,||\, 3n+1$,*

$$a_3\left(p^k n + \left(\frac{p^k-1}{3}\right)\right) = \frac{\alpha+k+1}{\alpha+1} a_3(n).$$

Corollary 8. *Let $p \equiv 2 \pmod 3$ be prime and let k be a positive* **even** *integer. Then, for all $n \geq 0$,*

$$a_3\left(p^k n + \left(\frac{p^k-1}{3}\right)\right) = a_3(n).$$

Corollary 9. *Let $p \equiv 2 \pmod 3$ be prime. For each $1 \leq k \leq p-1$, let*

$$r \equiv \frac{p^2-1}{3} + kp \pmod{p^2}.$$

Then, for all $n \geq 0$,

$$a_3\left(p^2 n + r\right) = 0.$$

The case $p = 2, k = 2$ of Corollary 8 appeared very recently in a work of Baruah and Berndt [1, Theorem 4.1]. As noted in our introduction, the proof techniques used in that paper are relatively deep, relying on manipulations of a modular equation of Ramanujan. It should also be noted that the methods used in [1] do not appear to be easily generalizable in order to prove the infinite families of results we obtained above.

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