FURTHER RESULTS FOR PARTITIONS INTO FOUR SQUARES OF EQUAL PARITY

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Abstract

We prove several results dealing with various counting functions for partitions of an integer into four squares of equal parity. Some are easy consequences of earlier work, but two are new and surprising. That is, we show that the number of partitions of $72n+60$ into four odd squares (distinct or not) is even.

1. Introduction

In this article, we consider a number of counting functions related to partitions of the positive integer n into four squares. We will consider relationships involving the following functions:

$p_{4o}(n)$, the number of partitions of n into four odd squares

$p^d_{4o}(n)$, the number of partitions of n into four distinct odd squares

$p_{4e}(n)$, the number of partitions of n into four even squares

$p^+_{4e}(n)$, the number of partitions of n into four positive even squares

$p^d_{4e}(n)$, the number of partitions of n into four distinct even squares

$p^{d+}_{4e}(n)$, the number of partitions of n into four positive distinct even squares

In particular, we will prove the following theorems:

Theorem 1. For all $n \geq 0$,

$$p_{4o}(8n+4) = p_{4e}(8n+4) + p^+_{4e}(8n+4) \tag{1}$$

and

$$p^d_{4o}(8n+4) = p^d_{4e}(8n+4) + p^{d+}_{4e}(8n+4). \tag{2}$$

Theorem 2. For all $n \geq 0$,

$$p_{4o}(32n+28) = 2p_{4e}(32n+28) \tag{3}$$

and

$$p^d_{4o}(32n+28) = 2p^d_{4e}(32n+28). \tag{4}$$

Theorem 3. For all $n \geq 0$,

$$p_{4o}(72n+60) \equiv 0 \pmod 2 \tag{5}$$

and

$$p^d_{4o}(72n+60) \equiv 0 \pmod 2. \tag{6}$$

Theorems 1 and 2 are consequences of two straightforward lemmas and some of our earlier results on partitions into four squares [6]. Also, it should be noted that (2) was previously proved by Hirschhorn [3].

2. Proofs of Theorems 1 and 2

We begin with the following straightforward observations.

Lemma 1. For all $n \geq 0$,

$$\begin{aligned} p_{4e}(4n) &= p_{4\square}(n) \\ \text{and} \quad p^{+}_{4e}(4n) &= p^{+}_{4\square}(n). \end{aligned}$$

Lemma 2. For all $n \geq 0$,

$$p_{4\square}(8n+4) = p_{4o}(8n+4) + p_{4e}(8n+4).$$

Proof. If $8n+4$ is written as the sum of four squares, then the squares are all odd or all even. This is because any odd square is congruent to 1 modulo 4. ∎

Lemma 3. For all $n \geq 0$,

$$\begin{aligned} p_{4\square}(8n+4) &= 2p_{4\square}(2n+1) + p^{+}_{4\square}(2n+1) \\ p^{d}_{4\square}(8n+4) &= 2p^{d}_{4\square}(2n+1) + p^{d+}_{4\square}(2n+1) \\ p_{4\square}(32n+28) &= 3p_{4\square}(8n+7) \\ \text{and} \quad p^{d}_{4\square}(32n+28) &= 3p^{d}_{4\square}(8n+7) \end{aligned}$$

Proof. These four equalities are proven in [6, Theorem 3]. The third of these equalities was also proven even earlier in Hirschhorn and Sellers [5].∎

With Lemmas 1, 2, and 3 in hand, we are now in a position to prove both Theorems 1 and 2.

Proof of Theorem 1: For all $n \geq 0$,

$$\begin{aligned} p_{4o}(8n+4) &= p_{4\square}(8n+4) - p_{4e}(8n+4) \quad \text{by Lemma 2} \\ &= 2p_{4\square}(2n+1) + p^{+}_{4\square}(2n+1) - p_{4\square}(2n+1) \\ &= p_{4\square}(2n+1) + p^{+}_{4\square}(2n+1) \\ &= p_{4e}(8n+4) + p^{+}_{4e}(8n+4) \quad \text{by Lemma 1} \end{aligned}$$

which is (1). The same line of reasoning can be used to prove (2). ■

Proof of Theorem 2: For all $n \geq 0$,

$$\begin{aligned} p_{4o}(32n+28) &= p_{4\square}(32n+28) - p_{4e}(32n+28) \quad \text{by Lemma 2} \\ &= 3p_{4\square}(8n+7) - p_{4\square}(8n+7) \quad \text{by Lemmas 3 and 1} \\ &= 2p_{4\square}(8n+7) \\ &= 2p_{4e}(32n+28) \quad \text{by Lemma 1} \end{aligned}$$

which is (3). Equality (4) can be obtained similarly. ■

3. A Generating Function Proof of Theorem 3

We now move to a proof of Theorem 3. Following Ramanujan [2], we define

$$\psi(q) = \sum_{n\geq 0} q^{(n^2+n)/2}.$$

We first prove a number of generating function identities for partitions into four odd squares.

Lemma 4.

$$\begin{aligned} \sum_{n\geq 0} p_{4o}(n)q^n &= \frac{1}{24}\left(q^4\psi(q^8)^4 + 6q^4\psi(q^8)^2\psi(q^{16}) + 3q^4\psi(q^{16})^2 \right. \\ &\quad \left. + 8q^4\psi(q^8)\psi(q^{24}) + 6q^4\psi(q^{32})\right), \end{aligned} \tag{7}$$

$$\begin{aligned} \sum_{n\geq 0} p^d_{4o}(n)q^n &= \frac{1}{24}\left(q^4\psi(q^8)^4 - 6q^4\psi(q^8)^2\psi(q^{16}) + 3q^4\psi(q^{16})^2 \right. \\ &\quad \left. + 8q^4\psi(q^8)\psi(q^{24}) - 6q^4\psi(q^{32})\right), \end{aligned} \tag{8}$$

$$\begin{aligned} \sum_{n\geq 0} p_{4o}(8n+4)q^n &= \frac{1}{24}\left(\psi(q)^4 + 6\psi(q)^2\psi(q^2) + 3\psi(q^2)^2 \right. \\ &\quad \left. + 8\psi(q)\psi(q^3) + 6\psi(q^4)\right), \end{aligned} \tag{9}$$

and

$$\sum_{n\geq 0} p^d_{4o}(8n+4)q^n = \frac{1}{24}\left(\psi(q)^4 - 6\psi(q)^2\psi(q^2) + 3\psi(q^2)^2 + 8\psi(q)\psi(q^3) - 6\psi(q^4)\right). \tag{10}$$

Proof: Here we use the methods of [6]. As in [6], we denote the generating function for the number of partitions of n of the form $n = a^2+b^2+c^2+d^2$ by $F(a^2+b^2+c^2+d^2, q)$, and use similar notation to define related generating functions. (So, for example, $F(a^2+a^2+b^2, q)$ is the generating function for the number of partitions of n into the sum of twice one square plus another (different) square.) Remembering that all squares are odd in the context of this lemma, we have

$$F(a^2, q) = \sum_{a \text{ odd, positive}} q^{a^2} = q\psi(q^8).$$

Then a number of related results follow:

$$\begin{aligned}
F(a^2+a^2, q) &= q^2\psi(q^{16}),\\
F(a^2+a^2+a^2, q) &= q^3\psi(q^{24}),\\
F(a^2+a^2+a^2+a^2, q) &= q^4\psi(q^{32}),\\
F(a^2+b^2, q) &= \frac{1}{2}\left(F(a^2,q)^2 - F(a^2+a^2,q)\right)\\
&= \frac{1}{2}q^2\left(\psi(q^8)^2 - \psi(q^{16})\right),\\
F(a^2+a^2+b^2+b^2, q) &= \frac{1}{2}q^4\left(\psi(q^{16})^2 - \psi(q^{32})\right),\\
F(a^2+a^2+b^2, q) &= F(a^2+a^2,q)F(a^2,q) - F(a^2+a^2+a^2,q)\\
&= q^3\left(\psi(q^8)\psi(q^{16}) - \psi(q^{24})\right),\\
F(a^2+a^2+a^2+b^2, q) &= F(a^2+a^2+a^2,q)F(a^2,q)\\
&\quad - F(a^2+a^2+a^2+a^2,q)
\end{aligned}$$

$$
\begin{aligned}
&= q^4\left(\psi(q^8)\psi(q^{24})-\psi(q^{32})\right),\\
F(a^2+b^2+c^2,q) &= \frac{1}{3}\left(F(a^2+b^2,q)F(a^2,q)-F(a^2+a^2+b^2,q)\right)\\
&= \frac{1}{6}q^3\left(\psi(q^8)^3-3\psi(q^8)\psi(q^{16})+2\psi(q^{24})\right),\\
F(a^2+a^2+b^2+c^2,q) &= F(a^2+a^2,q)F(a^2+b^2,q)\\
&\qquad -F(a^2+a^2+a^2+b^2,q)\\
&= \frac{1}{2}q^4\left(\psi(q^8)^2\psi(q^{16})-\psi(q^{16})^2\right.\\
&\qquad \left.-2\psi(q^8\psi(q^{24})+2\psi(q^{32})\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&F(a^2+b^2+c^2+d^2,q)\\
&\quad=\frac{1}{4}\left(F(a^2+b^2+c^2,q)F(a^2,q)-F(a^2+a^2+b^2+c^2,q)\right)\\
&\quad=\frac{1}{24}q^4\left(\psi(q^8)^4-6\psi(q^8)^2\psi(q^{16})+3\psi(q^{16})^2+8\psi(q^8)\psi(q^{24})-6\psi(q^{32})\right)
\end{aligned}
$$

From these we obtain

$$
\begin{aligned}
\sum_{n\geq 0}p_{4o}(n)q^n &= F(a^2+b^2+c^2+d^2,q)+F(a^2+a^2+b^2+c^2,q)\\
&\quad+F(a^2+a^2+b^2+b^2,q)+F(a^2+a^2+a^2+b^2,q)\\
&\quad+F(a^2+a^2+a^2+a^2,q)\\
&=\frac{1}{24}q^4\left(\psi(q^8)^4+6\psi(q^8)^2\psi(q^{16})+3\psi(q^{16})^2\right.\\
&\qquad\left.+8\psi(q^8)\psi(q^{24})+6\psi(q^{32})\right),
\end{aligned}
$$

which is (7). Next,

$$
\begin{aligned}
\sum_{n\geq 0}p^d_{4o}(n)q^n &= F(a^2+b^2+c^2+d^2,q)\\
&=\frac{1}{24}q^4\left(\psi(q^8)^4-6\psi(q^8)^2\psi(q^{16})+3\psi(q^{16})^2\right.
\end{aligned}
$$

$$+ 8\psi(q^8)\psi(q^{24}) - 6\psi(q^{32})\big) ,$$

which is (8). To prove (9), we simply divide the right hand side of (7) by q^4 and replace q^8 by q to obtain

$$\sum_{n\geq 0} p_{4o}(8n+4)q^n = \frac{1}{24}\big(\psi(q)^4 + 6\psi(q)^2\psi(q^2) + 3\psi(q^2)^2$$
$$+ 8\psi(q)\psi(q^3) + 6\psi(q^4)\big) .$$

Result (10) follows from (8) in a similar fashion. ■

We now turn our attention to Theorem 3. We prove Theorem 3 by verifying the following two generating function identities.

Theorem 4.

$$\sum_{n\geq 0} p_{4o}(72n+60)q^n = \psi(q)\psi(q^3)\left\{\frac{(q^3)_\infty^3}{(q)_\infty} + \sum_{r=-\infty}^{\infty} q^{3r^2-2r}\right\} \tag{11}$$

and

$$\sum_{n\geq 0} p_{4o}^d(72n+60)q^n = \psi(q)\psi(q^3)\left\{\frac{(q^3)_\infty^3}{(q)_\infty} - \sum_{r=-\infty}^{\infty} q^{3r^2-2r}\right\} \tag{12}$$

where $(q)_\infty = \prod_{n\geq 1}(1-q^n)$.

In order to prove Theorem 4, we require a number of definitions and lemmas. We define

$$P(q) = \sum_{r=-\infty}^{\infty} q^{(3r^2-r)/2}, \quad H(q) = \sum_{n=-\infty}^{\infty} q^{(9n^2-n)/2},$$

$$I(q) = \sum_{n=-\infty}^{\infty} q^{(9n^2-5n)/2} \quad \text{and} \quad J(q) = \sum_{n=-\infty}^{\infty} q^{(9n^2-7n)/2}.$$

We now state and prove a number of lemmas involving various combinations of the functions above.

Lemma 5.

$$\psi(q) = P(q^3) + q\psi(q^9)$$
$$\text{and} \quad P(q) = H(q^3) + qI(q^3) + q^2J(q^3).$$

Proof: These are straightforward 3–dissections. ■

Lemma 6.

$$H(q)I(q^2) + qI(q)J(q^2) + J(q)H(q^2) = 2\psi(q^3) \sum_{r=-\infty}^{\infty} q^{3r^2-2r} \tag{13}$$

Proof:

Let $G_1(q) = H(q)I(q^2) + qI(q)J(q^2) + J(q)H(q^2)$, the left hand side of (13). Then

$$\begin{aligned}
q^{51}G_1(q^{72}) &= \sum_{m=-\infty}^{\infty} q^{(18m-1)^2+2(18m+5)^2} \\
&\quad + \sum_{m=-\infty}^{\infty} q^{(18m+5)^2+2(18m-7)^2} \\
&\quad + \sum_{m=-\infty}^{\infty} q^{(18m-7)^2+2(18m-1)^2} \\
&= \sum_{(a,b)\equiv(0,1),(1,-1) \text{ or } (-1,0) \pmod 3} q^{(6a-1)^2+2(6b-1)^2} \\
&= \sum_{a-b\equiv -1 \pmod 3} q^{(6a-1)^2+2(6b-1)^2} \\
&= \sum_{r,s=-\infty}^{\infty} q^{(12r+6s-7)^2+2(6s-6r-1)^2} \\
&= q^{51} \sum_{r,s=-\infty}^{\infty} q^{216r^2-144r+108s^2-108s} \\
&= 2q^{51}\psi(q^{216}) \sum_{-\infty}^{\infty} q^{216r^2-144r}.
\end{aligned}$$

Therefore,

$$G_1(q) = 2\psi(q^3) \sum_{r=-\infty}^{\infty} q^{3r^2-2r},$$

which completes the proof of Lemma 6. ■

Lemma 7.

$$H(q)I(q)^2 + qI(q)J(q)^2 + J(q)H(q)^2 = 2\psi(q^3)\frac{(q^3)_\infty^3}{(q)_\infty} \tag{14}$$

Proof:

Let $G_2(q) = H(q)I(q)^2 + qI(q)J(q)^2 + J(q)H(q)^2$, the left hand side of (14). Then

$$\begin{aligned}
q^{51}G_2(q^{72}) &= \sum_{m=-\infty}^{\infty} q^{(18m-1)^2}\left(\sum_{n=-\infty}^{\infty} q^{(18n+5)^2}\right)^2 \\
&\quad + \sum_{m=-\infty}^{\infty} q^{(18m+5)^2}\left(\sum_{n=-\infty}^{\infty} q^{(18n-7)^2}\right)^2 \\
&\quad + \sum_{m=-\infty}^{\infty} q^{(18m-7)^2}\left(\sum_{n=-\infty}^{\infty} q^{(18n-1)^2}\right)^2 \\
&= \sum_{(a,b,c)\equiv(0,1,1),(1,-1,-1) \text{ or } (-1,0,0) \pmod 3} q^{(6a-1)^2+(6b-1)^2+(6c-1)^2} \\
&= \frac{1}{3}\sum_{a+b+c=-1} q^{(6a-1)^2+(6b-1)^2+(6c-1)^2} \\
&= \frac{1}{3}\sum_{r=-\infty}^{\infty}\sum_{s+t+u=-1} q^{36\left((r+s)^2+(r+t)^2+(r+u)^2\right)-12(3r-1)+3} \\
&= \frac{1}{3}\sum_{r=-\infty}^{\infty} q^{108r^2-108r+15}\sum_{s+t+u=-1} q^{36(s^2+t^2+u^2)} \\
&= \frac{2}{3}q^{15}\psi(q^{216})\sum_{s,t=-\infty}^{\infty} q^{36\left(s^2+t^2+(s+t+1)^2\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3}q^{51}\psi(q^{216})\sum_{s,t=-\infty}^{\infty} q^{72(s^2+t^2+st+s+t)} \\
&= \frac{2}{3}q^{51}\psi(q^{216})\cdot 3\frac{(q^{216})_\infty^3}{(q^{72})_\infty} \\
&= 2q^{51}\psi(q^{216})\frac{(q^{216})_\infty^3}{(q^{72})_\infty}.
\end{aligned}
$$

Thus,

$$G_2(q) = 2\psi(q^3)\frac{(q^3)_\infty^3}{(q)_\infty}.$$

Here we have used the fact that

$$\sum_{s,t=-\infty}^{\infty} q^{s^2+t^2+st+s+t} = 3\frac{(q^3)_\infty^3}{(q)_\infty},$$

a proof of which is found in [4]. ■

We are now prepared to prove Theorem 4.

Proof of Theorem 4:

We note that, by Lemma 5,

$$
\begin{aligned}
\sum_{n\geq 0} p_{4o}(8n+4)q^n &= \frac{1}{24}\left(\psi(q)^4 + 6\psi(q)^2\psi(q^2) + 3\psi(q^2)^2\right. \\
&\qquad \left. + 8\psi(q)\psi(q^3) + 6\psi(q^4)\right) \\
&= \frac{1}{24}\left\{\left(P(q^3)+q\psi(q^9)\right)^4\right. \\
&\quad + 6\left(P(q^3)+q\psi(q^9)\right)^2\left(P(q^6)+q^2\psi(q^{18})\right) \\
&\quad +3\left(P(q^6)+q^2\psi(q^{18})\right)^2 + 8\left(P(q^3)+q\psi(q^9)\right)\psi(q^3) \\
&\quad \left.+6\left(P(q^{12})+q^4\psi(q^{36})\right)\right\}.
\end{aligned}
$$

It follows that

$$\sum_{n\geq 0} p_{4o}(24n+12)q^n = \frac{1}{24}\left\{\left(4P(q)^3\psi(q^3) + q\psi(q^3)^4\right)\right.$$

$$
\begin{aligned}
&+6\left(2P(q)P(q^2)\psi(q^3)+q\psi(q^3)^2\psi(q^6)\right)\\
&+3q\psi(q^6)^2+8\psi(q)\psi(q^3)+6q\psi(q^{12})\big\}\\
=\ &\frac{1}{24}\Big\{4\psi(q^3)\left(H(q^3)+qI(q^3)+q^2J(q^3)\right)^3\\
&+12\psi(q^3)\left(H(q^3)+qI(q^3)+q^2J(q^3)\right)\times\\
&\quad\left(H(q^6)+q^2I(q^6)+q^4J(q^6)\right)\\
&+8\psi(q^3)\left(P(q^3)+q\psi(q^9)\right)\\
&+q\psi(q^3)^4+6q\psi(q^3)^2\psi(q^6)+3q\psi(q^6)^2+6q\psi(q^{12})\big\}\,.
\end{aligned}
$$

With the aid of Lemmas 6 and 7, one further dissection yields

$$
\begin{aligned}
\sum_{n\geq 0}p_{4o}(72n+60)q^n &= \frac{1}{24}\big\{4\psi(q)\left(3H(q)I(q)^2+3qI(q)J(q)^2+3J(q)H(q)^2\right)\\
&\quad+12\psi(q)\left(H(q)I(q^2)+qI(q)J(q^2)+J(q)H(q^2)\right)\big\}\\
&=\frac{1}{2}\psi(q)\left\{\left(H(q)I(q)^2+qI(q)J(q)^2+J(q)H(q)^2\right)\right.\\
&\qquad\left.+\left(H(q)I(q^2)+qI(q)J(q^2)+J(q)H(q^2)\right)\right\}\\
&=\frac{1}{2}\psi(q)\left\{2\psi(q^3)\frac{(q^3)_\infty^3}{(q)_\infty}+2\psi(q^3)\sum_{r=-\infty}^{\infty}q^{3r^2-2r}\right\}\\
&=\psi(q)\psi(q^3)\left\{\frac{(q^3)_\infty^3}{(q)_\infty}+\sum_{r=-\infty}^{\infty}q^{3r^2-2r}\right\},
\end{aligned}
$$

which is (11). The same approach can be used to prove (12). ■

Theorem 3 now follows from Theorem 4. Indeed, we have

$$
\begin{aligned}
\frac{(q^3)_\infty^3}{(q)_\infty} &= \frac{(q^3)_\infty^4}{(q)_\infty(q^3)_\infty}\\
&\equiv \frac{(q^6)_\infty^2}{(q)_\infty(q^3)_\infty} \pmod 2\\
&\equiv \frac{(q)_\infty(q^6)_\infty^2}{(q^2)_\infty(q^3)_\infty} \pmod 2\\
&= \prod_{n\geq 1}\frac{(1-q^{2n-1})(1-q^{6n})}{(1-q^{6n-3})}
\end{aligned}
$$

$$\equiv \prod_{n\geq 1} \frac{(1+q^{2n-1})(1-q^{6n})}{(1+q^{6n-3})} \pmod 2$$
$$= \sum_{r=-\infty}^{\infty} q^{3r^2-2r}$$

by Jacobi's triple product identity [1, Theorem 2.8].

4. A Combinatorial Proof of Theorem 3

Suppose

$$72n+60 = k^2+l^2+m^2+p^2$$

with $k,\ l,\ m$ and p odd. Then, modulo 6,

$$k^2+l^2+m^2+p^2 \equiv 0.$$

If we consider all possibilities modulo 6, we find that precisely one of $k,\ l,\ m$ and p is $3 \pmod 6$. Suppose without loss of generality that $p = 3q$. Then

$$72n+60-9q^2 = k^2+l^2+m^2.$$

Modulo 18, this becomes

$$k^2+l^2+m^2 \equiv 15.$$

If we consider all possibilities modulo 18, we find that this has the solutions (with permutations)

$$(k,l,m) \equiv (\pm 1, \pm 1, \pm 7),\ (\pm 5, \pm 5, \pm 1) \text{ or } (\pm 7, \pm 7, \pm 5).$$

We can suppose without loss of generality (allowing $(k,\ l,\ m)$ to be negative) that

$$(k,l,m) \equiv (1,1,7),\ (-5,-5,1) \text{ or } (7,7,-5) \pmod{18}.$$

Set

$$\begin{pmatrix} k' \\ l' \\ m' \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} k \\ l \\ m \end{pmatrix}.$$

Then

$$(k', l', m') \equiv (1, 1, 1) \pmod 6,$$

$$k' + l' + m' = -(k + l + m)$$

and

$$(k')^2 + (l')^2 + (m')^2 = k^2 + l^2 + m^2 = 72n + 60 - 9q^2.$$

We now show that $(k')^2 + (l')^2 + (m')^2$ is not the same partition of $72n + 60 - 9q^2$ as $k^2 + l^2 + m^2$. For if $\begin{pmatrix} k \\ l \\ m \end{pmatrix}$ is a fixed point, we have $\{k', l', m'\} = \{k, l, m\}$, and so $k' + l' + m' = k + l + m = 0$. But this is impossible as $k + l + m \equiv 3 \pmod 6$. Thus the involution on the set of solutions of $72n + 60 - 9q^2 = k^2 + l^2 + m^2$ has no fixed points, and $p_{4o}(72n + 60)$ is even. ■

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