# Congruences modulo 11 for broken 5-diamond partitions


Eric H. Liu[1], James A. Sellers[2] and Ernest X.W. Xia[3]

[1]School of Business Information
Shanghai University of International Business and Economics
Shanghai, 201620, P. R. China

[2]Department of Mathematics
Penn State University, University Park
PA, 16802, USA

[3]Department of Mathematics
Jiangsu University
Jiangsu, Zhenjiang 212013, P. R. China

email: [1]liuhai@suibe.edu.cn, [2]sellersj@psu.edu, [3]ernestxwxia@163.com



**Abstract.** The notion of broken $k$-diamond partitions was introduced by Andrews and Paule in 2007. For a fixed positive integer $k$, let $\Delta_k(n)$ denote the number of broken $k$-diamond partitions of $n$. Recently, Paule and Radu conjectured two relations on $\Delta_5(n)$ which were proved by Xiong and Jameson respectively. In this paper, employing these relations, we prove that for any prime $p$ with $p \equiv 1 \pmod 4$, there exists an integer $\lambda(p) \in \{2,\ 3,\ 5,\ 6,\ 11\}$ such that for $n,\ \alpha \geq 0$, if $p \nmid (2n+1)$, then

$$\Delta_5\left(11p^{\lambda(p)(\alpha+1)-1}n + \frac{11p^{\lambda(p)(\alpha+1)-1}+1}{2}\right) \equiv 0 \pmod{11}.$$

Moreover, some non-standard congruences modulo 11 for $\Delta_5(n)$ are deduced. For example, we prove that for $\alpha \geq 0$, $\Delta_5\left(\frac{11\times 5^{5\alpha}+1}{2}\right) \equiv 7 \pmod{11}$.



**Keywords:** broken $k$-diamond partition, congruence, theta function.

**AMS Subject Classification:** 11P83, 05A17


## 1 Introduction

A combinatorial study guided by MacMahon's Partition Analysis led Andrews and Paule [1] to the construction of a new class of directed graphs called broken $k$-diamond partitions. Let $\Delta_k(n)$ denote the number of broken $k$-diamond partitions of $n$ for a fixed positive integer $k$. Andrews and Paule [1] established the following generating function of $\Delta_k(n)$:

$$\sum_{n=0}^{\infty} \Delta_k(n)q^n = \frac{(q^2;q^2)_\infty (q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty^3 (q^{4k+2};q^{4k+2})_\infty},$$

where

$$(q;q)_\infty := \prod_{n=1}^{\infty}(1-q^n).$$

Employing generating function manipulations, Andrews and Paule [1] proved that for all integers $n \geq 0$,

$$\Delta_1(2n+1) \equiv 0 \pmod{3}.$$

Moreover, they gave three conjectures modulo 2, 5 and 25 for $\Delta_k(n)$. Since then, a number of congruences satisfied by $\Delta_k(n)$ for small values of $k$ have been proved, see for example, Chan [2], Cui and Gu [3], Hirschhorn and Sellers [4], Lin [6], Lin and Wang [7], Radu and Sellers [9, 10], Wang and Yao [16], Xia [11, 12, 13] and Yao [15]. Recently, Paule and Radu [8] discovered some non-standard congruences modulo 5 for $\Delta_2(n)$. Moreover, they presented two conjectures on $\Delta_5(n)$ which were proved by Xiong [14] and Jameson [5] respectively.

In this paper, we establish infinite families of congruences and non-standard congruences modulo 11 for $\Delta_5(n)$ by utilizing the two relations on $\Delta_5(n)$ which were conjectured by Paule and Radu and proved by Xiong [14] and Jameson [5].

In order to state the main results of this paper, we first give some definitions. In this paper, we always define

$$S_0 := \{(0,1),\ (0,3),\ (0,4),\ (0,5),\ (0,9)\}$$

and for $1 \leq k \leq 5$,

$$S_k := \{(r,s)|1 \leq r \leq 10,\ 1 \leq s \leq 9,\ s \equiv k^2r^2 \pmod{11}\}.$$

Moreover, we let $p$ be a prime with $p \equiv 1 \pmod 4$ and assume that $c\left(\frac{p-1}{2}\right) \equiv r \pmod{11}$ and $p^8 \equiv s \pmod{11}$ with $0 \leq r \leq 10$ and $s \in \{1,\ 3,\ 4,\ 5,\ 9\}$ (the quadratic residues modulo 11). Define

$$\lambda(p) := \begin{cases} 2, & \text{if } (r,s) \in S_0, \\ 3, & \text{if } (r,s) \in S_1, \\ 6, & \text{if } (r,s) \in S_2, \\ 5, & \text{if } (r,s) \in S_3 \cup S_4, \\ 11, & \text{if } (r,s) \in S_5, \end{cases} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} c(n)q^n := (q;q)_\infty^8 (q^2;q^2)_\infty^2 \left(1 + 240\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\right). \tag{1.2}$$

The coefficients $c(n)$ are of interest here since they are related to broken 5-diamond partitions in the following congruence relation:

$$c(n) \equiv 8\Delta_5(11n+6) \pmod{11}. \tag{1.3}$$

Congruence (1.3) was conjectured by Paule and Radu [8] and was proved by Xiong [14] by using the theory of modular forms.

The infinite families of congruences modulo 11 for $\Delta_5(n)$ can be stated as follows.

**Theorem 1.1** *Let $p$ be a prime with $p \equiv 1 \pmod 4$. For $n,\ \alpha \geq 0$, if $p \nmid (2n+1)$, then*

$$\Delta_5\left(11p^{\lambda(p)(\alpha+1)-1}n + \frac{11p^{\lambda(p)(\alpha+1)-1}+1}{2}\right) \equiv 0 \pmod{11}, \tag{1.4}$$

*where $\lambda(p)$ is defined by* (1.1).

The non-standard congruences modulo 11 for $\Delta_5(n)$ can be stated as follows.

**Theorem 1.2** *Let $p$ be a prime with $p \equiv 1 \pmod 4$. For $\alpha \geq 0$,*

$$\Delta_5\left(\frac{11p^{\lambda(p)\alpha}+1}{2}\right) \equiv 7V(r,s)^{\alpha} \pmod{11}, \tag{1.5}$$

*where*

$$V(r,s) := \begin{cases} -s, & \text{if } (r,s) \in S_0, \\ -r^3, & \text{if } (r,s) \in S_1, \\ 2r^6, & \text{if } (r,s) \in S_2, \\ -r^5, & \text{if } (r,s) \in S_3, \\ r^5, & \text{if } (r,s) \in S_4, \\ 6r^{11}, & \text{if } (r,s) \in S_5. \end{cases} \tag{1.6}$$

For example, let $p = 5$ in Theorem 1.1. It is easy to check that $c(2) = 258 \equiv 5 = r \pmod{11}$ and $5^8 \equiv 4 = s \pmod{11}$. Therefore, $\lambda(5) = 5$. From Theorem 1.1, we see that if $5 \nmid (2n+1)$, then for $\alpha \geq 0$,

$$\Delta_5\left(11 \cdot 5^{5\alpha+4}n + \frac{11 \cdot 5^{5\alpha+4}+1}{2}\right) \equiv 0 \pmod{11}.$$

From Theorem 1.2, we deduce that for $\alpha \geq 0$,

$$\Delta_5\left(\frac{11 \times 5^{5\alpha}+1}{2}\right) \equiv 7 \pmod{11}.$$

## 2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we first prove two lemmas.

**Lemma 2.1** *Let* $(r,s) \in \bigcup_{k=0}^{5} S_k$ *and define*

$$S_{r,s} := \{p|p \text{ is a prime, } p \equiv 1 \pmod 4,\ c\left(\frac{p-1}{2}\right) \equiv r \pmod{11} \text{ and } p^8 \equiv s \pmod{11}\}.$$

*If* $p \in S_{r,s}$, *then for* $n,\ \alpha \geq 0$,

$$c\left(p^\alpha n + \frac{p^\alpha - 1}{2}\right) \equiv A_{r,s}(\alpha) c\left(pn + \frac{p-1}{2}\right) + B_{r,s}(\alpha)c(n) \pmod{11}, \tag{2.1}$$

*where* $c(n)$ *is defined by* (1.2), $A_{r,s}(\alpha)$ *and* $B_{r,s}(\alpha)$ *are defined by*

$$A_{r,s}(\alpha+2) = rA_{r,s}(\alpha+1) - sA_{r,s}(\alpha), \tag{2.2}$$

$$B_{r,s}(\alpha+2) = rB_{r,s}(\alpha+1) - sB_{r,s}(\alpha), \tag{2.3}$$

*with* $B_{r,s}(0) = A_{r,s}(1) = 1$ *and* $B_{r,s}(1) = A_{r,s}(0) = 0$.

*Proof.* We prove (2.1) by induction on $\alpha$. It is routine to check that (2.1) holds when $\alpha = 0$ and $\alpha = 1$ since $A_{r,s}(1) = B_{r,s}(0) = 1$ and $A_{r,s}(0) = B_{r,s}(1) = 0$ for $(r,s) \in \bigcup_{k=0}^{5} S_k$. Suppose that (2.1) holds when $\alpha = m$ and $\alpha = m+1$ ($m \geq 0$), that is,

$$c\left(p^m n + \frac{p^m - 1}{2}\right) \equiv A_{r,s}(m) c\left(pn + \frac{p-1}{2}\right) + B_{r,s}(m)c(n) \pmod{11}, \tag{2.4}$$

and

$$c\left(p^{m+1} n + \frac{p^{m+1} - 1}{2}\right) \equiv A_{r,s}(m+1) c\left(pn + \frac{p-1}{2}\right) + B_{r,s}(m+1)c(n) \pmod{11}, \tag{2.5}$$

where $p \in S_{r,s}$. From the definition of $S_{r,s}$,

$$c\left(\frac{p-1}{2}\right) \equiv r \pmod{11} \quad \text{and} \quad p^8 \equiv s \pmod{11}. \tag{2.6}$$

Jameson [5] proved that there exists an integer $y(p)$ such that for $n \geq 0$,

$$c\left(pn + \frac{p-1}{2}\right) = y(p)c(n) - p^8 c\left(\frac{n-(p-1)/2}{p}\right), \tag{2.7}$$

where $p$ is a prime with $p \equiv 1 \pmod 4$. Identity (2.7) was conjectured by Paule and Radu [8]. Taking $n = 0$ in (2.7) and using the facts that $c(0) = 1$ and $c\left(\frac{-(p-1)/2}{p}\right) = 0$, we deduce that

$$y(p) = c\left(\frac{p-1}{2}\right). \tag{2.8}$$

Replacing $n$ by $pn + \frac{p-1}{2}$ in (2.7) and using (2.8) yields

$$c\left(p^2 n + \frac{p^2-1}{2}\right) = c\left(\frac{p-1}{2}\right) c\left(pn + \frac{p-1}{2}\right) - p^8 c(n). \tag{2.9}$$

Thanks to (2.6) and (2.9),

$$c\left(p^2 n + \frac{p^2-1}{2}\right) \equiv rc\left(pn + \frac{p-1}{2}\right) - sc(n) \pmod{11}, \tag{2.10}$$

where $p \in S_{r,s}$. Replacing $n$ by $p^m n + \frac{p^m-1}{2}$ in (2.10) and utilizing (2.2)–(2.5) yields

$$\begin{aligned}
c\left(p^{m+2} n + \frac{p^{m+2}-1}{2}\right) &\equiv rc\left(p^{m+1} n + \frac{p^{m+1}-1}{2}\right) - sc\left(p^m n + \frac{p^m-1}{2}\right)\\
&\equiv rA_{r,s}(m+1)c\left(pn + \frac{p-1}{2}\right) + rB_{r,s}(m+1)c(n)\\
&\quad - sA_{r,s}(m)c\left(pn + \frac{p-1}{2}\right) - sB_{r,s}(m)c(n)\\
&\equiv (rA_{r,s}(m+1) - sA_{r,s}(m))c\left(pn + \frac{p-1}{2}\right)\\
&\quad + (rB_{r,s}(m+1) - sB_{r,s}(m))c(n)\\
&\equiv A_{r,s}(m+2)c\left(pn + \frac{p-1}{2}\right)\\
&\qquad + B_{r,s}(m+2)c(n) \pmod{11},
\end{aligned}$$

which implies that (2.1) is true when $\alpha = m+2$. Congruence (2.1) is proved by induction and this completes the proof of Lemma 2.1. ∎

**Lemma 2.2** *If* $(r, s) \in \bigcup_{k=0}^{5} S_k$, *then for* $\alpha \geq 0$,

$$rA_{r,s}(\lambda(p)(\alpha+1)-1) + B_{r,s}(\lambda(p)(\alpha+1)-1) \equiv 0 \pmod{11}, \tag{2.11}$$

*where* $\lambda(p)$, $A_{r,s}(\alpha)$ *and* $B_{r,s}(\alpha)$ *are defined by* (1.1), (2.2) *and* (2.3), *respectively.*

*Proof.* We also prove (2.11) by induction on $\alpha$. It is easy to verify that (2.11) holds when $\alpha = 0$ for all $(r,s) \in S_k$ with $0 \le k \le 5$. Assume that (2.11) is true when $\alpha = m$ ($m \ge 0$), namely,

$$rA_{r,s}(\lambda(p)m + \lambda(p) - 1) + B_{r,s}(\lambda(p)m + \lambda(p) - 1) \equiv 0 \pmod{11}, \tag{2.12}$$

where $(r,s) \in \bigcup_{k=0}^{5} S_k$. Based on (2.2) and (2.3),

$$\begin{aligned} &rA_{r,s}(\lambda(p)m + 2\lambda(p) - 1) + B_{r,s}(\lambda(p)m + 2\lambda(p) - 1) \\ \equiv &V(r,s)(rA_{r,s}(\lambda(p)m + \lambda(p) - 1) + B_{r,s}(\lambda(p)m + \lambda(p) - 1)) \pmod{11}, \end{aligned} \tag{2.13}$$

where $V(r,s)$ is defined by (1.6).

Combining (2.12) and (2.13), we see that for $(r,s) \in \bigcup_{k=0}^{5} S_k$,

$$rA_{r,s}(\lambda(p)m + 2\lambda(p) - 1) + B_{r,s}(\lambda(p)m + 2\lambda(p) - 1) \equiv 0 \pmod{11},$$

which implies that (2.11) is true when $\alpha = m+1$ and (2.11) is proved by induction. This completes the proof of this lemma. ∎

Now, we turn to prove Theorem 1.1.

Let $p$ be a prime with $p \equiv 1 \pmod 4$. Assume that $c\left(\frac{p-1}{2}\right) \equiv r \pmod{11}$ and $p^8 \equiv s \pmod{11}$ with $0 \le r \le 10$ and $s \in \{1,\ 3,\ 4,\ 5,\ 9\}$. Thus, for any prime $p$, there exists a pair $(r,s) \in \bigcup_{k=0}^{5} S_k$ such that $p \in S_{r,s}$, where $S_{r,s}$ is defined in Lemma 2.1. In view of (2.1), (2.7) and (2.8),

$$\begin{aligned} c\left(p^\alpha n + \frac{p^\alpha - 1}{2}\right) &\equiv A_{r,s}(\alpha)\left(c\left(\frac{p-1}{2}\right)c(n) - p^8 c\left(\frac{n-(p-1)/2}{p}\right)\right) + B_{r,s}(\alpha)c(n) \\ &\equiv (rA_{r,s}(\alpha) + B_{r,s}(\alpha))c(n) \\ &\qquad - sA_{r,s}(\alpha)c\left(\frac{n-(p-1)/2}{p}\right) \pmod{11}. \end{aligned} \tag{2.14}$$

Replacing $\alpha$ by $\lambda(p)(\alpha+1)-1$ in (2.14) and utilizing (2.11), we obtain

$$\begin{aligned} &c\left(p^{\lambda(p)(\alpha+1)-1}n + \frac{p^{\lambda(p)(\alpha+1)-1} - 1}{2}\right) \\ \equiv &- sA_{r,s}(\lambda(p)(\alpha+1)-1)c\left(\frac{n-(p-1)/2}{p}\right) \pmod{11}. \end{aligned} \tag{2.15}$$

Note that if $p \nmid (2n+1)$, then $\frac{n-(p-1)/2}{p}$ is not an integer and

$$c\left(\frac{n-(p-1)/2}{p}\right) = 0. \tag{2.16}$$

Combining (2.15) and (2.16), we deduce that if $p \nmid (2n+1)$, then for $\alpha \geq 0$,

$$c\left(p^{\lambda(p)(\alpha+1)-1}n + \frac{p^{\lambda(p)(\alpha+1)-1}-1}{2}\right) \equiv 0 \pmod{11}. \tag{2.17}$$

Replacing $n$ by $p^{\lambda(p)(\alpha+1)-1}n + \frac{p^{\lambda(p)(\alpha+1)-1}-1}{2}$ in (1.3) and using (2.17), we arrive at (1.4). This completes the proof of Theorem 1.1. ∎

# 3 Proof of Theorem 1.2

In order to prove Theorem 1.2, we first prove the following two lemmas.

**Lemma 3.1** *Let $(r,s) \in \bigcup_{k=0}^{5} S_k$ and let $S_{r,s}$ be defined in Lemma 2.1. If $p \in S_{r,s}$, then for $\alpha \geq 0$,*

$$A_{r,s}(\lambda(p)\alpha) \equiv 0 \pmod{11}, \tag{3.1}$$

*where $A_{r,s}(\alpha)$ is defined by (2.2) respectively.*

*Proof.* We prove (3.1) by induction on $\alpha$. It is easy to see that (3.1) holds when $\alpha = 0$ since $A_{r,s}(0) = 0$. Suppose that (3.1) holds when $\alpha = m$ $(m \geq 0)$, namely,

$$A_{r,s}(\lambda(p)m) \equiv 0 \pmod{11}. \tag{3.2}$$

Thanks to (2.2),

$$A_{r,s}(\lambda(p)m + \lambda(p)) \equiv V(r,s)A_{r,s}(\lambda(p)m) \pmod{11}, \tag{3.3}$$

where $V(r,s)$ is defined by (1.6). Because of (3.2) and (3.3), we see that (3.1) is true when $\alpha = m+1$ and Lemma 3.1 is proved by induction. ∎

**Lemma 3.2** *Let $(r,s) \in \bigcup_{k=0}^{5} S_k$ and let $S_{r,s}$ be defined in Lemma 2.1. If $p \in S_{r,s}$, then for $\alpha \geq 0$,*

$$B_{r,s}(\lambda(p)\alpha) \equiv V(r,s)^{\alpha} \pmod{11}, \tag{3.4}$$

*where $V(r,s)$ and $B_{r,s}(\alpha)$ are defined by (1.6) and (2.3) respectively.*

*Proof.* We also prove (3.4) by induction. It is easy to see that (3.4) is true when $\alpha = 0$ since $B_{r,s}(0) = 1$. Suppose that (3.4) holds when $\alpha = m$ $(m \geq 0)$, that is,

$$B_{r,s}(\lambda(p)m) \equiv V(r,s)^{m} \pmod{11}. \tag{3.5}$$

In view of (2.3),

$$B_{r,s}(\lambda(p)m+\lambda(p)) \equiv V(r,s)B_{r,s}(\lambda(p)m) \pmod{11}. \tag{3.6}$$

It follows from (3.5) and (3.6) that Lemma 3.2 is true when $\alpha = m+1$. This lemma is proved by induction. ∎

Now, we are ready to prove Theorem 1.2.

Setting $n=0$ in (2.1) and using the fact that $c(0)=1$, we get

$$c\left(\frac{p^\alpha-1}{2}\right) \equiv A_{r,s}(\alpha)c\left(\frac{p-1}{2}\right)+B_{r,s}(\alpha) \pmod{11}, \tag{3.7}$$

Replacing $\alpha$ by $\lambda(p)\alpha$ in (3.7) and employing (3.1) and (3.4),

$$c\left(\frac{p^{\lambda(p)\alpha}-1}{2}\right) \equiv V(r,s)^\alpha \pmod{11}. \tag{3.8}$$

It follows from (1.3) that for $n \geq 0$,

$$\Delta_5(11n+6) \equiv 7c(n) \pmod{11}. \tag{3.9}$$

Replacing $n$ by $\frac{p^{\lambda(p)\alpha}-1}{2}$ in (3.9) and using (3.8), we arrive at (1.5). This completes the proof. ∎

**Acknowledgments.** This work was supported by the National Natural Science Foundation of China (11401260 and 11571143).

# References

[1] G.E. Andrews and P. Paule, MacMahon's partition analysis XI: broken diamonds and modular forms, Acta Arith. 126 (2007) 281–294.

[2] S.H. Chan, Some congruences for Andrews-Paule's broken 2-diamond partitions, Discrete Math. 308 (2008) 5735–5741.

[3] S.P. Cui and N.S.S. Gu, Congruences for broken 3-diamond and 7 dots bracelet partitions, Ramanujan J. 35 (2014) 165–178.

[4] M.D. Hirschhorn and J.A. Sellers, On recent congruence results of Andrews and Paule, Bull. Austral. Math. Soc. 75 (2007) 121–126.

[5] M. Jameson, Congruences for broken $k$-diamond partitions, Ann. Combin. 17 (2013) 333–338.

[6] B.L.S. Lin, Elementary proofs of parity results for broken 3-diamond partitions, J. Number Theory 135 (2014) 1–7.

[7] B.L.S. Lin and A.Y.Z. Wang, Elementary proofs of Radu and Sellers' results for broken 2-diamond partitions, Ramanujan J. 37 (2015) 291–297.

[8] P. Paule and S. Radu, Infinite families of strange partition congruences for broken 2-diamonds, Ramanujan J. 23 (2010) 409–416.

[9] S. Radu and J.A. Sellers, Parity results for broken $k$-diamond partitions and $(2k+1)$-cores, Acta Arith. 146 (2011) 43–52.

[10] S. Radu and J.A. Sellers, An extensive analysis of the parity of broken 3-diamond partitions, J. Number Theory 133 (2013) 3703–3716.

[11] E.X.W. Xia, New congruences modulo powers of 2 for broken 3-diamond partitions and 7-core partitions, J. Number Theory 141 (2014) 119–135.

[12] E.X.W. Xia, Infinite families of congruences modulo 7 for broken 3-diamond partitions, Ramanujan J. 40 (2016) 389–403.

[13] E.X.W. Xia, More infinite families of congruences modulo 5 for broken 2-diamond partitions, J. Number Theory, to appear (10.1016/j.jnt.2016.04.023).

[14] X. Xiong, Two congruences involving Andrews-Paule's broken 3-diamond partitions and 5-diamond partitions, Proc. Japan Acad. Ser. A (Math. Sci.) 87 (2011) 65–68.

[15] O.X.M. Yao, New parity results for broken 11-diamond partitions, J. Number Theory 140 (2014) 267–276.

[16] O.X.M. Yao and Y.J. Wang, Newman's identity and infinite families of congruences modulo 7 for broken 3-diamond partitions, Ramanujan J., to appear (DOI: 10.1007/s11139-016-9801-5).