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An infinite family of internal congruences modulo powers of 2 for partitions into odd parts with designated summands

by

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Abstract. In 2002, Andrews, Lewis, and Lovejoy introduced the combinatorial objects which they called *partitions with designated summands*. These are built by taking unrestricted integer partitions and designating exactly one part of each size. In the same work, Andrews, Lewis, and Lovejoy also studied such partitions wherein all parts must be odd, and they denoted the number of such partitions of size n by $\mathrm{PDO}(n)$. Since then, numerous authors have proven a variety of divisibility properties satisfied by $\mathrm{PDO}(n)$. Recently, the second author proved the following internal congruences satisfied by $\mathrm{PDO}(n)$: *For all* $n \geq 0$,

$$\mathrm{PDO}(4n) \equiv \mathrm{PDO}(n) \pmod 4,$$
$$\mathrm{PDO}(16n) \equiv \mathrm{PDO}(4n) \pmod 8.$$

In the present work, we significantly extend these results by proving the following new infinite family of congruences: *For all* $k \geq 0$ *and all* $n \geq 0$,

$$\mathrm{PDO}(2^{2k+3}n) \equiv \mathrm{PDO}(2^{2k+1}n) \pmod{2^{2k+3}}.$$

To do so, we utilize several classical tools, including generating function dissections via the unitizing operator of degree 2, various modular relations and recurrences involving a Hauptmodul on the classical modular curve $X_0(6)$, and an induction argument which provides the final step in proving the necessary divisibilities. It is notable that the construction of each 2-dissection slice of our generating function bears an entirely different nature to those studied in the past literature.

1. Introduction. In 2002, Andrews, Lewis, and Lovejoy [2] introduced the combinatorial objects which they called *partitions with designated summands*. These are built by taking unrestricted integer partitions and designating exactly one part of each size. For example, there are ten partitions

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of 4 with designated summands:

$$4', \quad 3'+1', \quad 2'+2, \quad 2+2', \quad 2'+1'+1, \quad 2'+1+1',$$
$$1'+1+1+1, \quad 1+1'+1+1, \quad 1+1+1'+1, \quad 1+1+1+1'.$$

Andrews, Lewis, and Lovejoy denoted the number of partitions of n with designated summands by $\mathrm{PD}(n)$. Hence, using this notation and the example above, we know $\mathrm{PD}(4) = 10$.

In [2] are also considered the restricted partitions with designated summands wherein all parts must be odd; the corresponding enumeration function is denoted by $\mathrm{PDO}(n)$. Thus, from the example above, we see that $\mathrm{PDO}(4) = 5$, where we have counted the following five objects:

$$3'+1', \quad 1'+1+1+1, \quad 1+1'+1+1, \quad 1+1+1'+1, \quad 1+1+1+1'.$$

It is noted in [2, eq. (1.6)] that the generating function for $\mathrm{PDO}(n)$ is given by

$$\sum_{n=0}^{\infty} \mathrm{PDO}(n) q^n = \frac{E(q^4)E(q^6)^2}{E(q)E(q^3)E(q^{12})}, \tag{1.1}$$

where

$$E(q) = (q;q)_\infty := (1-q)(1-q^2)(1-q^3)(1-q^4)\cdots$$

is the usual *q-Pochhammer symbol*.

Beginning with [2], a wide variety of Ramanujan-like congruences have been proven for $\mathrm{PD}(n)$ and $\mathrm{PDO}(n)$ under different moduli [1, 3, 4, 5, 6, 7, 10, 12, 18]. Recently, Sellers [15] proved a number of arithmetic properties satisfied by $\mathrm{PDO}(n)$ modulo powers of 2 in response to a conjecture of Herden et al. [11]. As part of that work, Sellers proved the following internal congruences on the way to proving infinite families of divisibility properties satisfied by $\mathrm{PDO}(n)$: *For all $n \geq 0$,*

$$\mathrm{PDO}(4n) \equiv \mathrm{PDO}(n) \pmod{4}, \tag{1.2}$$
$$\mathrm{PDO}(16n) \equiv \mathrm{PDO}(4n) \pmod{8}. \tag{1.3}$$

While searching for such internal congruences computationally, it became clear that the above internal congruences are part of a much larger family. The ultimate goal of this paper is to prove the following:

Theorem 1.1. *For all $k \geq 0$ and all $n \geq 0$,*

$$\mathrm{PDO}(2^{2k+3}n) \equiv \mathrm{PDO}(2^{2k+1}n) \pmod{2^{2k+3}}. \tag{1.4}$$

Replacing n by $2n$ in the above immediately yields the following corollary:

Corollary 1.2. *For all $k \geq 0$ and all $n \geq 0$,*

$$\mathrm{PDO}(2^{2k+4}n) \equiv \mathrm{PDO}(2^{2k+2}n) \pmod{2^{2k+3}}. \tag{1.5}$$

The $k = 0$ case of this corollary states that, for all $n \geq 0$,

$$\mathrm{PDO}(16n) \equiv \mathrm{PDO}(4n) \pmod 8,$$

which is (1.3). We point out that the above families of congruences are not optimal in several isolated cases, especially when k is small. For instance, we will show in (6.1) that

$$\mathrm{PDO}(32n) \equiv \mathrm{PDO}(8n) \pmod{64},$$

which also yields

$$\mathrm{PDO}(64n) \equiv \mathrm{PDO}(16n) \pmod{64}.$$

To prove Theorem 1.1, we introduce the following auxiliary functions:

$$\delta = \delta(q) := \frac{E(q^4)E(q^6)^2}{E(q)E(q^3)E(q^{12})}, \qquad \xi = \xi(q) := \frac{E(q^2)^5 E(q^6)}{E(q)E(q^3)^5},$$
$$\gamma = \gamma(q) := \frac{E(q)^5 E(q^2)^5 E(q^6)^5}{E(q^3)^{15}}, \qquad \kappa = \kappa(q) := \frac{\gamma(q^2)^2}{\gamma(q)}.$$

Note from (1.1) that

$$\delta(q) = \sum_{n=0}^{\infty} \mathrm{PDO}(n) q^n.$$

We further define, for $k \geq 2$,

$$\Lambda_k = \Lambda_k(q) := \gamma(q)^{2^{k-2}} \sum_{n=0}^{\infty} \mathrm{PDO}(2^k n) q^n. \tag{1.6}$$

Finally, let U be the *unitizing operator of degree* 2, given by

$$U\Big(\sum_n a_n q^n\Big) := \sum_n a_{2n} q^n.$$

These will allow us to represent each 2-dissection slice of the generating function of $\mathrm{PDO}(n)$, accompanied by a certain multiplier:

$$\lambda_k \sum_{n=0}^{\infty} \mathrm{PDO}(2^k n) q^n,$$

as a polynomial in the Hauptmodul ξ on the classical modular curve $X_0(6)$ of genus 0. We then complete our analysis of these functions in order to prove Theorem 1.1.

This paper is organized as follows: Section 2 is devoted to proving several modular equations. In particular, Section 2.2 concerns the representation of the degree 2 unitizations $\zeta_{i,j} = U(\kappa^i \xi^j)$ in $\mathbb{Z}[\xi]$ for arbitrary exponents i and j, where κ is one of the auxiliary functions we introduced and ξ is the aforementioned Hauptmodul. Sections 3 and 4 are devoted to the 2-adic behavior of these $\zeta_{i,j}$ series. Recall that in Theorem 1.1, we are indeed considering internal congruences for PDO(n). Hence, we introduce another family of auxiliary functions in Section 5:

$$\lambda_k' \sum_{n=0}^{\infty} (\mathrm{PDO}(2^{k+2}n) - \mathrm{PDO}(2^k n))q^n,$$

so as to capture this internal nature. Here the new multipliers λ_k' are closely related to the original λ_k. We then move on to the divisibility properties for the above new family of series, and in particular we offer our proof of Theorem 1.1 in Section 6. It is notable that the construction of each 2-dissection slice of our generating function bears an entirely different nature to those studied in the past literature, in the sense that the multipliers λ_k are pairwise distinct. This makes our 2-adic analysis far more complicated but in the meantime very unique. We present a discussion of the difference between our machinery and past work in Section 7.

2. Modular equations. In this section, we collect a number of modular relations that will be utilized subsequently.

2.1. Modular relations for γ and ξ. It is clear that both γ^6 and ξ are modular functions on the classical modular curve $X_0(6)$.

THEOREM 2.1. *We have*

$$\gamma^6 = 59049\xi^{10} - 262440\xi^{11} + 466560\xi^{12} - 414720\xi^{13} + 184320\xi^{14} - 32768\xi^{15}. \tag{2.1}$$

Proof. We only need to analyze the order of γ^6 and ξ at each of the cusps (cf. [13, p. 18, Theorem 1.65]):

$$\begin{aligned} \mathrm{ord}_0\, \gamma^6 &= 5, & \mathrm{ord}_0\, \xi &= 0,\\ \mathrm{ord}_{1/2}\, \gamma^6 &= 10, & \mathrm{ord}_{1/2}\, \xi &= 1,\\ \mathrm{ord}_{1/3}\, \gamma^6 &= -15, & \mathrm{ord}_{1/3}\, \xi &= -1,\\ \mathrm{ord}_{\infty}\, \gamma^6 &= 0, & \mathrm{ord}_{\infty}\, \xi &= 0. \end{aligned}$$

We see that both functions have a pole at the cusp $\left[\frac{1}{3}\right]_6$, with the pole for ξ being simple. Since $X_0(6)$ has genus 0, we can write γ^6 as a polynomial in ξ of degree 15, as given above. A more streamlined computer-aided analysis could be achieved automatically by Garvan's Maple package ETA [8]. ∎

2.2. Modular relations for κ and ξ. Our objective in this subsection is to prove the following:

THEOREM 2.2. *For any $i, j \geq 0$,*

$$U(\kappa^i \xi^j) \in \mathbb{Z}[\xi]. \tag{2.2}$$

We briefly postpone the proof in order to complete some necessary analysis.

2.2.1. *Initial cases.* Let us focus on the following five initial relations:

THEOREM 2.3. *We have*

$$U(\kappa) = 5\xi^3 - 20\xi^4 + 16\xi^5, \tag{2.3}$$

$$U(\xi) = 5\xi - 4\xi^2, \tag{2.4}$$

$$U(\kappa^2) = -\xi^5 + 50\xi^6 - 400\xi^7 + 1120\xi^8 - 1280\xi^9 + 512\xi^{10}, \tag{2.5}$$

$$U(\kappa\xi) = 3\xi^3 - 18\xi^4 + 16\xi^5, \tag{2.6}$$

$$U(\xi^2) = -9\xi + 58\xi^2 - 80\xi^3 + 32\xi^4. \tag{2.7}$$

Proof. These results can be shown by standard techniques from the theory of modular cusp analysis. Here we shall turn to a computer-aided proof by applying Smoot's Mathematica implementation `RaduRK` [16] of the Radu–Kolberg algorithm [14]. For example, with `RaduRK`, we can express $U(\xi)$ as a multiplier times a polynomial in a Hauptmodul $t = t(q)$ on $X_0(12)$ that has a pole only at the cusp $[\infty]_{12}$:

$$U(\xi) \cdot \frac{E(q^3)^{12}E(q^4)^4}{q^4E(q)^4E(q^{12})^{12}} = 15t - t^2 + t^3 + t^4,$$

where

$$t = \frac{E(q^4)^4E(q^6)^2}{qE(q^2)^2E(q^{12})^4}.$$

Now to show that the above expression equals $5\xi - 4\xi^2$ as claimed in (2.4), we only need to examine that the alleged linear combination of eta-products is identical to 0, i.e.,

$$(5\xi - 4\xi^2) - (15t - t^2 + t^3 + t^4) \cdot \frac{q^4E(q)^4E(q^{12})^{12}}{E(q^3)^{12}E(q^4)^4} = 0.$$

This task can be performed readily by Garvan's Maple package `ETA` [8], as pointed out in the proof of Theorem 2.1. The remaining identities can be proven in the same way. ∎

2.2.2. *Recurrences.* We start by considering two generic formal power series $\alpha = \alpha(q)$ and $\beta = \beta(q)$ such that each of $U(\alpha)$ and $U(\alpha^2)$ is representable as a *polynomial* with *rational* (usually *integer*) coefficients in a generic formal power series $\rho = \rho(q)$. Now we define

$$\begin{aligned}\sigma_{\alpha,1} &:= \alpha(q) + \alpha(-q) = 2U(\alpha),\\ \sigma_{\alpha,2} &:= \alpha(q)\alpha(-q) = \tfrac{1}{2}[(\alpha(q) + \alpha(-q))^2 - (\alpha(q)^2 + \alpha(-q)^2)]\\ &= 2U(\alpha)^2 - U(\alpha^2).\end{aligned}$$

Clearly, both $\sigma_{\alpha,1}$ and $\sigma_{\alpha,2}$ are in $\mathbb{Q}[\rho]$. Further, writing $\alpha_0 = \alpha(q)$ and $\alpha_1 = \alpha(-q)$, we see that α_0 and α_1 are the two roots of

$$(X - \alpha(q))(X - \alpha(-q)) = X^2 - \sigma_{\alpha,1} X + \sigma_{\alpha,2}.$$

That is, for $t = 0, 1$,

$$\alpha_t^2 - \sigma_{\alpha,1}\alpha_t + \sigma_{\alpha,2} = 0.$$

Let $\beta_0 = \beta(q)$ and $\beta_1 = \beta(-q)$. Now we see that, for $k \geq 2$,

$$\begin{aligned} 2U(\alpha^k\beta) &= \alpha_0^k\beta_0 + \alpha_1^k\beta_1 \\ &= (\sigma_{\alpha,1}\alpha_0^{k-1} - \sigma_{\alpha,2}\alpha_0^{k-2})\beta_0 + (\sigma_{\alpha,1}\alpha_1^{k-1} - \sigma_{\alpha,2}\alpha_1^{k-2})\beta_1 \\ &= 2\sigma_{\alpha,1}U(\alpha^{k-1}\beta) - 2\sigma_{\alpha,2}U(\alpha^{k-2}\beta), \end{aligned}$$

so

$$U(\alpha^k\beta) = \sigma_{\alpha,1}U(\alpha^{k-1}\beta) - \sigma_{\alpha,2}U(\alpha^{k-2}\beta).$$

Hence, if $U(\beta)$ and $U(\alpha\beta)$ are also in $\mathbb{Q}[\rho]$, we deduce recursively that $U(\alpha^k\beta) \in \mathbb{Q}[\rho]$ for all $k \geq 0$.

Now moving back to our scenario, for $i, j \geq 0$ we define

$$\zeta_{i,j} := U(\kappa^i\xi^j). \tag{2.8}$$

THEOREM 2.4. *For any $i \geq 2$ and $j \geq 0$,*

$$\zeta_{i,j} = (10\xi^3 - 40\xi^4 + 32\xi^5) \cdot \zeta_{i-1,j} - (\xi^5) \cdot \zeta_{i-2,j}. \tag{2.9}$$

Also, for any $i \geq 0$ and $j \geq 2$,

$$\zeta_{i,j} = (10\xi - 8\xi^2) \cdot \zeta_{i,j-1} - (9\xi - 8\xi^2) \cdot \zeta_{i,j-2}. \tag{2.10}$$

REMARK 2.5. We may combine the two recurrences (2.9) and (2.10) and derive, for $i, j \geq 2$,

$$\begin{aligned} \zeta_{i,j} &= (10\xi - 8\xi^2)(10\xi^3 - 40\xi^4 + 32\xi^5) \cdot \zeta_{i-1,j-1} \\ &\quad - (9\xi - 8\xi^2)(10\xi^3 - 40\xi^4 + 32\xi^5) \cdot \zeta_{i-1,j-2} \\ &\quad - (10\xi - 8\xi^2)(\xi^5) \cdot \zeta_{i-2,j-1} + (9\xi - 8\xi^2)(\xi^5) \cdot \zeta_{i-2,j-2}. \end{aligned} \tag{2.11}$$

Proof of Theorem 2.4. Let us first choose $(\alpha, \beta, \xi) \mapsto (\kappa, \xi^j, \xi)$ with $j \geq 0$. It is clear from (2.3) and (2.5) that

$$\sigma_{\kappa,1} = 10\xi^3 - 40\xi^4 + 32\xi^5, \qquad \sigma_{\kappa,2} = \xi^5.$$

Hence, (2.9) follows. In the same fashion, we choose $(\alpha, \beta, \xi) \mapsto (\xi, \kappa^i, \xi)$ with $i \geq 0$. By (2.3) and (2.5), we compute that

$$\sigma_{\xi,1} = 10\xi - 8\xi^2, \qquad \sigma_{\xi,2} = 9\xi - 8\xi^2,$$

and therefore (2.10) is true. ■

Proof of Theorem 2.2. With Theorem 2.3 in hand, we currently have two sets of initial relations for the recurrence (2.9): $\{\zeta_{0,0}, \zeta_{1,0}\}$ gives us the representation of each $\zeta_{i,0}$ in terms of ξ, while $\{\zeta_{0,1}, \zeta_{1,1}\}$ gives the representation of each $\zeta_{i,1}$. With these recipes, (2.10) further produces the representation of each $\zeta_{i,j}$ in $\mathbb{Z}[\xi]$, thereby confirming (2.2). ∎

2.3. Modular relations for Λ and ξ. We begin with a relation connecting $\gamma(q^2)\delta(q)^2$ and $\xi(q)$.

THEOREM 2.6. *We have*

$$U(\gamma(q^2)\delta(q)^2) = 3\xi(q)^2 - 2\xi(q)^3. \tag{2.12}$$

Proof. This relation can be shown in a way similar to that for Theorem 2.3. ∎

Now we move to relations for Λ_k and ξ for each $k \geq 2$.

THEOREM 2.7. *For any $k \geq 2$,*

$$\Lambda_k \in \mathbb{Z}[\xi]. \tag{2.13}$$

More precisely, if we write

$$\Lambda_k := \sum_m c_k(m)\xi^m, \tag{2.14}$$

then

$$\Lambda_2 = 3\xi^2 - 2\xi^3, \tag{2.15}$$

and for $k \geq 3$, we recursively have

$$\Lambda_k = \sum_\ell c_{k-1}(\ell) \cdot \zeta_{2^{k-3},\ell}, \tag{2.16}$$

where $\zeta_{2^{k-3},\ell}$ is given by (2.8).

Proof. We begin with the proof of (2.15). It was already shown in [2, Theorem 21] that

$$\sum_{n=0}^{\infty} \mathrm{PDO}(2n)q^n = \delta(q)^2.$$

Thus,

$$\Lambda_2 = \gamma(q)\sum_{n=0}^{\infty} \mathrm{PDO}(4n)q^n = U\Big(\gamma(q^2)\sum_{n=0}^{\infty} \mathrm{PDO}(2n)q^n\Big) = U(\gamma(q^2)\delta(q)^2).$$

Invoking (2.12) gives the claimed expression for Λ_2.

For (2.16), we make use of the fact that, for $k \geq 3$,

$$\begin{aligned} \Lambda_k &= \gamma(q)^{2^{k-2}} \sum_{n=0}^{\infty} \mathrm{PDO}(2^k n) q^n = U\Big(\gamma(q^2)^{2^{k-2}} \sum_{n=0}^{\infty} \mathrm{PDO}(2^{k-1} n) q^n\Big) \\ &= U\bigg(\bigg(\frac{\gamma(q^2)^2}{\gamma(q)}\bigg)^{2^{k-3}} \gamma(q)^{2^{k-3}} \sum_{n=0}^{\infty} \mathrm{PDO}(2^{k-1} n) q^n\bigg) \\ &= U\bigg(\bigg(\frac{\gamma(q^2)^2}{\gamma(q)}\bigg)^{2^{k-3}} \Lambda_{k-1}\bigg). \end{aligned}$$

Noting that $\kappa = \gamma(q^2)^2/\gamma(q)$ and writing Λ_{k-1} in the above as a polynomial in ξ by virtue of (2.14), we finally obtain

$$\Lambda_k = \sum_{\ell} c_{k-1}(\ell) U(\kappa^{2^{k-3}} \xi^{\ell}).$$

Recalling (2.8), the required result follows. ∎

Example 2.8. We give a few examples to illustrate what Λ_k looks like when $k \geq 3$:

(1) In light of the recurrences in Section 2.2.2, it is plain that

$$\zeta_{1,2} = -15\xi^4 + 16\xi^5, \qquad \zeta_{1,3} = -27\xi^4 + 36\xi^5 - 8\xi^6.$$

Hence, from the relation that $\Lambda_2 = 3\xi^2 - 2\xi^3$, we have

$$\Lambda_3 = 3\zeta_{1,2} - 2\zeta_{1,3} = 3(-15\xi^4 + 16\xi^5) - 2(-27\xi^4 + 36\xi^5 - 8\xi^6),$$

which gives

$$\Lambda_3 = 9\xi^4 - 24\xi^5 + 16\xi^6. \tag{2.17}$$

(2) In the same vein,

$$\begin{aligned} \zeta_{2,4} &= -81\xi^7 + 594\xi^8 - 1024\xi^9 + 512\xi^{10}, \\ \zeta_{2,5} &= 405\xi^8 - 900\xi^9 + 496\xi^{10}, \\ \zeta_{2,6} &= 729\xi^8 - 1944\xi^9 + 1728\xi^{10} - 640\xi^{11} + 128\xi^{12}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Lambda_4 = -729\xi^7 + 7290\xi^8 - 18720\xi^9 + 20352\xi^{10} \\ - 10240\xi^{11} + 2048\xi^{12}. \end{aligned} \tag{2.18}$$

(3) With a lengthier computation, we have

$$\begin{aligned} \Lambda_5 = 34543665\xi^{14} - 400588416\xi^{15} + 2073171024\xi^{16} \\ - 6214952448\xi^{17} + 11906611200\xi^{18} - 15261990912\xi^{19} \\ + 13313703936\xi^{20} - 7841251328\xi^{21} + 2994733056\xi^{22} \\ - 671088640\xi^{23} + 67108864\xi^{24}. \end{aligned} \tag{2.19}$$

3. Minimal ξ-power in ζ. For all $i, j \geq 0$, let the coefficients $Z_{i,j}(m)$ with $m \geq 0$ be such that

$$\zeta_{i,j} := \sum_m Z_{i,j}(m)\xi^m.$$

Since $Z_{i,j}(m)$ eventually vanishes as $\zeta_{i,j} \in \mathbb{Z}[\xi]$, the above summation is indeed finite.

From the evaluations in Section 2.2, we have also seen that for each $\zeta_{i,j}$ as a polynomial in ξ, the terms ξ^m with a lower degree usually vanish. In the next theorem, we characterize the minimal ξ-power in the polynomial expression of $\zeta_{i,j}$.

THEOREM 3.1. *For any $i, j \geq 0$, define*

$$d_{i,j} := \begin{cases} 5I + J & \text{if } (i,j) = (2I, 2J), \\ 5I + J + 1 & \text{if } (i,j) = (2I, 2J+1), \\ 5I + J + 3 & \text{if } (i,j) = (2I+1, 2J), \\ 5I + J + 3 & \text{if } (i,j) = (2I+1, 2J+1). \end{cases}$$

Then for any m with $0 \leq m < d_{i,j}$, we have

$$Z_{i,j}(m) = 0.$$

Furthermore, the coefficient $Z_{i,j}(d_{i,j})$ is an odd integer.

Proof. Clearly, $\zeta_{0,0} = 1$, from which we find that $\zeta_{0,0}$ starts with the power ξ^0 with $Z_{0,0}(d_{0,0}) = Z_{0,0}(0) = 1$ being odd. We further know from (2.3) that $\zeta_{1,0}$ starts with $5\xi^3$, where $Z_{1,0}(d_{1,0}) = Z_{1,0}(3) = 5$ is also odd. Inductively, it follows from the recurrence (2.9) that $\zeta_{2I+2,0}$ starts with $Z_{2I+2,0}(5I+5)\xi^{5I+5}$ with

$$Z_{2I+2,0}(d_{2I+2,0}) = Z_{2I+2,0}(5I+5) = -Z_{2I,0}(5I),$$

which is an odd integer. Similarly, we deduce by the same recurrence that $\zeta_{2I+3,0}$ starts with $Z_{2I+3,0}(5I+8)\xi^{5I+8}$ with

$$Z_{2I+3,0}(d_{2I+3,0}) = Z_{2I+3,0}(5I+8) = 10Z_{2I+2,0}(5I+5) - Z_{2I+1,0}(5I+3),$$

again being odd.

In the same fashion, we find from (2.4) that $\zeta_{0,1}$ starts with $5\xi^1$ with an odd coefficient 5, and from (2.6) that $\zeta_{1,1}$ starts with $3\xi^3$ also with its coefficient 3 being odd. By induction under the rule of the recurrence (2.9), it follows that $\zeta_{2I+2,1}$ starts with $Z_{2I+2,1}(5I+6)\xi^{5I+6}$, where the coefficient is

$$Z_{2I+2,1}(5I+6) = 10Z_{2I+1,1}(5I+3) - Z_{2I,1}(5I+1),$$

and it is an odd integer. Also, $\zeta_{2I+3,1}$ starts with $Z_{2I+3,1}(5I+8)\xi^{5I+8}$, where the coefficient is the odd integer

$$Z_{2I+3,1}(5I+8) = -Z_{2I+1,1}(5I+3).$$

Now we have shown the desired result for each $\zeta_{i,0}$ and $\zeta_{i,1}$, and we will then apply induction on j. In light of the recurrence (2.10), we find that $\zeta_{2I,2J+2}$ starts with $Z_{2I,2J+2}(5I+J+1)\xi^{5I+J+1}$, where the coefficient is an odd integer given by

$$Z_{2I,2J+2}(5I+J+1) = -9Z_{2I,2J}(5I+J).$$

Likewise, $\zeta_{2I,2J+3}$ starts with $Z_{2I,2J+3}(5I+J+2)\xi^{5I+J+2}$, where the coefficient is an odd integer given by

$$Z_{2I,2J+3}(5I+J+2) = 10Z_{2I,2J+2}(5I+J+1) - 9Z_{2I,2J+1}(5I+J+1).$$

Moreover, $\zeta_{2I+1,2J+2}$ starts with $Z_{2I+1,2J+2}(5I+J+4)\xi^{5I+J+4}$, where the coefficient is

$$Z_{2I+1,2J+2}(5I+J+4) = 10Z_{2I+1,2J+1}(5I+J+3) - 9Z_{2I+1,2J}(5I+J+3),$$

which is odd. Meanwhile, $\zeta_{2I+1,2J+3}$ starts with $Z_{2I+1,2J+3}(5I+J+4)\xi^{5I+J+4}$, where the coefficient is

$$Z_{2I+1,2J+3}(5I+J+4) = -9Z_{2I+1,2J+1}(5I+J+3),$$

once again being odd. ∎

4. 2-Adic analysis for ζ. With all of the above preparations in hand, we are now in a position to begin thinking about the divisibility of various objects by powers of 2. Throughout the remainder of this work, we denote by $\nu(n)$ the 2-*adic evaluation* of n, that is, $\nu(n)$ is the *largest* nonnegative integer α such that $2^\alpha \mid n$. We also adopt the convention that $\nu(0) = \infty$.

The following trivial result on 2-adic evaluations will be frequently used:

Lemma 4.1. *Let a and b be two integers. Then*

$$\nu(a+b) \begin{cases} = \min\{\nu(a), \nu(b)\} & \text{if } \nu(a) \neq \nu(b), \\ \geq 2\nu(a) = 2\nu(b) & \text{if } \nu(a) = \nu(b). \end{cases} \tag{4.1}$$

Let us start by analyzing the 2-adic behavior of the coefficients $Z_{i,0}(m)$ for each $i \geq 0$ whenever $m \geq d_{i,0}$.

Theorem 4.2. *For any $i \geq 0$, we have*

$$\nu(Z_{i,0}(d_{i,0})) = 0 \quad \text{and} \quad \nu(Z_{i,0}(d_{i,0}+1)) \begin{cases} = 1 & \text{if } i \equiv 2 \pmod 4, \\ \geq 2 & \text{if } i \not\equiv 2 \pmod 4. \end{cases}$$

Furthermore, for $M \geq 2$,

$$\nu(Z_{i,0}(d_{i,0}+M)) \geq M+1.$$

Proof. We first note that the theorem is true for $i = 0, 1$ by the fact that $\zeta_{0,0} = 1$ and the relation given in (2.3), respectively. Hence, we may apply induction to prove the required result for $i = 4I+2, 4I+3, 4I+4, 4I+5$ with the assumption that it holds for $i = 4I+0, 4I+1$.

The $i = 4I + 2$ case can be illustrated by the following table:

$m - d_{4I+0,0}$	0	1	2	3	4	5	6	7	8	9	M
$\nu(Z_{4I+0,0}(m))$	0	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 9	≥ 10	$\geq M+1$
$\nu(Z_{4I+1,0}(m))$	∞	∞	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	$\geq M-2$
	∞	∞	∞	∞	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M-4$
	∞	∞	∞	∞	∞	∞	1	≥ 3	≥ 4	≥ 5	$\geq M-4$
$\nu(Z_{4I+2,0}(m))$	∞	∞	∞	∞	∞	0	1	≥ 3	≥ 4	≥ 5	$\geq M-4$

Here the last column holds for any $M \geq 10$. The recurrence (2.9) gives

$$\zeta_{4I+2,0} = (10\xi^3 - 40\xi^4 + 32\xi^5) \cdot \zeta_{4I+1,0} - (\xi^5) \cdot \zeta_{4I+0,0}.$$

Hence, the first line in the group for $\nu(Z_{4I+2,0}(m))$ provides the 2-adic evaluations of the coefficients $Z_{4I+2,0}$ contributed from $(\xi^5) \cdot \zeta_{4I+0,0}$, while the second line in this group provides the contribution from $(10\xi^3 - 40\xi^4 + 32\xi^5) \cdot \zeta_{4I+1,0}$. Combining the two lines then gives the required 2-adic evaluations $\nu(Z_{4I+2,0}(m))$ as shown in the third line of the group. For example, the 2-adic evaluation of the coefficient of the $\xi^{d_{4I+0,0}+5}$ term in $(\xi^5) \cdot \zeta_{4I+0,0}$ is clearly 0, while in $(10\xi^3 - 40\xi^4 + 32\xi^5) \cdot \zeta_{4I+1,0}$, the power $\xi^{d_{4I+0,0}+5}$ vanishes, thereby having coefficient 0 and 2-adic evaluation ∞. Consequently,

$$\nu(Z_{4I+2,0}(d_{4I+0,0} + 5)) = 0.$$

By Theorem 3.1, we have $d_{4I+2,0} = d_{4I+0,0} + 5$, so

$$\nu(Z_{4I+2,0}(d_{4I+2,0})) = 0.$$

Likewise, the 2-adic evaluation of the coefficient of the $\xi^{d_{4I+0,0}+6}$ term in $(\xi^5) \cdot \zeta_{4I+0,0}$ is ≥ 2, while in $(10\xi^3 - 40\xi^4 + 32\xi^5) \cdot \zeta_{4I+1,0}$, the corresponding 2-adic evaluation is 1. So we have

$$\nu(Z_{4I+2,0}(d_{4I+0,0} + 6)) = \min\{\geq 2, 1\} = 1,$$

thereby yielding

$$\nu(Z_{4I+2,0}(d_{4I+2,0} + 1)) = 1.$$

This process can be continued to all remaining coefficients $Z_{4I+2,0}$.

For $i = 4I + 3$, we shall make use of the following table and argue in the same vein:

$m - d_{4I+1,0}$	0	1	2	3	4	5	6	7	8	9	M
$\nu(Z_{4I+1,0}(m))$	0	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 9	≥ 10	$\geq M+1$
$\nu(Z_{4I+2,0}(m))$	∞	∞	0	1	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	$\geq M-1$
	∞	∞	∞	∞	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M-4$
	∞	∞	∞	∞	∞	1	2	≥ 4	≥ 5	≥ 6	$\geq M-3$
$\nu(Z_{4I+3,0}(m))$	∞	∞	∞	∞	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M-4$

For $i = 4I + 4$, the required table is

$m - d_{4I+2,0}$	0	1	2	3	4	5	6	7	8	9	M
$\nu(Z_{4I+2,0}(m))$	0	1	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 9	≥ 10	$\geq M+1$
$\nu(Z_{4I+3,0}(m))$	∞	∞	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	$\geq M-2$
	∞	∞	∞	∞	∞	0	1	≥ 3	≥ 4	≥ 5	$\geq M-4$
	∞	∞	∞	∞	∞	∞	1	≥ 3	≥ 4	≥ 5	$\geq M-4$
$\nu(Z_{4I+4,0}(m))$	∞	∞	∞	∞	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M-4$

It is notable that for the 2-adic evaluation of

$$\nu(Z_{4I+4,0}(d_{4I+4,0}+1)) = \nu(Z_{4I+4,0}(d_{4I+2,0}+6)),$$

we shall use the second case of (4.1) to get

$$\nu(Z_{4I+4,0}(d_{4I+2,0}+6)) \geq 2 \cdot 1 = 2,$$

as given in the table.

Finally, for $i = 4I + 5$, we require this table:

$m - d_{4I+3,0}$	0	1	2	3	4	5	6	7	8	9	M
$\nu(Z_{4I+3,0}(m))$	0	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 9	≥ 10	$\geq M+1$
$\nu(Z_{4I+4,0}(m))$	∞	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	$\geq M-1$
	∞	∞	∞	∞	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M-4$
	∞	∞	∞	∞	∞	1	≥ 3	≥ 4	≥ 5	≥ 6	$\geq M-3$
$\nu(Z_{4I+5,0}(m))$	∞	∞	∞	∞	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M-4$

and then perform a similar analysis to that for the above $i = 4I + 2$ case. ∎

In the same fashion, we have parallel results for $Z_{i,1}(m)$.

Theorem 4.3. *For any $i \geq 0$, we have*

$$\nu(Z_{i,1}(d_{i,1})) = 0 \quad \textit{and} \quad \nu(Z_{i,1}(d_{i,1}+1)) \begin{cases} = 1 & \textit{if } i \equiv 1 \pmod 4, \\ \geq 2 & \textit{if } i \not\equiv 1 \pmod 4. \end{cases}$$

Furthermore, for $M \geq 2$,

$$\nu(Z_{i,1}(d_{i,1}+M)) \geq M+1.$$

Proof. By (2.4) and (2.6), the theorem holds true for $i = 0, 1$. We may then apply a similar inductive argument to that for Theorem 4.2. ∎

Now we are ready to perform the 2-adic evaluations for the coefficients $Z_{2^k,j}(m)$ for each $k \geq 0$ and $j \geq 0$ whenever $m \geq d_{2^k,j}$.

THEOREM 4.4. *For any $k \geq 2$ and $j \geq 0$, we have*

$$\nu(Z_{2^k,j}(d_{2^k,j})) = 0$$

and

$$\nu(Z_{2^k,j}(d_{2^k,j}+1)) \begin{cases} = 1 & \text{if } j \equiv 2 \pmod 4, \\ \geq 2 & \text{if } j \not\equiv 2 \pmod 4. \end{cases}$$

Furthermore, for $M \geq 2$,

$$\nu(Z_{2^k,j}(d_{2^k,j}+M)) \geq M+1.$$

Proof. In view of Theorems 4.2 and 4.3, it is known that the results are true for $Z_{2^k,0}(m)$ and $Z_{2^k,1}(m)$ with any $k \geq 2$. Now we apply induction on j and argue for $j = 4J+2, 4J+3, 4J+4, 4J+5$ under the assumption of validity for $j = 4J+0, 4J+1$. Here a similar strategy to that for Theorem 4.2 will be used, with (2.10) being invoked:

$$\zeta_{2^k,j} = (10\xi - 8\xi^2)\cdot\zeta_{2^k,j-1} - (9\xi - 8\xi^2)\cdot\zeta_{2^k,j-2}.$$

For $j = 4J+2$, we require this table:

$m - d_{2^k,4J+0}$	0	1	2	3	4	5	M
$\nu(Z_{2^k,4J+0}(m))$	0	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6	$\geq M+1$
$\nu(Z_{2^k,4J+1}(m))$	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M+0$
	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M+0$
	∞	∞	1	≥ 3	≥ 4	≥ 5	$\geq M+0$
$\nu(Z_{2^k,4J+2}(m))$	∞	0	1	≥ 3	≥ 4	≥ 5	$\geq M+0$

For $j = 4J+3$, we require this table:

$m - d_{2^k,4J+1}$	0	1	2	3	4	5	M
$\nu(Z_{2^k,4J+1}(m))$	0	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6	$\geq M+1$
$\nu(Z_{2^k,4J+2}(m))$	0	1	≥ 3	≥ 4	≥ 5	≥ 6	$\geq M+1$
	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M+0$
	∞	1	2	≥ 4	≥ 5	≥ 6	$\geq M+1$
$\nu(Z_{2^k,4J+3}(m))$	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M+0$

For $j = 4J+4$, we require this table:

$m - d_{2^k,4J+2}$	0	1	2	3	4	5	M
$\nu(Z_{2^k,4J+2}(m))$	0	1	≥ 3	≥ 4	≥ 5	≥ 6	$\geq M+1$
$\nu(Z_{2^k,4J+3}(m))$	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M+0$
	∞	0	1	≥ 3	≥ 4	≥ 5	$\geq M+0$
	∞	∞	1	≥ 3	≥ 4	≥ 5	$\geq M+0$
$\nu(Z_{2^k,4J+4}(m))$	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M+0$

For $j = 4J + 5$, we require this table:

$m - d_{2^k,4J+3}$	0	1	2	3	4	5	M
$\nu(Z_{2^k,4J+3}(m))$	0	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6	$\geq M + 1$
$\nu(Z_{2^k,4J+4}(m))$	0	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6	$\geq M + 1$
	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M + 0$
	∞	1	≥ 3	≥ 4	≥ 5	≥ 6	$\geq M + 1$
$\nu(Z_{2^k,4J+5}(m))$	∞	0	≥ 2	≥ 3	≥ 4	≥ 5	$\geq M + 0$

Concrete analyses can be mimicked by consulting the $i = 4I + 2$ case in the proof of Theorem 4.2, and we will omit the details. ∎

5. New auxiliary functions and the associated minimal ξ-powers. All of the work above has revolved around the generating function for the function $\mathrm{PDO}(n)$. However, we keep in mind that Theorem 1.1 is really focused on the internal congruences for the PDO function. To capture this nature, let us introduce a new family of auxiliary functions for $k \geq 3$:

$$\Phi_k = \Phi_k(q) := \gamma(q)^{2^k} \Big(\sum_{n=0}^{\infty} \mathrm{PDO}(2^{k+2}n) q^n - \sum_{n=0}^{\infty} \mathrm{PDO}(2^k n) q^n \Big).$$

In light of (1.6), we have

$$\Phi_k = \Lambda_{k+2} - \gamma^{3 \cdot 2^{k-2}} \Lambda_k = \Lambda_{k+2} - (\gamma^6)^{2^{k-3}} \Lambda_k.$$

THEOREM 5.1. *For any $k \geq 3$,*

$$\Phi_k \in \mathbb{Z}[\xi]. \tag{5.1}$$

More precisely, if we write

$$\Phi_k := \sum_m F_k(m) \xi^m, \tag{5.2}$$

then

$$\begin{aligned} \Phi_3 = {} & 34012224\xi^{14} - 396809280\xi^{15} + 2061728640\xi^{16} \\ & - 6195823488\xi^{17} + 11887534080\xi^{18} - 15250636800\xi^{19} \\ & + 13309968384\xi^{20} - 7840727040\xi^{21} + 2994733056\xi^{22} \\ & - 671088640\xi^{23} + 67108864\xi^{24}, \end{aligned} \tag{5.3}$$

and for $k \geq 4$, we recursively have

$$\Phi_k = \sum_\ell F_{k-1}(\ell) \cdot \zeta_{2^{k-1},\ell}, \tag{5.4}$$

where $\zeta_{2^{k-1},\ell}$ is given by (2.8).

Proof. Since both Λ_k and Λ_{k+2} are in $\mathbb{Z}[\xi]$ by (2.13), while γ^6 is also in $\mathbb{Z}[\xi]$ as we have shown in (2.1), it follows that $\Phi_k \in \mathbb{Z}[\xi]$. In particular, $\Phi_3 = \Lambda_5 - \gamma^6\Lambda_3$. Applying (2.17) and (2.19) together with (2.1) gives (5.3). For (5.4), we note that

$$\begin{aligned}\Phi_k &= U\Big(\gamma(q^2)^{2^k}\Big(\sum_{n=0}^{\infty}\mathrm{PDO}(2^{k+1}n)q^n - \sum_{n=0}^{\infty}\mathrm{PDO}(2^{k-1}n)q^n\Big)\Big)\\ &= U\bigg(\bigg(\frac{\gamma(q^2)^2}{\gamma(q)}\bigg)^{2^{k-1}}\gamma(q)^{2^{k-1}}\Big(\sum_{n=0}^{\infty}\mathrm{PDO}(2^{k+1}n)q^n - \sum_{n=0}^{\infty}\mathrm{PDO}(2^{k-1}n)q^n\Big)\bigg)\\ &= U(\kappa^{2^{k-1}}\Phi_{k-1}).\end{aligned}$$

Finally, we recall (5.2) and invoke (2.8) to obtain the desired recurrence. ∎

Next, in analogy to Theorem 3.1, we show that the lower powers of ξ in each Φ_k shall vanish.

THEOREM 5.2. *For any $k \geq 3$, define*

$$\tau_k := \begin{cases} 7\cdot 2^{2K-3} - \frac{2}{3}(4^{K-2}-1) & \text{if } k = 2K-1,\\ 7\cdot 2^{2K-2} - \frac{1}{3}(4^{K-1}-1) & \text{if } k = 2K.\end{cases}$$

Then for any m with $0 \leq m < \tau_k$, we have

$$F_k(m) = 0.$$

REMARK 5.3. A straightforward computation reveals that

$$\tau_{2K-1} \equiv 2 \pmod 4, \tag{5.5}$$

$$\tau_{2K} \equiv 3 \pmod 4, \tag{5.6}$$

for any choice of $K \geq 2$.

Proof of Theorem 5.2. From (5.3), we see that Φ_3 starts with the power ξ^{14}, while

$$14 = 7\cdot 2^1 - \tfrac{2}{3}(4^0 - 1).$$

Now we assume that the theorem is true for a certain $k = 2K-1$, and we proceed inductively for $k = 2K$ and $k = 2K+1$.

For $k = 2K$, we know from (5.4) that

$$\Phi_{2K} = F_{2K-1}(\tau_{2K-1})\cdot \zeta_{2^{2K-1},\tau_{2K-1}} + \cdots. \tag{5.7}$$

By virtue of Theorem 3.1, $\zeta_{2^{2K-1},\tau_{2K-1}}$ starts with the power ξ^d where

$$\begin{aligned}d = d_{2^{2K-1},\tau_{2K-1}} &= 5\cdot 2^{2K-2} + \tfrac{1}{2}\tau_{2K-1} = 5\cdot 2^{2K-2} + 7\cdot 2^{2K-4} - \tfrac{1}{3}(4^{K-2}-1)\\ &= 7\cdot 2^{2K-2} - \tfrac{1}{3}(4^{K-1}-1) = \tau_{2K}.\end{aligned}$$

Here we make use of the fact that both 2^{2K-1} and τ_{2K-1} are even. Further, each of the $\zeta \in \mathbb{Z}[\xi]$ components in the remaining summands in (5.7) starts with a power higher than ξ^d. Hence, Φ_{2K} starts with *at least* $\xi^{\tau_{2K}}$.

For $k = 2K + 1$, we also deduce from (5.4) that

$$\Phi_{2K+1} = F_{2K}(\tau_{2K}) \cdot \zeta_{2^{2K},\tau_{2K}} + F_{2K}(\tau_{2K}+1) \cdot \zeta_{2^{2K},\tau_{2K}+1} + \cdots . \tag{5.8}$$

Invoking Theorem 3.1 and noting that τ_{2K} is odd, it follows that $\zeta_{2^{2K},\tau_{2K}}$ starts with the power ξ^d where

$$\begin{aligned} d = d_{2^{2K},\tau_{2K}} &= 5 \cdot 2^{2K-1} + \tfrac{1}{2}(\tau_{2K} - 1) + 1 \\ &= 5 \cdot 2^{2K-1} + 7 \cdot 2^{2K-3} - \tfrac{1}{6}(4^{K-1} - 1) + \tfrac{1}{2} \\ &= 7 \cdot 2^{2K-1} - \tfrac{2}{3}(4^{K-1} - 1) = \tau_{2K+1}. \end{aligned}$$

For the second summand in (5.8), we find that $\zeta_{2^{2K},\tau_{2K}+1}$ starts with the power $\xi^{d'}$ where

$$d' = d_{2^{2K},\tau_{2K}+1} = 5 \cdot 2^{2K-1} + \tfrac{1}{2}(\tau_{2K} + 1) = \tau_{2K+1},$$

according to a similar computation. Furthermore, each of the $\zeta \in \mathbb{Z}[\xi]$ components in the remaining summands in (5.8) starts with a power higher than $\xi^d = \xi^{d'}$. Hence, we can claim that Φ_{2K+1} starts with *at least* $\xi^{\tau_{2K+1}}$. ∎

6. 2-Adic analysis for Φ. We are now in a position to prove our main result, Theorem 1.1. To do so, we only need to confirm that for each $K \geq 1$,

$$\nu(F_{2K+1}(m)) \geq 2K + 3$$

whenever $m \geq \tau_{2K+1}$. In what follows, we first manually analyze the initial cases where $K \in \{1, 2\}$ and then move on to general K by induction.

6.1. Initial cases. Recall that Φ_3 was already given in (5.3). Further, we may obtain an explicit expression in $\mathbb{Z}[\xi]$ (containing 43 terms!) for Φ_5 by applying the recurrence (5.4) twice. Consequently, we get the following 2-adic evaluations, where τ stands for $\tau_3 = 14$ or $\tau_5 = 54$ accordingly:

$m - \tau$	0	1	2	M
$\nu(F_3(m))$	6	6	7	$\geq M + 4$
$\nu(F_5(m))$	7	7	9	$\geq M + 8$

In the above table, the last column is true for all $M \geq 3$. So we indeed have the following two congruences, with the former being even stronger than the general scenario in Theorem 1.1:

$$\mathrm{PDO}(2^3 n) \equiv \mathrm{PDO}(2^5 n) \pmod{2^6}, \tag{6.1}$$

$$\mathrm{PDO}(2^5 n) \equiv \mathrm{PDO}(2^7 n) \pmod{2^7}. \tag{6.2}$$

6.2. Induction. Now we perform induction on $K \geq 2$ and establish the following lower bounds for the 2-adic evaluations.

THEOREM 6.1. *For any* $K \geq 2$,

$$\nu(F_{2K+1}(\tau_{2K+1})) \geq 2K+3, \tag{6.3}$$

and for $M \geq 1$,

$$\nu(F_{2K+1}(\tau_{2K+1}+M)) \geq 2K+M+2. \tag{6.4}$$

Note that the $K=2$ case was already covered in Section 6.1. So in what follows, we first prove (6.4) for $K \geq 3$ under the inductive assumption that it is true for $K-1$. For convenience, let us denote

$$\tau'' := \tau_{2K-1}, \qquad \tau' := \tau_{2K}, \qquad \tau := \tau_{2K+1}.$$

Also, we write

$$\{Z''(j,m) : j,m \geq 0\} := \{Z_{2^{2K-1},j}(m) : j,m \geq 0\},$$
$$\{Z'(j,m) : j,m \geq 0\} := \{Z_{2^{2K},j}(m) : j,m \geq 0\}.$$

Proof of (6.4). To produce Φ_{2K+1} from Φ_{2K-1}, we need to employ the recurrence (5.4) twice. Writing

$$\Phi_{2K-1} = \sum_{M'' \geq 0} F_{2K-1}(\tau''+M'')\xi^{\tau''+M''},$$

we deduce from (5.4) that

$$\Phi_{2K} = \sum_{M'' \geq 0} F_{2K-1}(\tau''+M'')\zeta_{2^{2K-1},\tau''+M''}.$$

Recall that τ'' is even according to (5.5). It follows from Theorems 3.1 and 5.2 that the minimal degree of the ξ-powers in $\zeta_{2^{2K-1},\tau''+M''}$ is $\tau'+\lfloor (M''+1)/2 \rfloor$, i.e.,

$$d_{2^{2K-1},\tau''+M''} = \tau' + \lfloor (M''+1)/2 \rfloor. \tag{6.5}$$

Hence,

$$\Phi_{2K} = \sum_{\substack{M'' \geq 0 \\ M' \geq \lfloor (M''+1)/2 \rfloor}} F_{2K-1}(\tau''+M'')Z''(\tau''+M'',\tau'+M')\xi^{\tau'+M'}.$$

Applying the recurrence (5.4) once again gives

$$\Phi_{2K+1} = \sum_{\substack{M'' \geq 0 \\ M' \geq \lfloor (M''+1)/2 \rfloor}} F_{2K-1}(\tau''+M'')Z''(\tau''+M'',\tau'+M')\zeta_{2^{2K},\tau'+M'}.$$

Noting that τ' is odd according to (5.6), Theorems 3.1 and 5.2 then imply that the minimal degree of the ξ-powers in $\zeta_{2^{2K},\tau'+M'}$ is $\tau + \lfloor M'/2 \rfloor$, i.e.,

$$d_{2^{2K},\tau'+M'} = \tau' + \lfloor M'/2 \rfloor. \tag{6.6}$$

Hence,

$$\Phi_{2K+1} = \sum_{\substack{M''\geq 0\\ M'\geq\lfloor (M''+1)/2\rfloor\\ M\geq \lfloor M'/2\rfloor}} C(M,M'',M'')\xi^{\tau+M},$$

where

$$C(M,M'',M'') \\ := F_{2K-1}(\tau''+M'')Z''(\tau''+M'',\tau'+M')Z'(\tau'+M',\tau+M).$$

To complete the inductive step, it is sufficient to show that

$$\nu(C(M,M'',M'')) \geq 2K+M+2 \tag{6.7}$$

for every tuple (M,M'',M'') such that $M''\geq 0$, $M'\geq \lfloor (M''+1)/2\rfloor$ and $M\geq \lfloor M'/2\rfloor$.

We first consider two generic cases: (i) $M''\geq 2$ and (ii) $M''=0$ or 1 and $M'\geq 3$. By our inductive assumption,

$$\nu(F_{2K-1}(\tau''+M''))\geq 2K+M''.$$

Also, by a weaker form of Theorem 4.4 with recourse to (6.5) and (6.6), respectively, we have

$$\nu(Z''(\tau''+M'',\tau'+M'))\geq M'-\left\lfloor \tfrac{M''+1}{2}\right\rfloor,$$
$$\nu(Z'(\tau'+M',\tau+M))\geq M-\left\lfloor \tfrac{M'}{2}\right\rfloor.$$

It follows that

$$\nu(C(M,M'',M''))\geq 2K+M+\left(M''-\left\lfloor \tfrac{M''+1}{2}\right\rfloor\right)+\left(M'-\left\lfloor \tfrac{M'}{2}\right\rfloor\right).$$

Now (i) if $M''\geq 2$, then $M'\geq 1$ so that $M''-\lfloor (M''+1)/2\rfloor\geq 1$ and $M'-\lfloor M'/2\rfloor\geq 1$; (ii) if $M''=0$ or 1 and $M'\geq 3$, then $M''-\lfloor (M''+1)/2\rfloor = 0$ but $M'-\lfloor M'/2\rfloor\geq 2$. Thus, in both cases the inequality (6.7) holds.

Now there are five sporadic cases to be investigated, and we work on them separately. Note that for (6.4), we have the condition that $M\geq 1$.

(a) $(M',M'')=(0,0)$: In this case, by the inductive assumption,

$$\nu(F_{2K-1}(\tau''+0))\geq 2K+1;$$

by Theorem 4.4,

$$\nu(Z''(\tau''+0,\tau'+0))=0;$$

and also by Theorem 4.4 (noting that $\tau'+0\equiv 3 \pmod 4$),

$$\nu(Z'(\tau'+0,\tau+M))\geq M+1.$$

(b) $(M',M'')=(1,0)$: In this case, by the inductive assumption,

$$\nu(F_{2K-1}(\tau''+0))\geq 2K+1;$$

by Theorem 4.4 (noting that $\tau'' + 0 \equiv 2 \pmod 4$),

$$\nu(Z''(\tau'' + 0, \tau' + 1)) = 1;$$

and also by Theorem 4.4 (noting that $\tau' + 1 \equiv 0 \pmod 4$),

$$\nu(Z'(\tau' + 1, \tau + M)) \geq M + 1.$$

(c) $(M', M'') = (2, 0)$: In this case, by the inductive assumption,

$$\nu(F_{2K-1}(\tau'' + 0)) \geq 2K + 1;$$

by Theorem 4.4,

$$\nu(Z''(\tau'' + 0, \tau' + 2)) \geq 3;$$

and by a weaker form of Theorem 4.4,

$$\nu(Z'(\tau' + 2, \tau + M)) \geq M - 1.$$

(d) $(M', M'') = (1, 1)$: In this case, by the inductive assumption,

$$\nu(F_{2K-1}(\tau'' + 1)) \geq 2K + 1;$$

by Theorem 4.4,

$$\nu(Z''(\tau'' + 1, \tau' + 1)) = 0;$$

and also by Theorem 4.4 (noting that $\tau' + 1 \equiv 0 \pmod 4$),

$$\nu(Z'(\tau' + 1, \tau + M)) \geq M + 1.$$

(e) $(M', M'') = (2, 1)$: In this case, by the inductive assumption,

$$\nu(F_{2K-1}(\tau'' + 1)) \geq 2K + 1;$$

by Theorem 4.4 (noting that $\tau'' + 1 \equiv 3 \pmod 4$),

$$\nu(Z''(\tau'' + 1, \tau' + 2)) \geq 2;$$

and by a weaker form of Theorem 4.4,

$$\nu(Z'(\tau' + 2, \tau + M)) \geq M - 1.$$

We find that in each of these cases, inequality (6.7) is also valid, thereby closing our proof of (6.4). ■

Finally, for (6.3), we shall make use of completely different logic without consulting the relevant recurrences.

Proof of (6.3). Recall that we have shown in the proof of (6.4) that the 2-adic evaluations satisfy

$$\nu(F_{2K+1}(\tau_{2K+1} + M)) \geq 2K + 3$$

whenever $M \geq 1$. Modulo 2^{2K+3}, we have

$$\Phi_{2K+1} \equiv F_{2K+1}(\tau_{2K+1}) \cdot \xi^{\tau_{2K+1}} \pmod{2^{2K+3}}. \tag{6.8}$$

Note that the constant term on the left-hand side is 0 since

$$\Phi_{2K+1} = \gamma(q)^{2^{2K+1}}\Big(\sum_{n=0}^{\infty} \mathrm{PDO}(2^{2K+3}n)q^n - \sum_{n=0}^{\infty} \mathrm{PDO}(2^{2K+1}n)q^n\Big),$$

while within the parentheses it is clear that

$$\mathrm{PDO}(2^{2K+3}\cdot 0)q^0 - \mathrm{PDO}(2^{2K+1}\cdot 0)q^0 = 0.$$

Further, the constant term on the right-hand side of (6.8) is $F_{2K+1}(\tau_{2K+1})$ as the constant term of $\xi(q)$ is 1. Since the congruence (6.8) is valid, it must be the case that their constant terms are also congruent under the same modulus, so

$$F_{2K+1}(\tau_{2K+1}) \equiv 0 \pmod{2^{2K+3}},$$

which is equivalent to (6.3). ■

7. Conclusion. We close with two sets of comments. First, we remind the reader of one of the original goals for proving such internal congruences. As was mentioned above, the second author [15] used two corollaries of Theorem 1.1 to prove the following Ramanujan-like congruences by induction on α:

THEOREM 7.1. *For all $\alpha \geq 0$ and all $n \geq 0$,*

$$\mathrm{PDO}(2^{\alpha}(4n+3)) \equiv 0 \pmod{4},$$
$$\mathrm{PDO}(2^{\alpha}(8n+7)) \equiv 0 \pmod{8}.$$

It would be gratifying to see other cases of Theorem 1.1 used to assist in proving divisibility properties satisfied by $\mathrm{PDO}(n)$ for higher powers of 2.

Secondly, in the course of proving congruences modulo arbitrary powers for the coefficients of an eta-product

$$\mathrm{H}(q) = \sum_{n=0}^{\infty} h(n)q^n,$$

the usual strategy is to find a suitable basis $\{\xi_1, \dots, \xi_L\}$ of the corresponding modular space such that each dissection slice, accompanied by a certain multiplier (usually an eta-product),

$$\lambda_m \sum_{n=0}^{\infty} h(p^m n + t_m)q^n,$$

can be represented as a polynomial in $\mathbb{Z}[\xi_1, \dots, \xi_L]$. For example, when proving the congruences modulo powers of 5 for the partition function [17] or the congruences modulo powers of 7 for the distinct partition function [9], two specific multipliers typically take turns showing up, i.e., $\lambda_{2M-1} = \lambda$ and $\lambda_{2M} = \lambda'$ for two certain series λ and λ'. However, in our study here, the multipliers $\gamma, \gamma^2, \gamma^4, \gamma^8, \ldots$ never overlap. Further, an important outcome of cycling the multipliers in the previous studies is that it is typically sufficient to represent each degree p unitization $U_p(\kappa^i \xi^j)$ as a polynomial in ξ for a certain series κ, with the exponent i restricted to $\{0, 1\}$; here we use the case where the basis of the modular space is given by $\{\xi\}$ as an illustration. However, as shown in Section 2.2, when there are endless possibilities for the multipliers, we have to extend the consideration of i to infinity, thereby substantially increasing the amount of required p-adic analysis. These striking facts distinguish our work from the past literature.

References

[1] P. Adansie, S. Chern, and E. X. W. Xia, *New infinite families of congruences for the number of tagged parts over partitions with designated summands*, Int. J. Number Theory 14 (2018), 1935–1942.

[2] G. E. Andrews, R. P. Lewis, and J. Lovejoy, *Partitions with designated summands*, Acta Arith. 105 (2002), 51–66.

[3] N. D. Baruah and M. Kaur, *A note on some recent results of da Silva and Sellers on congruences for k-regular partitions with designated summands*, Integers 20 (2020), art. A74, 6 pp.

[4] N. D. Baruah and K. K. Ojah, *Partitions with designated summands in which all parts are odd*, Integers 15 (2015), art. A9, 16 pp.

[5] W. Y. C. Chen, K. Q. Ji, H.-T. Jin, and E. Y. Y. Shen, *On the number of partitions with designated summands*, J. Number Theory 133 (2013), 2929–2938.

[6] R. da Silva and J. A. Sellers, *New congruences for 3-regular partitions with designated summands*, Integers 20A (2020), art. A6, 11 pp.

[7] R. da Silva and J. A. Sellers, *Infinitely many congruences for k-regular partitions with designated summands*, Bull. Braz. Math. Soc. (N.S.) 51 (2020), 357–370.

[8] F. Garvan, *A q-product tutorial for a q-series Maple package*, Sém. Lothar. Combin. 42 (1999), art. B42d, 27 pp.

[9] B. Gordon and K. Hughes, *Ramanujan congruences for $q(n)$*, in: Analytic Number Theory (Philadelphia, PA, 1980), Lecture Notes in Math. 899, Springer, Berlin, 1981, 333–359.

[10] B. Hemanthkumar, H. S. Sumanth Bharadwaj, and M. S. Mahadeva Naika, *Congruences modulo small powers of 2 and 3 for partitions into odd designated summands*, J. Integer Seq. 20 (2017), art. 17.4.3, 25 pp.

[11] D. Herden, M. R. Sepanski, J. Stanfill, C. C. Hammon, J. Henningsen, H. Ickes, and I. Ruiz, *Partitions with designated summands not divisible by 2^ℓ, 2, and 3^ℓ modulo 2, 4, and 3*, Integers 23 (2023), art. A43, 23 pp.

[12] M. S. Mahadeva Naika and D. S. Gireesh, *Congruences for 3-regular partitions with designated summands*, Integers 16 (2016), art. A25, 14 pp.

[13] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series*, Amer. Math. Soc., Providence, RI, 2004.

[14] C.-S. Radu, *An algorithmic approach to Ramanujan–Kolberg identities*, J. Symbolic Comput. 68 (2015), Part 1, 225–253.

[15] J. A. Sellers, *New infinite families of congruences modulo powers of 2 for 2-regular partitions with designated summands*, Integers 24 (2024), art. A16, 17 pp.

[16] N. A. Smoot, *On the computation of identities relating partition numbers in arithmetic progressions with eta quotients: an implementation of Radu's algorithm*, J. Symbolic Comput. 104 (2021), 276–311.

[17] G. N. Watson, *Ramanujans Vermutung über Zerfällungszahlen*, J. Reine Angew. Math. 179 (1938), 97–128.

[18] E. X. W. Xia, *Arithmetic properties of partitions with designated summands*, J. Number Theory 159 (2016), 160–175.

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