ELEMENTARY PROOFS OF CONGRUENCES FOR DRAKE'S VARIANT OF 2-COLORED GENERALIZED FROBENIUS PARTITIONS

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Abstract. In his 1984 AMS Memoir, George Andrews defined the family of k–colored generalized Frobenius partition functions. These functions count two-rowed arrays of positive integers of equal length which naturally generalize Frobenius symbols of integer partitions. These are denoted by $c\phi_k(n)$ where $k \geq 1$ is the number of colors in question. In that Memoir, Andrews proved (among many other things) that, for all $n \geq 0$, $c\phi_2(5n+3) \equiv 0 \pmod 5$. Soon after, many authors proved congruence properties for various k–colored generalized Frobenius partition functions.

In 2009, Drake considered a further generalization of the Frobenius symbol and defined a new family of colored generalized Frobenius partition functions which count variants of the two–rowed arrays that Andrews defined in 1984. The key difference between this new family of functions and those defined by Andrews is that the variant allows for the rows of each array to be of different length.

In this work, we focus specifically on Drake's 2–colored generalized Frobenius partition function, and we prove a number of congruences satisfied by this particular function. Our proofs are truly elementary, relying on the corresponding generating function, classical q–series results, and elementary generating function manipulations.

1. Introduction

A partition λ of a positive integer n is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. We denote the number of partitions of n by the function $p(n)$.

The Ferrers graph associated with a partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ is generally represented as a set of left–justified rows of dots where the i^{th} row contains λ_i dots. For example, the Ferrers graph of the partition $7+7+5+3+2+2$ is given by the following:

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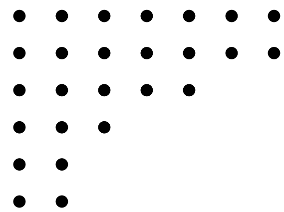

Given the Ferrers graph of a partition, note that the rows of dots strictly to the right of the diagonal elements can be enumerated to provide one strictly decreasing sequence of nonnegative integers. The remaining dots strictly below the main diagonal can be enumerated by columns to provide a second strictly decreasing sequence of nonnegative integers. The resulting two sequences are then written in the form of a two–rowed array, which we call the Frobenius symbol of the corresponding partition. For example, the partition $7+7+5+3+2+2$ of 26 mentioned above is represented by the Frobenius symbol

$$\begin{pmatrix} 6 & 5 & 2 \\ 5 & 4 & 1 \end{pmatrix}.$$

In his 1984 AMS Memoir, George Andrews [1] generalized this idea of the Frobenius symbol by defining the family of k–colored generalized Frobenius partition functions which are enumerated by $c\phi_k(n)$ where $k \geq 1$ is the number of colors in question. These combinatorial objects serve as a natural generalization of ordinary integer partitions. We provide a brief explanation here.

Consider k copies of the nonnegative integers written j_i where $j \geq 0$ and $1 \leq i \leq k$ (where we think of the subscript i as the "color" assigned to the part of size j). We then say that $j_i < l_m$ precisely when $j < l$ or $j = l$ and $i < m$. Moreover, j_i is equal to l_m if and only if $j = l$ and $i = m$.

We then say that the function $c\phi_k(n)$ counts the number of generalized Frobenius partitions of n under the conditions that the parts are decreasing (using the ordering above). Note that, for all n, $c\phi_1(n) = p(n)$.

For example, $c\phi_2(3) = 20$ thanks to the following generalized Frobenius symbols with 2 colors:

$$\begin{pmatrix} 1_1 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 1_1 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 1_2 \end{pmatrix},$$

$$\begin{pmatrix} 2_1 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 2_2 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 2_1 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 2_2 \\ 0_2 \end{pmatrix},$$

$$\begin{pmatrix} 0_1 \\ 2_1 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 2_2 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 2_1 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 2_2 \end{pmatrix},$$

$$\begin{pmatrix} 1_1 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}, \begin{pmatrix} 1_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}, \begin{pmatrix} 1_1 & 0_2 \\ 0_2 & 0_1 \end{pmatrix}, \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & 0_1 \end{pmatrix},$$

$$\begin{pmatrix} 0_2 & 0_1 \\ 1_1 & 0_1 \end{pmatrix}, \begin{pmatrix} 0_2 & 0_1 \\ 1_2 & 0_1 \end{pmatrix}, \begin{pmatrix} 0_2 & 0_1 \\ 1_1 & 0_2 \end{pmatrix}, \begin{pmatrix} 0_2 & 0_1 \\ 1_2 & 0_2 \end{pmatrix}$$

Among many things, Andrews [1, Corollary 10.1] proved that, for all $n \geq 0$, $c\phi_2(5n+3) \equiv 0 \pmod 5$. Since then, numerous authors have proved similar congruence properties for various k–colored generalized Frobenius partition functions, typically for a small number of colors k. See, for example, [2, 3, 5, 6, 7, 8, 9, 10, 14, 15, 16, 18, 20, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

In 2009, Drake [13] considered a further generalization of the Frobenius symbol and defined a new family of colored generalized Frobenius partition functions (which count variants of the two–rowed arrays that Andrews defined in 1984).

Very recently, Jiang, Rolen, and Woodbury [21] synthesized the definitions of generalized Frobenius partitions of Andrews and Drake into the following:

Definition 1.1. *(Jiang–Rolen–Woodbury) Let n, k be positive integers and $\beta \in \mathbb{Z} + \frac{k}{2}$ be nonnegative. A (k, β)-colored generalized Frobenius partition of n is an array of the form*

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_s \end{pmatrix}$$

satisfying the following:

- *Each a_i, b_j is a nonnegative integer, and any integer occurs at most k times,*
- *Each row is decreasing with respect to lexicographic ordering,*
- $r + s \neq 0$
- $r - s = \beta - \frac{k}{2}$
- $n = r + \sum_{1 \leq i \leq r} a_i + \sum_{1 \leq j \leq s} b_j.$

The number of such arrays is denoted by $c\psi_{k,\beta}(n)$, and the associated generating function in the variable q is denoted $\mathrm{C}\Psi_{k,\beta}(q)$.

Our focus in this paper is on the instances where $k = 2$, and in those cases, we have explicit expressions for the generating functions in question. Thanks to Andrews [1], we know

$$\mathrm{C}\Psi_{2,1}(q) = \sum_{n=0}^{\infty} c\phi_2(n) q^n = \sum_{n=0}^{\infty} c\psi_{2,1}(n) q^n = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^4 (q^4;q^4)_\infty^2},$$

where the q–Pochhammer symbol is defined as

$$(A;q)_\infty = \prod_{k=0}^{\infty}(1-Aq^k).$$

Moreover, from Drake [13], we have

$$\mathrm{C}\Psi_{2,0}(q) = \sum_{n=0}^{\infty} c\psi_2(n)q^n = \sum_{n=0}^{\infty} c\psi_{2,0}(n)q^n = \frac{2(q^4;q^4)_\infty^2}{(q;q)_\infty^2(q^2;q^2)_\infty}.$$

In order to simplify our notation even further, we define the function $f_k := (q^k;q^k)_\infty$ for any $k \geq 1$. Then we see that our generating functions above can be written as

$$\sum_{n=0}^{\infty} c\phi_2(n)q^n = \sum_{n=0}^{\infty} c\psi_{2,1}(n)q^n = \frac{f_2^5}{f_1^4 f_4^2} \tag{1}$$

and

$$\sum_{n=0}^{\infty} c\psi_2(n)q^n = \sum_{n=0}^{\infty} c\psi_{2,0}(n)q^n = \frac{2f_4^2}{f_1^2 f_2}. \tag{2}$$

Earlier, we noted that $c\phi_2(3) = 20$, and we shared the corresponding generalized Frobenius partitions. In contrast, we know that $c\psi_2(3) = 24$, where the objects counted here are as follows:

$$\begin{pmatrix} 1_1 & \\ 1_1 & 0_1 \end{pmatrix}, \begin{pmatrix} 1_2 & \\ 1_1 & 0_1 \end{pmatrix}, \begin{pmatrix} 1_1 & \\ 1_2 & 0_1 \end{pmatrix}, \begin{pmatrix} 1_2 & \\ 1_2 & 0_1 \end{pmatrix},$$

$$\begin{pmatrix} 1_1 & \\ 1_1 & 0_2 \end{pmatrix}, \begin{pmatrix} 1_2 & \\ 1_1 & 0_2 \end{pmatrix}, \begin{pmatrix} 1_1 & \\ 1_2 & 0_2 \end{pmatrix}, \begin{pmatrix} 1_2 & \\ 1_2 & 0_2 \end{pmatrix},$$

$$\begin{pmatrix} 0_1 & \\ 2_1 & 0_1 \end{pmatrix}, \begin{pmatrix} 0_2 & \\ 2_1 & 0_1 \end{pmatrix}, \begin{pmatrix} 0_1 & \\ 2_2 & 0_1 \end{pmatrix}, \begin{pmatrix} 0_2 & \\ 2_2 & 0_1 \end{pmatrix},$$

$$\begin{pmatrix} 0_1 & \\ 2_1 & 0_2 \end{pmatrix}, \begin{pmatrix} 0_2 & \\ 2_1 & 0_2 \end{pmatrix}, \begin{pmatrix} 0_1 & \\ 2_2 & 0_2 \end{pmatrix}, \begin{pmatrix} 0_2 & \\ 2_2 & 0_2 \end{pmatrix},$$

$$\begin{pmatrix} 0_1 & \\ 1_2 & 1_1 \end{pmatrix}, \begin{pmatrix} 0_2 & \\ 1_2 & 1_1 \end{pmatrix}, \begin{pmatrix} 2_1 & \\ 0_2 & 0_1 \end{pmatrix}, \begin{pmatrix} 2_2 & \\ 0_2 & 0_1 \end{pmatrix},$$

$$\begin{pmatrix} 0_2 & 0_1 & \\ 1_1 & 0_2 & 0_1 \end{pmatrix}, \begin{pmatrix} 0_2 & 0_1 & \\ 1_2 & 0_2 & 0_1 \end{pmatrix}, \begin{pmatrix} \\ 3_1 \end{pmatrix}, \begin{pmatrix} \\ 3_2 \end{pmatrix}$$

As was noted above, Andrews' 2-colored Frobenius partition function $c\phi_2(n)$ satisfies numerous congruences for various arithmetic progressions and moduli. So, given the relationship between $c\phi_2(n)$ and $c\psi_2(n)$, it seems natural to ask whether the latter function also satisfies any such congruences. Indeed, this is the case. In the work below, we prove a number of congruence properties satisfied by $c\psi_2(n)$.

2. Preliminaries

All of the proofs that we provide below will follow from elementary generating function manipulations. As such, we will need a few preliminary facts which we collect here.

The first result that we mention is often referred to as Euler's Pentagonal Number Theorem.

Lemma 2.1. *We have*

$$f_1 = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3k^2-k}{2}}.$$

Proof. See Hirschhorn [19, (1.6.1)]. □

Next, we mention a result which involves the cube of the left–hand side of Lemma 2.1 and is usually attributed to Jacobi.

Lemma 2.2. *We have*

$$f_1^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}.$$

Proof. See Hirschhorn [19, (1.7.1)]. □

We require two additional q-series identities which we mention here.

Lemma 2.3. *We have*

$$\frac{f_2^2}{f_1} = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}.$$

Proof. See Hirschhorn [19, (1.5.3)]. □

Lemma 2.4. *We have*

$$\frac{f_1^2 f_6}{f_2 f_3} = \sum_{t=0}^{\infty} \left(q^{\frac{t(t+1)}{2}} - 3q^{\frac{(3t+1)(3t+2)}{2}}\right).$$

Proof. See [12] as well as [4, p. 30] and [23, Theorem 1.2]. □

We will also require the following 2–dissection results.

Lemma 2.5. *We have*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}.$$

Proof. See da Silva and Sellers [11, Lemma 1, Equation (11)]. □

Lemma 2.6. *We have*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}.$$

Proof. See da Silva and Sellers [11, Lemma 1, Equation (12)]. □

Lastly, we will need the following well-known fact which follows, in essence, from the binomial theorem.

Lemma 2.7. *For a prime p, and positive integers k and l, we have*

$$f_k^{p^l} \equiv f_{pk}^{p^{l-1}} \pmod{p^l}.$$

3. Congruences for Drake's Variant of $c\phi_2(n)$

With the above tools in hand, we now transition to our elementary proofs of various Ramanujan-like congruences.

From (2), it is clear that, for all n, $c\psi_2(n) \equiv 0 \pmod 2$. Thus, our next goal is to consider $c\psi_2(n) \pmod 4$. Indeed, we have the following characterization.

Theorem 3.1. *For all $n \geq 0$,*

$$c\psi_2(n) \equiv \begin{cases} 2 \pmod 4 & \text{if } n = 4\left(\frac{3k^2-k}{2}\right) \text{ for some integer } k, \\ 0 \pmod 4 & \text{otherwise.} \end{cases}$$

Proof. Beginning with (2), we have

$$\begin{aligned} \sum_{n=0}^{\infty} c\psi_2(n)q^n &= \frac{2f_4^2}{f_1^2 f_2} \\ &\equiv \frac{2f_4 f_1^4}{f_1^2 f_1^2} \pmod 4 \quad \text{(thanks to Lemma 2.7)} \\ &= 2f_4. \end{aligned}$$

The proof then follows from Lemma 2.1. □

We next consider arithmetic properties modulo 8 satisfied by $c\psi_2(n)$. Our first goal is to prove the following theorem.

Theorem 3.2. *For all $n \geq 0, c\psi_2(4n+3) \equiv 0 \pmod 8$.*

Proof. Note that

$$\begin{aligned} \sum_{n=0}^{\infty} c\psi_2(n)q^n &= 2\frac{f_4^2}{f_1^2 f_2} \\ &\equiv 2\frac{f_2^4}{f_1^2 f_2} \pmod 8 \quad \text{(thanks to Lemma 2.7)} \end{aligned}$$

$$\begin{aligned} &= 2\frac{f_1^2 f_2^3}{f_1^4} \\ &\equiv 2\frac{f_1^2 f_2^3}{f_2^2} \pmod 8 \\ &= 2f_1^2 f_2. \end{aligned} \tag{3}$$

By Lemma 2.6, we know

$$2f_1^2 f_2 = 2f_2\left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}\right),$$

which means

$$\sum_{n=0}^{\infty} c\psi_2(2n+1)q^{2n+1} \equiv -4q\frac{f_2^2 f_{16}^2}{f_8} \pmod 8$$

or

$$\sum_{n=0}^{\infty} c\psi_2(2n+1)q^{n} \equiv -4\frac{f_1^2 f_8^2}{f_4} \pmod 8.$$

Applying Lemma 2.6 a second time implies

$$\sum_{n=0}^{\infty} c\psi_2(4n+3)q^{2n+1} \equiv 8q\frac{f_2 f_8 f_{16}^2}{f_4} \equiv 0 \pmod 8.$$

□

In fact, much more can be said about $c\psi_2(n)$ modulo 8.

Theorem 3.3. *For all $n \geq 0$,*

$$c\psi_2(4n) \equiv \begin{cases} 2 \pmod 8 & \text{if } n = \frac{3k^2-k}{2} \text{ and } k \equiv 0,1 \pmod 4, \\ 6 \pmod 8 & \text{if } n = \frac{3k^2-k}{2} \text{ and } k \equiv 2,3 \pmod 4, \\ 0 \pmod 8 & \text{otherwise.} \end{cases}$$

Proof. Recall from (3) that

$$\sum_{n=0}^{\infty} c\psi_2(n)q^{n} \equiv 2f_1^2 f_2 \pmod 8.$$

Applying Lemma 2.6 gives

$$\sum_{n=0}^{\infty} c\psi_2(n)q^{n} \equiv 2f_2\left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}\right) \pmod 8,$$

which then yields

$$\sum_{n=0}^{\infty} c\psi_2(2n)q^{2n} \equiv 2\frac{f_2^2 f_8^5}{f_4^2 f_{16}^2} \pmod 8$$

$$\begin{aligned}
&\equiv 2\frac{f_2^2 f_8^5}{f_2^4 f_8^4} \pmod{8} \\
&\equiv 2\frac{f_8}{f_2^2} \pmod{8}.
\end{aligned}$$

This means

$$\sum_{n=0}^{\infty} c\psi_2(2n)q^n \equiv 2\frac{f_4}{f_1^2} \pmod{8}.$$

Next, thanks to Lemma 2.5, we know

$$\begin{aligned}
\sum_{n=0}^{\infty} c\psi_2(2n)q^n &\equiv 2\frac{f_4}{f_1^2} \pmod{8} \\
&\equiv 2f_4\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right) \pmod{8}
\end{aligned}$$

and this implies that

$$\begin{aligned}
\sum_{n=0}^{\infty} c\psi_2(4n)q^{2n} &\equiv 2\frac{f_4 f_8^5}{f_2^5 f_{16}^2} \pmod{8} \\
&\equiv 2\frac{f_4 f_8^5}{f_2 f_2^4 f_8^4} \pmod{8} \\
&\equiv 2\frac{f_4 f_8}{f_2 f_4^2} \pmod{8} \\
&\equiv 2\frac{f_8}{f_2 f_4} \pmod{8}.
\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} c\psi_2(4n)q^n \equiv 2\frac{f_4}{f_1 f_2} \pmod{8}.$$

Note then that

$$\begin{aligned}
\sum_{n=0}^{\infty} c\psi_2(4n)(-q)^n &\equiv 2\frac{f_4}{f_2}\prod_{i=1}^{\infty}\frac{1}{(1-(-q)^i)} \pmod{8} \\
&= 2\frac{f_4}{f_2}\prod_{i=1}^{\infty}\frac{1}{(1-q^{2i})(1+q^{2i-1})} \\
&= 2\frac{f_4}{f_2^2}\prod_{i=1}^{\infty}\frac{(1+q^{2i})}{(1+q^i)} \\
&= 2\frac{f_4}{f_2^2}\prod_{i=1}^{\infty}\frac{(1-q^i)(1-q^{4i})}{(1-q^{2i})^2}
\end{aligned}$$

$$
\begin{aligned}
&= 2\frac{f_1 f_4^2}{f_2^4} \\
&\equiv 2\frac{f_1 f_2^4}{f_2^4} \pmod 8 \\
&= 2f_1.
\end{aligned}
$$

By Lemma 2.1, we know

$$\sum_{n=0}^{\infty} c\psi_2(4n)(-q)^n \equiv 2\sum_{k=-\infty}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}} \pmod 8. \tag{4}$$

From (4), we immediately see that, if $n \neq \frac{k(3k-1)}{2}$, then $c\psi_2(4n) \equiv 0 \pmod 8$. We then return to (4) and consider the four cases when $n = \frac{k(3k-1)}{2}$ with $k \equiv 0, 1, 2, 3 \pmod 4$ in order to complete the proof. $\square$

We quickly note that Theorem 3.3 can be used to write down infinitely many Ramanujan-like congruences modulo 8 satisfied by $c\psi_2(n)$.

Corollary 3.4. *If $p \geq 5$ is prime and $1 \leq r \leq p-1$ such that $24r+1$ is a quadratic nonresidue modulo p, then for all $n \geq 0$,*

$$c\psi_2(4(pn+r)) \equiv 0 \pmod 8.$$

Proof. By Theorem 3.3, it suffices to show that there are no solutions to the equation $pn + r = \frac{k(3k-1)}{2}$ for any integers k. Note that if this equation does have solutions for integers n and k, then completing the square yields $24pn + 24r + 1 = (6k-1)^2$ which would imply $24r + 1 \equiv (6k-1)^2 \pmod p$. However, we have assumed that $24r+1$ is a quadratic nonresidue modulo p, and this completes the proof. $\square$

We next prove a congruence that is analogous to Ramanujan's congruence that, for all $n \geq 0$, $p(5n+4) \equiv 0 \pmod 5$.

Theorem 3.5. *For all $n \geq 0, c\psi_2(5n+4) \equiv 0 \pmod 5$.*

Proof. Note that

$$
\begin{aligned}
\sum_{n=0}^{\infty} c\psi_2(n)q^n &= 2\frac{f_4^2}{f_1^2 f_2} \\
&= 2\frac{f_1^3 f_4^2}{f_1^5 f_2} \\
&\equiv 2\frac{f_1^3 f_4^2}{f_5 f_2} \pmod 5.
\end{aligned}
$$

Thanks to Lemma 2.2 and Lemma 2.3, we have

$$\sum_{n=0}^{\infty} c\psi_2(n)q^n \equiv \frac{2}{f_5}\sum_{k\geq 0}\sum_{l\geq 0}(-1)^k(2k+1)q^{\frac{k(k+1)}{2}+l(l+1)} \pmod 5.$$

We now consider solutions of the equation

$$\frac{k(k+1)}{2}+l(l+1)=5n+4,$$

which means we wish to find values of k and l such that

$$\frac{k(k+1)}{2}+l(l+1)\equiv 4 \pmod 5.$$

A straightforward set of computations shows that this only occurs when $k\equiv l\equiv 2 \pmod 5$. Hence, the factor of $2k+1$ which appears in the double sum above must be divisible by 5. This implies that, for all $n\geq 0$, $c\psi_2(5n+4)\equiv 0 \pmod 5$. □

4. Congruences for Drake's Variant of $\phi_2(n)$

In [1], Andrews defines a second family of generalized Frobenius partition functions, which he denotes $\phi_k(n)$. In this setting, the Frobenius symbols for ordinary partitions are generalized simply by allowing the parts of each of the two rows to repeat up to k times. (Note that the elements of the two-rowed arrays here are simply nonnegative integers, not the colored versions that we discussed in Section 1.) It is clear from this description that, for all $n\geq 0$, $\phi_1(n)=p(n)$. Thus, for example, $\phi_2(3)=5$, and the five two-rowed arrays in question are

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

When one reviews the literature on the subject of congruences satisfied by Andrews' generalized Frobenius partition functions, it seems clear that the functions $c\phi_k(n)$ satisfy many more congruences than their counterpart functions $\phi_k(n)$. One might argue that this is true "combinatorially" given the structure of the objects being counted by these functions (and symmetries that are inherent in the two–rowed arrays counted by $c\phi_k(n)$).

A similar phenomenon also appears to exist between $c\psi_2(n)$ and $\psi_2(n)$; namely, while we have seen numerous congruences satisfied by $c\psi_2(n)$ (in Section 3), it appears to be the case, computationally, that $\psi_2(n)$ simply does not satisfy as many arithmetic properties. Even so, for completeness' sake, we prove two such congruences here. To do so, we will use the following generating function result from Drake [13].

Theorem 4.1. *We have*

$$\sum_{n=0}^{\infty}\psi_2(n)q^n = \frac{f_2^2f_{12}}{f_1^2f_4f_6}.$$

We begin with a very straightforward proof of a parity result.

Theorem 4.2. *For all* $n\geq 0, \psi_2(2n+1)\equiv 0 \pmod{2}$.

Proof. From Theorem 4.1 and Lemma 2.7, we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_2(n)q^n &= \frac{f_2^2f_{12}}{f_1^2f_4f_6}\\ &\equiv \frac{f_2^2f_{12}}{f_2f_4f_6} \pmod{2}\\ &= \frac{f_2f_{12}}{f_4f_6}\end{aligned}$$

and this last expression is clearly an even function of q. Thus, we know that, for all $n\geq 0$, $\psi_2(2n+1)\equiv 0 \pmod{2}$. □

Next, we prove a clear companion to Theorem 3.5.

Theorem 4.3. *For all* $n\geq 0$, $\psi_2(5n+4)\equiv 0 \pmod{5}$.

Proof. Beginning with Theorem 4.1, we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_2(n)q^n &= \frac{f_2^2f_{12}}{f_1^2f_4f_6}\\ &= \frac{f_1^3f_2^2f_{12}}{f_1^5f_4f_6}\\ &\equiv \frac{1}{f_5}\frac{f_1^3f_2^2f_{12}}{f_4f_6} \pmod{5}.\end{aligned}$$

By Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_2(n)q^n &\equiv \frac{1}{f_5}\left(\sum_{k\geq 0}(-1)^k(2k+1)q^{\frac{k(k+1)}{2}}\right)\times\\ &\qquad \left(\sum_{t\geq 0}q^{t(t+1)}-3q^{(3t+1)(3t+2)}\right) \pmod{5}\\ &\equiv \frac{1}{f_5}\sum_{k\geq 0}\sum_{t\geq 0}(-1)^k(2k+1)\times\\ &\qquad \left(q^{\frac{k(k+1)}{2}+t(t+1)}-3q^{\frac{k(k+1)}{2}+(3t+1)(3t+2)}\right) \pmod{5}.\end{aligned}$$

We now need to ask whether there are solutions to each of the following two congruences:

$$\frac{k(k+1)}{2} + t(t+1) \equiv 4 \pmod 5, \quad \text{and}$$
$$\frac{k(k+1)}{2} + (3t+1)(3t+2) \equiv 4 \pmod 5.$$

As in the proof of Theorem 3.5, we find that both cases only occur when $k \equiv t \equiv 2 \pmod 5$. Since $2k+1 \equiv 0 \pmod 5$ when $k \equiv 2 \pmod 5$, our result follows. □

5. Concluding Remarks

It is worth noting that the second author has discovered an infinite family of congruences, modulo arbitrarily large powers of 5, which is satisfied by $c\psi_2(n)$ (and serves as a companion to a similar infinite family of congruences modulo powers of 5 satisfied by $c\phi_2(n)$ which was proven by Paule and Radu [27]). In this light, the following theorem was proved in recent work of Garvan, Sellers, and Smoot [17] using the theory of modular forms.

Theorem 5.1. *Let $n \geq 1$ and $\alpha \geq 1$ such that $6n \equiv -1 \pmod{5^\alpha}$. Then $c\psi_2(n) \equiv 0 \pmod{5^\alpha}$.*

Note that the case $\alpha = 1$ of Theorem 5.1 requires that $6n \equiv -1 \pmod 5$ or $n \equiv 4 \pmod 5$. This is Theorem 3.5 above.

As we close, we share three potential avenues for further research.

(1) Finding combinatorial proofs of the congruences provided above would be intriguing.
(2) It would be interesting to know whether other congruences are satisfied by $c\psi_2(n)$, especially for moduli different from those mentioned in Section 3 above.
(3) Drake's work provides an infinite family of functions related to Andrews' functions $c\phi_k(n)$, and for values of $k > 2$, relatively little work has been done to see if Drake's functions satisfy any divisibility properties. (The one work that has considered this avenue of research is [21].) Given the close relationship between $c\phi_k(n)$ and $c\psi_k(n)$ thanks to Definition 1.1, and given the rich set of congruences satisfied by $c\phi_k(n)$ for various k, we would encourage others to further such research.

Declarations

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References

[1] G. E. Andrews, *Generalized Frobenius Partitions*, Memoirs of the American Mathematical Society, Volume 301, Providence, RI, 1984

[2] N. D. Baruah and B. K. Sarmah, Congruences for generalized Frobenius partitions with 4 colors, *Discrete Math.* **311** (2011), 1892–1902

[3] N. D. Baruah and B. K. Sarmah, Generalized Frobenius partitions with 6 Colors, *Ramanujan J.* **38** (2015), 361–382

[4] J. H. Brunier, G. van der Geer, G. Harder, and D. Zagier, *The 1-2-3 of Modular Forms*, Universitext, Spring-Verlag, Berlin, 2008.

[5] H. H. Chan, L. Wang, and Y. Yang, Congruences modulo 5 and 7 for 4-colored generalized Frobenius partitions, *J. Aust. Math. Soc.* **103**, no. 2 (2017), 157–176

[6] H. H. Chan, L. Wang, and Y. Yang, Modular forms and k–colored generalized Frobenius Partitions, Transactions of the American Mathematical Society **371** (2020), 2159–2205

[7] S.-P. Cui and N. S. S. Gu, Congruences modulo powers of 2 for generalized Frobenius partitions with six colors, *Int. J. Number Theory* **15** (2019), 1173–1181

[8] S.-P. Cui and N. S. S. Gu, Congruences for generalized Frobenius partitions with 6 colors, *Rocky Mountain J. Math.* **52**, no. 3 (2022), 877–885

[9] S.-P. Cui, N. S. S. Gu, and A. X. Huang, Congruence properties for a certain kind of partition functions, *Adv. Math.* **290** (2016), 739–772

[10] S.-P. Cui, N. S. S. Gu, and D. Tang, The method of constant terms and k-colored generalized Frobenius partitions, *J. Combin. Theory Ser. A* **203** (2024), Paper No. 105837

[11] R. da Silva and J. A. Sellers, Congruences for the coefficients of the Gordon and McIntosh mock theta function $\xi(q)$, *Ramanujan J.* **58**, no. 3 (2022), 815–834

[12] D. Dockery, M. Jameson, J. A. Sellers, and S. Wilson, S., d–fold partition diamonds, *Discrete Math.* **347**, no. 12 (2024), Paper 114163

[13] B. Drake, Limits of areas under lattice paths, *Discrete Math.* **309**, no. 12 (2009), 3936–3953

[14] D. Eichhorn and J. A. Sellers, Computational proofs of congruences for 2-colored Frobenius partitions, *Int. J. Math. Math. Sci.*, **29**, no. 6 (2002), 333–340

[15] F. G. Garvan, Generalizations of Dyson's Rank, PhD Thesis, Pennsylvania State University, 1986

[16] F. G. Garvan and J. A. Sellers, Congruences for generalized Frobenius partitions with an arbitrarily large number of colors, *Integers* **14** (2014), Paper A7

[17] F. G. Garvan, J. A. Sellers, and N. A. Smoot, Old meets new: Connecting two infinite families of congruences modulo powers of 5 for generalized Frobenius partition functions, *Adv. Math.* **454** (2024), Paper 109866

[18] M. D. Hirschhorn, Some congruences for 6-colored generalized Frobenius partitions, *Ramanujan J.* **40**, no. 3 (2016), 463–471

[19] M. D. Hirschhorn, The Power of q: A Personal Journey, Developments in Mathematics, v. 49, Springer, 2017

[20] M. Jameson and M. Wieczorek, Congruences for modular forms and generalized Frobenius partitions, *Ramanujan J.* **52** (2020), 541–553

[21] Y. Jiang, L. Rolen, and M. Woodbury, Generalized Frobenius partitions, Motzkin paths, and Jacobi forms, *J. Combin. Theory Ser. A* **190** (2022), Paper 105633

[22] L. W. Kolitsch, An extension of a congruence by Andrews for generalized Frobenius partitions, *J. Comb. They. Ser. A* **45** (1987), 31–39

[23] R. J. Lemke Oliver, Eta-quotients and theta functions, *Adv. Math* **241** (2013), 1–17

[24] B. L. S. Lin, New Ramanujan type congruence modulo 7 for 4-colored generalized Frobenius partitions, *Int. J. Number Theory* **10**, no. 3 (2014), 637–639

[25] J. Lovejoy, Ramanujan-type congruences for three colored Frobenius partitions, *J. Number Theory* **85**, no. 2 (2000), 283–290

[26] K. Ono, Congruences for Frobenius partitions, *J. Number Theory* **57**, no. 1 (1996), 170–180

[27] P. Paule and C.-S. Radu, The Andrews-Sellers family of partition congruences, *Adv. Math.* **230**, no. 3 (2012), 819–838

[28] J. A. Sellers, Congruences involving F–partition functions, *Int. J. Math. Math. Sci.* **17**, no. 1 (1994), 187–188

[29] J. A. Sellers, An unexpected congruence modulo 5 for 4-colored generalized Frobenius partitions, *J. Indian Math. Soc. (N.S.)*, Special volume to commemorate the 125th birth anniversary of Srinivasa Ramanujan (2013), 97–103

[30] L. Wang, Congruences modulo powers of 3 for 3- and 9-colored generalized Frobenius partitions, *Discrete Math.* **341**, no. 12 (2018), 3370–3384

[31] C. Wang and W. Zhang, A unified proof of Ramanujan-type congruences modulo 5 for 4-colored generalized Frobenius partitions, *J. Math. Anal. Appl.* **447**, no. 1 (2017), 542–549

[32] E. X. W. Xia, Proof of a conjecture of Baruah and Sarmah on generalized Frobenius partitions with 6 colors, *J. Number Theory* **147** (2015), 852–860

[33] E. X. W. Xia, A Ramanujan-type congruence modulo 5 for 4-colored generalized Frobenius partitions, *Ramanujan J.* **39**, no. 3 (2016), 567–576

[34] W. Zhang and C. Wang, An unexpected Ramanujan-type congruence modulo 7 for 4-colored generalized Frobenius partitions, *Ramanujan J.* **44**, no. 1 (2017), 125–131

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