VARIATIONS ON A RESULT OF BRESSOUD

KAĞAN KURŞUNGÖZ AND JAMES A. SELLERS

Abstract. The well-known Rogers-Ramanujan identities have been a rich source of mathematical study over the last fifty years. In particular, Gordon's generalization in the early 1960s led to additional work by Andrews and Bressoud in subsequent years. Unfortunately, these results lacked a certain amount of uniformity in terms of combinatorial interpretation. In this work, we provide a single combinatorial interpretation of the series sides of these generating function results by using the concept of cluster parities. This unifies the aforementioned results of Andrews and Bressoud and also allows for a strikingly broader family of q–series results to be obtained. We close the paper by proving congruences for a "degenerate case" of Bressoud's theorem.

1. Introduction

A partition of a positive integer n is a sum of non-decreasing positive integers which sum to n. For instance, $1+1+1+3+4+5$ is a partition of 15. There are numerous ways to represent the same partition, one of which is the frequency notation where we write $n = f_1 \times 1 + f_2 \times 2 + f_3 \times 3 + \dots$ where f_i is the number of times the number i appears as a part. For the example above, $f_1 = 3$, $f_2 = 0$ (because there are no 2's as parts), $f_3 = f_4 = f_5 = 1$ and $f_i = 0$ for $i \geq 6$.

A well-known family of results involving such frequencies in partitions is the Rogers-Ramanujan-Gordon identities [11].

Theorem 1.1. *Given a positive integer k and an integer r such that $1 \leq r \leq k$, define $A_{k,r}(n)$ to be the number of partitions of n into parts $\not\equiv 0, \pm r \pmod{2k+1}$. Let $B_{k,r}(n)$ denote the number of partitions of n such that $f_1 < r$ and $f_i + f_{i+1} < k$. Then, $A_{k,r}(n) = B_{k,r}(n)$ for all n.*

Andrews [3] provided the generating function for $b_{k,r}(m,n)$, the number of partitions enumerated by $B_{k,r}(n)$ with m parts:

$$(1) \qquad \sum_{m,n \geq 0} b_{k,r}(m,n) x^m q^n = \sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \dots + N_{k-1}^2 + N_r + N_{r+1} + \dots + N_{k-1}} x^{N_1 + \dots + N_{k-1}}}{(q;q)_{n_1} (q;q)_{n_2} \cdots (q;q)_{n_{k-1}}},$$

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where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$, and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$.

Soon after, Bressoud [8] proved a closely related result:

Theorem 1.2. *Given a positive integer k and an integer r such that $1 \le r < k$, define $A_{k,r,2}(n)$ to be the number of partitions of n into parts $\not\equiv 0, \pm r \pmod{2k}$. Let $B_{k,r,2}(n)$ denote the number of partitions of n such that $f_1 < r$, $f_i + f_{i+1} < k$, and if $f_i + f_{i+1} = k-1$, then $if_i + (i+1)f_{i+1} \equiv r-1 \pmod 2$. Then, $A_{k,r,2}(n) = B_{k,r,2}(n)$ for all n.*

Notice that the congruence $if_i + (i+1)f_{i+1} \equiv r-1 \pmod 2$ when $f_i + f_{i+1} = k-1$ is equivalent to saying that for consecutive parts, f_{even} and f_{odd} have fixed parities that depend on k and r. In [9], Bressoud found the generating function for $b_{k,r,2}(m,n)$, the number of partitions enumerated by $B_{k,r,2}(n)$ which have exactly m parts:

$$\sum_{m,n\ge 0} b_{k,r,2}(m,n)x^m q^n = \sum_{n_1,\dots,n_{k-1}\ge 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_r+N_{r+1}+\cdots+N_{k-1}} x^{N_1+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-2}}(q^2;q^2)_{n_{k-1}}}, \tag{2}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$ as above.

Theorems 1.1 and 1.2 extend a number of classical results such as Euler's partition theorem [5, Corollary 1.2], and the Rogers-Ramanujan identities [5, Corollaries 7.6, and 7.7]. Moreover, for odd k, Theorem 1.2 was given by Andrews [2]. It is important to note at this stage that the case $r = k$ is excluded in the statement of Theorem 1.2; this will prove important later in this work as we address this particular case below.

In 2010, Andrews [6] found the following theorem which is in the same genre as Theorems 1.1 and 1.2.

Theorem 1.3. *Suppose $2 \le r \le k$ are integers with $k \equiv r \pmod 2$. Let $\mathcal{W}_{k,r}(n)$ denote the number of partitions enumerated by $B_{k,r}(n)$ with the added restriction that even parts appear an even number of times. If k and r are both even, let $G_{k,r}(n)$ denote the number of partitions of n in which no odd part is repeated and no even part $\equiv 0, \pm r \pmod{2k+2}$. If k and r are both odd, let $G_{k,r}(n)$ denote the number of partitions of n into parts that are neither $\equiv 2 \pmod 4$ nor $\equiv 0, \pm r \pmod{2k+2}$. Then $\mathcal{W}_{k,r}(n) = G_{k,r}(n)$ for all n.*

The generating function for $w_{k,r}(m,n)$, the number of partitions enumerated by $\mathcal{W}_{k,r}(n)$ with exactly m parts is also given in [6]:

$$\sum_{m,n\ge 0} w_{k,r}(m,n)x^m q^n = \sum_{n_1,\dots,n_{k-1}\ge 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+2N_r+2N_{r+2}+\cdots+2N_{k-2}} x^{N_1+\cdots+N_{k-1}}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}\cdots(q^2;q^2)_{n_{k-1}}}, \tag{3}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$, as above. Observe the similarity in the generating functions (1), (2) and (3).

Note that a series of the following form is not included in the above theorems:

$$\sum_{n_1,n_2,n_3\geq 0}\frac{q^{N_1^2+N_2^2+N_3^2+N_3}x^{N_1+N_2+N_3}}{(q^2;q^2)_{n_1}(q;q)_{n_2}(q^2;q^2)_{n_3}}. \tag{4}$$

That is to say, the above theorems require either zero, one (in particular, the last one), or all of the products in the denominator of the summand to be functions of q^2. Our hope is to address sums such as (4) whereby any number of the denominator products may be functions of q^2 while the others are simply functions of q.

The structure of the remaining part of this paper is as follows. In section 2, we provide some necessary preliminary material. In section 3, we recall the partition–theoretic interpretations of series such as (4) which are mentioned in [14]. We note that our interpretations appear to be different than those of Bressoud, although we will show their equivalence. In section 4, we examine the case $r = k$ in Theorem 1.2 (which was previously excluded by Bressoud). We explain why this case is "degenerate" in a sense, and prove curious congruences for it. Finally in section 5, we discuss possible directions for future research.

2. Background

Let $\lambda = \lambda_1 + \cdots + \lambda_m$ be a partition of n with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. The following definition appears in [15].

Definition 2.1. *The* Gordon marking *of a partition λ is an assignment of positive integers (*marks*) to λ such that*

- **i)** *equal or consecutive parts are assigned distinct marks,*
- **ii)** *smallest possible marks are used, and*
- **iii)** *parts are marked from smallest to largest.*

Let $\lambda^{(r)}$ denote the sub-partition of λ that consists of all r-marked parts.

For instance, if

$$\lambda = 4 + 5 + 5 + 6 + 6 + 6 + 7 + 8 + 8 + 9,$$

then its Gordon marking would be

$$\lambda = 4_1 + 5_2 + 5_3 + 6_1 + 6_4 + 6_5 + 7_2 + 8_1 + 8_3 + 9_2.$$

A more visual representation of the Gordon marking is a two-dimensional array. Columns specify the values of parts, and rows specify the marks. The 2-marked 5 (5_2), say, is in the fifth column from the left, and second row from the bottom.

$$\left\{\begin{array}{cccccc} & & 6 & & & \\ & & 6 & & & \\ & 5 & & & 8 & \\ & 5 & & 7 & & 9 \\ 4 & & 6 & & 8 & \end{array}\right\}$$

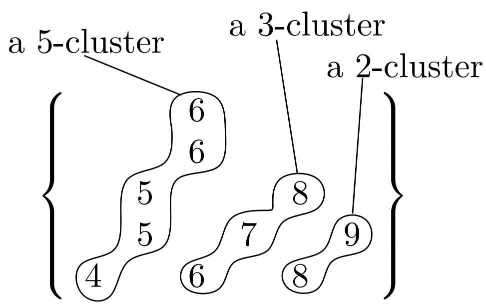
Here, $\lambda^{(2)}$ is the sub-partition $5 + 7 + 9$.

The following two definitions are from [14].

Definition 2.2. *An* r-cluster *in* $\lambda = \lambda_1 + \cdots + \lambda_m$ *is a sub-partition with* r *parts* $\lambda_{i_1} \leq \cdots \leq \lambda_{i_r}$ *such that*

i) λ_{i_j} *is* j*-marked for* $j = 1, \ldots, r$,
ii) $\lambda_{i_{j+1}} = \lambda_{i_j}$ *or* $\lambda_{i_j} + 1$, *and*
iii) *there are no* $(r+1)$*-marked parts equal to* λ_{i_r} *or* $\lambda_{i_r} + 1$.

It is not hard to show that the Gordon marking, and decomposition of any given partition into clusters, is unique [14, 15]. In the figure below, the clusters of the above partition λ are indicated.



Definition 2.3. *The* parity of an r-cluster *is the opposite parity of the number of even parts in that* r*-cluster.*

For instance, the 5-cluster in the above λ is an even cluster, because there are three even parts in it. The 3-cluster is an odd cluster, and the 2-cluster is an even one.

Definition 2.3 may seem a little awkward at first, but there are two constraints which lead to it. First, a 1-cluster is simply a number. We would like to keep its "traditional" parity as is. Next, we would like a single file of odd parts to be an odd cluster for obvious reasons, no matter how many parts.

3. Interpretations of the Series

Although the full theorem is stated and proven below (Theorem 3.3), we begin with a special case. This will give us the connection to Theorem 1.2. To facilitate the proof, we state a proposition first.

Proposition 3.1. *Given a partition* λ*, suppose the* r*-marked* i *determines an* r*-cluster. Then, among the* $1, \ldots, r$*- marked* $(i-1)$*'s and* i*'s, the number of even parts (respectively, odd parts) equals the number of even parts (respectively, odd parts) in the* r*-cluster.*

Proof. For a moment, discard all $(r+1)$ or greater marked parts from λ. Nothing in the r-clusters changes.

By the definition of Gordon marking, there must be $1, \ldots, (r-1)$-marked $(i-1)$'s or i's, exactly one for each mark. If there are $1-, \ldots, r-$marked i's in λ, then there are no r or smaller marked $(i-1)$'s, and all those i's are in the r-cluster. The proof is clear in this case. This provides the basis cases in the strong induction.

If there is an $(i-1)$ which is marked smaller than r, let s be the maximal mark such that there is an s-marked $(i-1)$ in the cluster. Then, the $(s+1), (s+2), \ldots, r$-marked i's are in the r-cluster. When we discard all of those i's, then the s-marked $(i-1)$ would determine an s-cluster $(s < r)$. This s-cluster consists precisely of the $1, \ldots, s$-marked parts of the former r-cluster. By the inductive hypothesis, the number of even parts (respectively, odd parts) in the latter s-cluster equals the number of even parts (respectively, odd parts) among the $1, \ldots, s$-marked $(i-2)$'s and $(i-1)$'s.

On the other hand, for $j < s$, if there is a j-marked $(i-2)$, then there are no j-marked $(i-1)$'s. But because there was an $(s+1)$-marked i before we discarded it, there must be a j-marked i thanks to the definition of Gordon marking. Thus, the number of $(i-2)$'s marked smaller than s equals the number of i's marked smaller than s. Noting that $i-2 \equiv i \pmod 2$, the proof is complete. $\square$

One could use direct arguments instead of induction in the proof of Proposition 3.1, sacrificing brevity for explicit construction. For such constructions, see [14].

Theorem 3.2. *Given $k \geq 2$, and $1 \leq r \leq k$, let ${}_{k-1}\tilde{b}_{k,r}(m,n)$ denote the number of partitions of n into m parts such that $f_1 < r$, $f_i + f_{i+1} < k$, and all $(k-1)$-clusters have the same parity as $(k-r+1)$.*

Let $b_{k,r,2}(m,n)$ be the number of partitions of n into m parts such that $f_1 < r$, $f_i + f_{i+1} < k$, and if $f_i + f_{i+1} = k-1$, then $if_i + (i+1)f_{i+1} \equiv r-1 \pmod 2$.

Then,

$$\sum_{m,n\geq 0} b_{k,r,2}(m,n)x^m q^n = \sum_{m,n\geq 0} {}_{k-1}\tilde{b}_{k,r}(m,n)x^m q^n$$

$$= \sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_r+N_{r+1}+\cdots+N_{k-1}} x^{N_1+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-2}}(q^2;q^2)_{n_{k-1}}}. \tag{5}$$

Proof. The fact that the first and the third sums are identical is shown in [9]. To see that the second and the third ones are the same, we put $y_1 = \cdots = y_{k-2} = 1$, $y_{k-1} = 0$ in [14, (3.2)], and observe that $(q^2;q^2)_n = (q;q)_n(-q;q)_n$. This gives a legitimate and complete proof.

However, we can also prove combinatorially that $b_{k,r,2}(m,n) = {}_{k-1}\tilde{b}_{k,r}(m,n)$.

By their respective definitions, partitions counted by $b_{k,r,2}(m,n)$ or ${}_{k-1}\tilde{b}_{k,r}(m,n)$ have m parts adding up to n. Either satisfy $f_1 < r$ and $f_i + f_{i+1} < k$.

We now show that if a partition enumerated by ${}_{k-1}\tilde{b}_{k,r}(m,n)$, then it is enumerated by $b_{k,r,2}(m,n)$ also. If there are no $(k-1)$-marked parts, then there is nothing to prove.

Otherwise, let i be any of the $(k-1)$-marked parts. Then, $f_{i-1} + f_i = k-1$ by the definition of Gordon marking. The $(k-1)$-marked i defines a $(k-1)$-cluster, the parity of which is $\not\equiv k-r \pmod 2$ by the hypothesis. In other words, the parity of the number of even parts in the cluster is $\equiv k-r \pmod 2$. There are a total of $(k-1)$ parts in the cluster, so the number of the remaining odd parts is $\equiv r-1 \pmod 2$. By Proposition 3.1, among the $(i-1)$'s and i's, the number of odd parts $\equiv r-1 \pmod 2$, and this implication is shown.

We next show that if a partition enumerated by $b_{k,r,2}(m,n)$, then it is enumerated by ${}_{k-1}\tilde{b}_{k,r}(m,n)$ also.

If $f_i + f_{i+1} = k-1$, then by the definition of Gordon marking, there will be a $(k-1)$-marked $(i+1)$ or i. In either case, the number of odd parts among i's and $(i+1)$'s $\equiv r-1 \pmod 2$.

If there is a $(k-1)$-marked $(i+1)$, then the number of odd parts in that cluster equals the number of odd parts among i's and $(i+1)$'s by Proposition 3.1, which is $\equiv r-1 \pmod 2$. Then, the number of even parts in the cluster is $\equiv k-r \pmod 2$, because there are a total of $(k-1)$ parts in the cluster. Hence the parity of the $(k-1)$-cluster is $\equiv k-r+1 \pmod 2$ by definition. The proof is complete in this case.

If there is a $(k-1)$-marked i, then $f_{i-1} + f_i = k-1$, by the definition of Gordon marking. Because of $f_i + f_{i+1} = k-1$, $f_{i-1} = f_{i+1}$. The previous paragraph with $(i-1)$ replacing i, together with the fact that $i-1 \equiv i+1 \pmod 2$ finishes the proof. $\square$

Next, we consider a significant generalization of Theorem 3.2 which requires the r-clusters defined above.

Theorem 3.3. *Given $k \geq 2$, $1 \leq r \leq k$, suppose $I = \{i_1, i_2, \ldots, i_s\}$ is a possibly empty subset of $[k-1] = \{1, 2, \ldots, k-1\}$. Let ${}_{i_1,\ldots,i_s}\tilde{b}_{k,r}(m,n)$ denote the number of partitions of n into m parts such that $f_1 < r$, $f_i + f_{i+1} < k$, and all i_j-clusters ...*

- **i)** *... are odd if $i_j < r$*
- **ii)** *... have the same parity as $i_j - r$ if $i_j \geq r$*

for $j = 1, \ldots, s$. Then,

$$\sum_{m,n\geq 0} {}_{i_1,\ldots,i_s}\tilde{b}_{k,r}(m,n)x^m q^n = \sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1}} x^{N_1+\cdots+N_{k-1}}}{\prod_{i\in I}(q^2;q^2)_{n_i} \quad \prod_{i\in[k-1]-I}(q;q)_{n_i}}. \tag{6}$$

Proof. We proceed as in the proof of [14, Theorem 3.7].

$$q^{N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1}} x^{N_1+\cdots+N_{k-1}}$$

gives us a base partition λ where

$$\begin{aligned}
\lambda^{(k-1)} &= 2+4+6+\cdots+2N_{k-1}, \\
&\vdots \\
\lambda^{(r)} &= 2+4+6+\cdots+2N_r, \\
\lambda^{(r-1)} &= 1+3+5+\cdots+2N_{r-1}-1, \\
&\vdots \\
\lambda^{(1)} &= 1+3+5+\cdots+2N_1-1.
\end{aligned}$$

Any j-cluster here is in the form

$$\begin{array}{cc} & 2 \\ & \vdots \\ & 2 \\ 1 & \\ \vdots & \\ 1 & \end{array}$$

with possibly no 2's. The parity of such a cluster is odd if $j < r$, and it is the same as the parity of $j - r + 1$ if $j \geq r$. To conclude the proof, we use [14, Theorem 3.6 (ii)]. Note that $(q^2; q^2)_{j_i}$ in the denominator will fix the parity of all j_i-clusters. $\square$

Thanks to cluster parities, Theorem 3.3 provides a combinatorial interpretation of generalizations of (1) where q is replaced by q^2 in an arbitrary selection of denominator factors. In Bressoud's result (Theorem 1.2), $I = \{k-1\}$. Choosing $I = \{1, 2, \ldots, k-1\}$ resembles (3). Notice that the powers of q in the numerator are different in (3) unless $k = r$.

As an example, let's utilize Theorem 3.3 to interpret (4).

$$\sum_{n_1, n_2, n_3 \geq 0} \frac{q^{N_1^2 + N_2^2 + N_3^2 + N_3} x^{N_1 + N_2 + N_3}}{(q^2; q^2)_{n_1} (q; q)_{n_2} (q^2; q^2)_{n_3}} = \sum_{m, n \geq 0} {}_{1,3}\tilde{b}_{4,3}(m, n) x^m q^n,$$

where ${}_{1,3}\tilde{b}_{4,3}(m, n)$ is the number of partitions of n into m parts such that $f_1 < 3$, $f_i + f_{i+1} < 4$, all 1-clusters are odd, and all 3-clusters are even.

4. A Degenerate Case and Congruences

When $r = k$ in Theorem 1.2, Bressoud's proof [8] still holds, i.e.

$$\sum_{n \geq 0} B_{k,k,2}(n) q^n = \frac{(q^k; q^{2k})_\infty^2 (q^{2k}; q^{2k})_\infty}{(q; q)_\infty} = \sum_{n \geq 0} A_{k,k,2}(n) q^n, \tag{7}$$

where $(a; q)_\infty = \lim_{n \to \infty} (a; q)_n$. However, we cannot interpret $A_{k,k,2}(n)$ as the number of partitions of n into parts that are $\not\equiv \pm k \pmod{2k}$ anymore. This is because $k \equiv -k \pmod{2k}$. Even so, it is possible to use standard q-series manipulations [5, Ch. 1] to simplify the infinite product in (7). We can then write $A_{k,k,2}(n)$ as a difference of cardinalities of certain classes of pairs of partitions. Yet, one sees that this interpretation is so unlike $A_{k,r,2}(n)$ described in Theorem 1.2 for $1 \leq r < k$. Our goal is rather to give congruence relations for $B_{k,k,2}(n)$.

Before moving to the congruence results satisfied by $B_{k,k,2}(n)$, we first note that the generating function in question can be rewritten as follows:

$$\begin{aligned}\sum_{n\geq 0} B_{k,k,2}(n)q^n &= \frac{(q^k;q^{2k})_\infty^2 (q^{2k};q^{2k})_\infty}{(q;q)_\infty}\\ &= \frac{(q^k;q^{2k})_\infty (q^k;q^k)_\infty}{(q;q)_\infty}\\ &= \frac{(q^k;q^k)_\infty^2}{(q;q)_\infty (q^{2k};q^{2k})_\infty} \qquad (8)\end{aligned}$$

From (8), we immediately see the following parity result for $B_{k,k,2}$:

Theorem 4.1. *For all $n \geq 0$ and for any k, $B_{k,k,2}(n) \equiv p(n) \pmod 2$ where $p(n)$ is the number of (unrestricted) partitions of n.*

Proof. The proof of this result is almost immediate:

$$\begin{aligned}\sum_{n\geq 0} B_{k,k,2}(n)q^n &= \frac{(q^k;q^k)_\infty^2}{(q;q)_\infty (q^{2k};q^{2k})_\infty}\\ &\equiv \frac{(q^{2k};q^{2k})_\infty}{(q;q)_\infty (q^{2k};q^{2k})_\infty} \pmod 2\\ &= \frac{1}{(q;q)_\infty}\\ &= \sum_{n\geq 0} p(n)q^n\end{aligned}$$

The result follows. □

The parity of $p(n)$ has been a topic of study for some time. The interested reader may wish to see [1, 7, 12, 13, 16, 17, 18] for a variety of works related to the parity of $p(n)$.

Again thanks to the form of the generating function for $B_{k,k,2}$ as seen in (8), we can prove an additional set of somewhat unexpected congruence results satisfied by $B_{k,k,2}$ for certain small values of k.

Theorem 4.2. *For all $n \geq 0$,*

$$\begin{aligned}B_{5,5,2}(5n+4) &\equiv 0 \pmod 5,\\ B_{7,7,2}(7n+5) &\equiv 0 \pmod 7, \quad \textit{and}\\ B_{11,11,2}(11n+6) &\equiv 0 \pmod{11}.\end{aligned}$$

Proof. The proof of this theorem is almost as elementary as the proof of Theorem 4.1. First, note that when written as power series, the terms $(q^k;q^k)_\infty^2$ and $1/(q^{2k};q^{2k})_\infty$ will be functions of q^k (for fixed $k = 5, 7,$ or 11). This means that every value $B_{k,k,2}(kn + r_k)$ for the pairs

$(k, r_k) = (5,4), (7,5),$ and $(11,6)$ will be a sum of terms each of which contains a factor of the form $p(kn+r_k)$. Lastly, since

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \quad \textit{and} \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

for all $n \geq 0$, our result follows. □

Actually, a bit more can be said thanks to the work of Ramanujan [19, Paper 25]. Namely, we have

$$\sum_{n\geq 0} B_{5,5,2}(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^4(q^2;q^2)_\infty}$$

and

$$\sum_{n\geq 0} B_{7,7,2}(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^2(q^2;q^2)_\infty} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^6(q^2;q^2)_\infty}.$$

5. Further Problems

One goal for the future would be to discover representations of general series like (4) as linear combinations of nice infinite products. Indeed, it is known that, given a q-series, there is a unique representation of the q-series as a **single** infinite product of powers of $(1-q^i)$. An algorithm for finding these powers is given in [10]. Sadly, a straightforward computer search shows us that there are no nice representations for series like (4) which are a **single** infinite product. So we must next attempt to find representations which are linear combinations of infinite products, but this is a much harder task. This is because there is no *a priori* reason for the representations of such series as linear combinations of infinite products to be unique.

Another obstacle in this study is that the r-clusters do not readily yield functional equations, as opposed to Andrews' [4] or Bressoud's [8] characterizations of classes of partitions, which use so many consecutive frequencies. Therefore, in order to obtain a partition identity relating multiplicity conditions (such as $f_i + f_{i+1} < k$) to conditions on residue classes of parts (such as $f_{5j} = f_{5j\pm 1} = 0$), or variants thereof, one has to come up with a way to make $r-$clusters work in functional equations. Otherwise, we need to devise a way to interpret general series of the form (4) using other mathematical machinery.

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Faculty of Engineering and Natural Sciences, Sabanci University, İstanbul, Turkey

E-mail address: `kursungoz@sabanciuniv.edu`

Department of Mathematics, The Pennsylvania State University, State College, PA 16802, United States

E-mail address: `sellersj@psu.edu`