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Infinite families of congruences modulo 5 and 9 for overpartitions

by

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Summary. Let $\bar{p}(n)$ denote the number of overpartitions of n. Recently, a number of congruences modulo 5 and powers of 3 for $\bar{p}(n)$ were established by a number of authors. In particular, Treneer proved that the generating function for $\bar{p}(5n)$ modulo 5 is $\sum_{n=0}^{\infty} \bar{p}(5n)q^n \equiv (q;q)_\infty^6/(q^2;q^2)_\infty^3 \pmod 5$. In this paper, employing elementary methods, we establish the generating function of $\bar{p}(5n)$ which yields the congruence due to Treneer. Furthermore, we prove some new congruences modulo 5 and 9 for $\bar{p}(n)$ by utilizing the fact that the generating functions for $\bar{p}(5n)$ modulo 5 and for $\bar{p}(3n)$ modulo 9 are eigenforms for half-integral weight Hecke operators.

1. Introduction. Recall that an overpartition of an integer is a partition in which the first occurrence of a part may be overlined. For instance, there are eight overpartitions of 3:

$$3,\ \bar{3},\ 2+1,\ \bar{2}+1,\ 2+\bar{1},\ \bar{2}+\bar{1},\ 1+1+1,\ \bar{1}+1+1.$$

As usual, let $\bar{p}(n)$ denote the number of overpartitions of n, and define $\bar{p}(0)$ to be 1. Therefore, $\bar{p}(3) = 8$. Since its introduction in [7], the overpartition function has been very popular, and has led to a number of studies in q-series, partition theory, number theory, modular and mock modular forms. The generating function for $\bar{p}(n)$ was given by Corteel and Lovejoy [7]:

$$\sum_{n=0}^{\infty} \bar{p}(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}, \tag{1.1}$$

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where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

Recently, a number of congruences modulo powers of 2, 3 and 5 for $\bar{p}(n)$ have been discovered. For congruences modulo powers of 2, we refer the reader to [3, 9, 12, 17, 21, 23]. For congruences modulo powers of 3, Hirschhorn and Sellers [13] proved that for $n, \beta \geq 0$,

$$\bar{p}(9^\beta(27n+18)) \equiv 0 \pmod{3}. \tag{1.2}$$

Congruences (1.2) were generalized by Lovejoy and Osburn [16]. Moreover, Xia [20] and Xia and Yao [21] established congruences modulo 9 and 27 for $\bar{p}(n)$. In a recent paper, Chen and Xia [5] proved that

$$\bar{p}(40n+35) \equiv 0 \pmod{40} \tag{1.3}$$

by using the (p,k)-parametrization of theta functions, which confirmed a conjecture given by Hirschhorn and Sellers [12]. Moreover, Lin [15] presented a new proof of (1.3). Hirschhorn [10] presented two simple proofs of (1.3). Earlier, by utilizing half-integral weight modular forms, Treneer [18] proved that

$$\sum_{n=0}^{\infty}\bar{p}(5n)q^n = \sum_{x,y,z\in\mathbb{Z}}(-1)^{x+y+z}q^{x^2+y^2+z^2} \equiv \frac{(q;q)_\infty^6}{(q^2;q^2)_\infty^3} \pmod{5}. \tag{1.4}$$

She used this to show that for a prime p such that $p \equiv 4 \pmod{5}$,

$$\bar{p}(5p^3n) \equiv 0 \pmod{5} \tag{1.5}$$

for all n coprime to p. Based on (1.4), many congruences modulo 5 for $\bar{p}(n)$ have been proved. Chen, Sun, Wang and Zhang [4] generalized (1.3) by proving that for $n, k \geq 0$,

$$\bar{p}(4^k(40n+35)) \equiv 0 \pmod{5}. \tag{1.6}$$

Moreover, they proved several infinite families of congruences modulo 5 for $\bar{p}(n)$; one of them is

$$\bar{p}(5\times 4^k p^2 n) \equiv 0 \pmod{5} \tag{1.7}$$

where $n,\ k$ are nonnegative integers, p is a prime with $p \equiv 3 \pmod{5}$ and $\left(\frac{-n}{p}\right) = -1$. Here $\left(\frac{a}{p}\right)$ denotes the Legendre symbol, which is defined by

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p \nmid a, \\ 0 & \text{if } p \mid a, \\ -1 & \text{if } a \text{ is a nonquadratic residue modulo } p. \end{cases}$$

Wang [19] gave a proof of (1.6) and established many congruences modulo 5 for $\bar{p}(n)$. Very recently, Dou and Lin [8] proved that for $n \geq 0$,

$$\bar{p}(80n+i) \equiv 0 \pmod{5},$$

where $i \in \{8, 52, 68, 72\}$. Hirschhorn [11] gave a simple proof of the above congruence.

The aim of this paper is to establish the generating function for $\bar{p}(5n)$ via elementary means and to prove new infinite families of congruences modulo 5 and 9 for $\bar{p}(n)$ by using the fact that the generating functions for $\bar{p}(5n)$ modulo 5 and for $\bar{p}(3n)$ modulo 9 are half-integral weight modular forms.

The main results of this paper can be stated as follows.

THEOREM 1.1. *We have*

$$\sum_{n=0}^{\infty} \bar{p}(5n) q^n = \frac{(q^2;q^2)_\infty^2 (q^5;q^5)_\infty^2}{(q;q)_\infty^4 (q^{10};q^{10})_\infty} - 5\frac{(q^2;q^2)_\infty^4 (q^5;q^5)_\infty^6}{(q;q)_\infty^8 (q^{10};q^{10})_\infty^3} + 5\frac{(q^2;q^2)_\infty^6 (q^5;q^5)_\infty^{10}}{(q;q)_\infty^{12} (q^{10};q^{10})_\infty^5}. \tag{1.8}$$

Note that (1.8) implies (1.4) after using the congruence

$$(q;q)_\infty^5 \equiv (q^5;q^5)_\infty \pmod{5}.$$

THEOREM 1.2. *Let $n \geq 1$, $k \geq 0$, $\alpha \geq 0$ be integers with $\left(\frac{-n}{p}\right) = -1$, where $p \geq 3$ is a prime with $p \equiv 1, 2, 3 \pmod{5}$.*

(1) *If $p \equiv 1 \pmod{5}$, then $\bar{p}(5 \times 4^k p^{10\alpha+4} n) \equiv 0 \pmod{5}$.*
(2) *If $p \equiv 2 \pmod{5}$, then $\bar{p}(5 \times 4^k p^{8\alpha+4} n) \equiv 0 \pmod{5}$.*
(3) *If $p \equiv 3 \pmod{5}$, then $\bar{p}(5 \times 4^k p^{8\alpha+2} n) \equiv 0 \pmod{5}$.*

Note that if we set $\alpha = 0$ and $p \equiv 3 \pmod{5}$ in Theorem 1.2, we get (1.7). Theorem 1.2 implies many congruences modulo 5 for $\bar{p}(n)$. For example, setting $k = \alpha = 0$ and $p = 7$, we see that for $n \geq 0$,

$$\bar{p}(5 \times 7^4 (7n+i)) \equiv 0 \pmod{5},$$

where $i \in \{1, 2, 4\}$.

THEOREM 1.3. *Let p be a prime with $p \equiv 4 \pmod{5}$. For $n, k, \alpha \geq 0$, if $p \nmid n$, then*

$$\bar{p}(5 \times 4^k p^{4\alpha+3} n) \equiv 0 \pmod{5}. \tag{1.9}$$

If we take $k = \alpha = 0$ in (1.9), we obtain (1.5). It should be noted that Wang [19] proved that for $n, \alpha \geq 0$, if $p \equiv 4 \pmod{5}$ is a prime with $p \nmid n$, then

$$\bar{p}(5p^{8\alpha+7} n) \equiv 0 \pmod{5},$$

which is a special case of (1.9).

THEOREM 1.4. *Let p be an odd prime and let n, k, α be nonnegative integers.*

(1) *If* $p \equiv 1 \pmod 5$, *then* $\bar{p}(4^k p^{10\alpha}(40n+35)) \equiv 0 \pmod 5$.
(2) *If* $p \equiv \pm 2 \pmod 5$, *then* $\bar{p}(4^k p^{8\alpha}(40n+35)) \equiv 0 \pmod 5$.
(3) *If* $p \equiv 4 \pmod 5$, *then* $\bar{p}(4^k p^{4\alpha}(40n+35)) \equiv 0 \pmod 5$.
(4) *If* $p = 5$, *then* $\bar{p}(4^k \times 5^{2\alpha+1}(8n+7)) \equiv 0 \pmod 5$.

From Theorem 1.4, we can deduce congruences modulo 5 for $\bar{p}(n)$. For example, taking $\alpha = 1$, $k = 0$ and $p = 11$ in Theorem 1.4, we see that for $n \geq 0$,

$$\bar{p}(11^{10}(40n+35)) \equiv 0 \pmod 5.$$

Moreover, we will prove the following congruences modulo 9 for $\bar{p}(n)$.

Theorem 1.5. *Let* n, α *be nonnegative integers and let* p *be a prime with* $p \equiv 1 \pmod 3$ *and* $\left(\frac{n}{p}\right) = -1$.

(1) *If* $p \equiv 1 \pmod 9$, *then* $\bar{p}(3p^{18\alpha+8}n) \equiv 0 \pmod 9$.
(2) *If* $p \equiv 4 \pmod 9$, *then* $\bar{p}(3p^{18\alpha+14}n) \equiv 0 \pmod 9$.
(3) *If* $p \equiv 7 \pmod 9$, *then* $\bar{p}(3p^{18\alpha+2}n) \equiv 0 \pmod 9$.

Theorem 1.6. *Let* p *be a prime with* $p \equiv 1 \pmod 3$. *For* $n, \alpha, \beta \geq 0$, *if* $p \mid (3n+2)$ *and* $p^2 \nmid (3n+2)$, *then*

$$\bar{p}(3^{2\beta+2} p^{6\alpha+4}(3n+2)) \equiv 0 \pmod 9. \tag{1.10}$$

Theorem 1.7. *Let* p *be a prime with* $p \equiv 1 \pmod 3$. *For* $\alpha, \beta, n \geq 0$, *if* $\left(\frac{9n+6}{p}\right) = -1$, *then*

$$\bar{p}(3^{2\beta+2} p^{6\alpha+2}(3n+2)) \equiv 0 \pmod 9. \tag{1.11}$$

2. Proof of Theorem 1.1. In this section, we establish the generating function for $\bar{p}(5n)$.

Replacing q by $-q$ in (1.1) and using the fact that

$$(-q;-q)_\infty = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty (q^4;q^4)_\infty}, \tag{2.1}$$

we have

$$\sum_{n=0}^{\infty} (-1)^n \bar{p}(n) q^n = \frac{1}{\phi(q)}, \tag{2.2}$$

where $\phi(q)$ is defined by

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}.$$

Let $\omega = e^{2\pi i/5}$. We rewrite (2.2) as

$$\sum_{n=0}^{\infty} (-1)^n \bar{p}(n) q^n = \frac{1}{\phi(q)} = \frac{\phi(q^{25})}{\phi(q^5)^6} \phi(\omega q)\phi(\omega^2 q)\phi(\omega^3 q)\phi(\omega^4 q). \tag{2.3}$$

The 5-dissection of $\phi(q)$ is [10, (36.3.2)]

$$\phi(q)=\phi(q^{25})+2qB(q^5)+2q^4C(q^5), \tag{2.4}$$

where

$$B(q)=(-q^3;q^{10})_\infty(-q^7;q^{10})_\infty(q^{10};q^{10})_\infty,$$
$$C(q)=(-q;q^{10})_\infty(-q^9;q^{10})_\infty(q^{10};q^{10})_\infty.$$

Substituting (2.4) into (2.3) and employing the fact that $\omega^5=1$ we get

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\bar{p}(n)q^n=\frac{\phi(q^{25})}{\phi(q^5)^6}&(\phi(q^{25})+2\omega qB(q^5)+2\omega^4q^4C(q^5))\\ &\times(\phi(q^{25})+2\omega^2qB(q^5)+2\omega^3q^4C(q^5))\\ &\times(\phi(q^{25})+2\omega^3qB(q^5)+2\omega^2q^4C(q^5))\\ &\times(\phi(q^{25})+2\omega^4qB(q^5)+2\omega q^4C(q^5)).\end{aligned} \tag{2.5}$$

Extracting the terms of the form q^{5n} in (2.5) and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty}(-1)^n\bar{p}(5n)q^n=\frac{\phi(q^5)}{\phi(q)^6}\big(\phi(q^5)^4-12q\phi(q^5)^2B(q)C(q)+16q^2B(q)^2C(q)^2\big).$$

The following identity can be found in Andrews and Berndt's book [1, (1.6.6)] and Hirschhorn's book [10, (34.1.20)]:

$$\phi(q)^2-\phi(q^5)^2=4qB(q)C(q). \tag{2.6}$$

Therefore, by (2.6),

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^n\bar{p}(5n)q^n\\ &\quad=\frac{\phi(q^5)}{\phi(q)^6}\big(\phi(q^5)^4-3\phi(q^5)^2(\phi(q)^2-\phi(q^5)^2)+(\phi(q)^2-\phi(q^5)^2)^2\big)\\ &\quad=\frac{\phi(q^5)}{\phi(q)^6}\big(\phi(q)^4-5\phi(q)^2\phi(q^5)^2+5\phi(q^5)^4\big).\end{aligned} \tag{2.7}$$

Replacing q by $-q$ in (2.7) and then applying the fact that

$$\phi(-q)=\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty},$$

we arrive at (1.8). ∎

3. Proofs of Theorems 1.2 and 1.3. We first prove the following lemma.

LEMMA 3.1. *Let p be an odd prime. Then*

$$\bar{p}(5p^{2\alpha}n) \equiv \frac{p^\alpha - 1}{p-1}\bar{p}(5p^2n) + \frac{p-p^\alpha}{p-1}\bar{p}(5n) \pmod 5. \tag{3.1}$$

Proof. It is easy to check that (3.1) is true for $\alpha = 0$ and $\alpha = 1$. Suppose (3.1) holds for two consecutive values, m and $m+1$ ($m \geq 0$), that is,

$$\bar{p}(5p^{2m}n) \equiv \frac{p^m - 1}{p-1}\bar{p}(5p^2n) + \frac{p-p^m}{p-1}\bar{p}(5n) \pmod 5, \tag{3.2}$$

$$\bar{p}(5p^{2m+2}n) \equiv \frac{p^{m+1} - 1}{p-1}\bar{p}(5p^2n) + \frac{p-p^{m+1}}{p-1}\bar{p}(5n) \pmod 5, \tag{3.3}$$

where p is an odd prime. In their nice paper [4], Chen, Sun, Wang and Zhang proved that if p is an odd prime, then

$$\bar{p}(5p^2n) \equiv \left(p+1-\left(\frac{-n}{p}\right)\right)\bar{p}(5n) - p\cdot\bar{p}\left(\frac{5n}{p^2}\right) \pmod 5. \tag{3.4}$$

Replacing n by $p^{2m+2}n$ ($m \geq 0$) in (3.4) and employing (3.2) and (3.3) yields

$$\begin{aligned}\bar{p}(5p^{2m+4}n) &\equiv (p+1)\bar{p}(5p^{2m+2}n) - p\cdot\bar{p}(5p^{2m}n) \\ &\equiv (p+1)\left(\frac{p^{m+1}-1}{p-1}\bar{p}(5p^2n) + \frac{p-p^{m+1}}{p-1}\bar{p}(5n)\right) \\ &\quad - p\left(\frac{p^m-1}{p-1}\bar{p}(5p^2n) + \frac{p-p^m}{p-1}\bar{p}(5n)\right) \\ &\equiv \frac{p^{m+2}-1}{p-1}\bar{p}(5p^2n) + \frac{p-p^{m+2}}{p-1}\bar{p}(5n) \pmod 5,\end{aligned} \tag{3.5}$$

which implies that (3.1) is true when $\alpha = m+2$. ■

Now, we are ready to prove Theorems 1.2 and 1.3.

In view of (3.1) and (3.4), we see that if $p \geq 3$ is a prime, then for $n, \alpha \geq 0$,

$$\begin{aligned}\bar{p}(5p^{2\alpha}n) &\equiv \frac{p^\alpha-1}{p-1}\left(\left(p+1-\left(\frac{-n}{p}\right)\right)\bar{p}(5n) - p\cdot\bar{p}(5np^2)\right) \\ &\quad + \frac{p-p^\alpha}{p-1}\bar{p}(5n) \\ &\equiv \lambda(p,\alpha,n)\bar{p}(5n) - \frac{p(p^\alpha-1)}{p-1}\bar{p}(5np^2) \pmod 5,\end{aligned} \tag{3.6}$$

where

$$\lambda(p,\alpha,n) = \frac{p^{\alpha+1} - \left(\frac{-n}{p}\right)p^\alpha - 1 + \left(\frac{-n}{p}\right)}{p-1}. \tag{3.7}$$

Note that if $p \geq 3$ is a prime with $p \neq 5$ and $\left(\frac{-n}{p}\right) = -1$, then $5n/p$ and

$5n/p^2$ are not integers and

$$\bar{p}\left(\frac{5n}{p^2}\right) = 0. \tag{3.8}$$

If $p \equiv 1 \pmod 5$, then

$$\begin{aligned}\lambda(p, 5\alpha+2, n) &= \frac{p^{5\alpha+3}+p^{5\alpha+2}-2}{p-1} = \frac{p^{5\alpha+3}-1}{p-1} + \frac{p^{5\alpha+2}-1}{p-1} \\ &= \sum_{j=0}^{5\alpha+2} p^j + \sum_{j=0}^{5\alpha+1} p^j \equiv 5\alpha+3+5\alpha+2 \equiv 0 \pmod 5.\end{aligned} \tag{3.9}$$

If $p \equiv 2 \pmod 5$, then

$$\begin{aligned}\lambda(p, 4\alpha+2, n) &= \frac{p^{4\alpha+3}+p^{4\alpha+2}-2}{p-1} = \frac{p^{4\alpha+3}-1}{p-1} + \frac{p^{4\alpha+2}-1}{p-1} \\ &= \sum_{j=0}^{4\alpha+2} p^j + \sum_{j=0}^{4\alpha+1} p^j \equiv \sum_{j=0}^{4\alpha+2} 2^j + \sum_{j=0}^{4\alpha+1} 2^j \\ &\equiv 2^{4\alpha+3} + 2^{4\alpha+2} - 2 \equiv 0 \pmod 5.\end{aligned} \tag{3.10}$$

If $p \equiv 3 \pmod 5$, then

$$\begin{aligned}\lambda(p, 4\alpha+1, n) &= \frac{p^{4\alpha+2}+p^{4\alpha+1}-2}{p-1} = \frac{p^{4\alpha+2}-1}{p-1} + \frac{p^{4\alpha+1}-1}{p-1} \\ &= \sum_{j=0}^{4\alpha+1} p^j + \sum_{j=0}^{4\alpha} p^j \equiv \sum_{j=0}^{4\alpha+1} 3^j + \sum_{j=0}^{4\alpha} 3^j \\ &\equiv \frac{3^{4\alpha+2}-1}{2} + \frac{3^{4\alpha+1}-1}{2} \equiv 0 \pmod 5.\end{aligned} \tag{3.11}$$

Combining (3.9)–(3.11), we deduce that

$$\lambda\left(p, \frac{\gamma(p)}{2}\alpha + \frac{\chi(q)}{2}, n\right) \equiv 0 \pmod 5, \tag{3.12}$$

where the pair $(\gamma(p), \chi(p))$ is defined by

$$(\gamma(p), \chi(p)) := \begin{cases} (10,4) & \text{if } p \equiv 1 \pmod 5, \\ (8,4) & \text{if } p \equiv 2 \pmod 5, \\ (8,2) & \text{if } p \equiv 3 \pmod 5. \end{cases}$$

Replacing α by $\frac{\gamma(p)}{2}\alpha + \frac{\chi(q)}{2}$ in (3.6) and using (3.12) yields

$$\bar{p}(5p^{\gamma(q)\alpha+\chi(q)}n) \equiv -\frac{p(p^\alpha-1)}{p-1}\bar{p}\left(\frac{5n}{p^2}\right) \pmod 5. \tag{3.13}$$

Combining (3.8) and (3.13), we see that if $p \equiv 1, 2, 3 \pmod 5$ and $\left(\frac{-n}{p}\right) = -1$, then

$$\bar{p}(5p^{\gamma(q)\alpha+\chi(q)}n) \equiv 0 \pmod{5}. \tag{3.14}$$

Chen, Sun, Wang and Zhang [4] proved that for $n \geq 0$,

$$\bar{p}(20n) \equiv (-1)^n \bar{p}(5n) \pmod{5}. \tag{3.15}$$

By (3.15) and induction, we see that for $n, k \geq 0$,

$$\bar{p}(5 \times 4^k n) \equiv (-1)^{ng(k)} \bar{p}(5n) \pmod{5}, \tag{3.16}$$

where $g(0) = 0$ and $g(k) = 1$ for $k \geq 1$. Formulas (3.14) and (3.16) complete the proof of Theorem 1.2. ∎

We now prove Theorem 1.3. Replacing n by pn in (3.6) yields

$$\bar{p}(5p^{2\alpha+1}n) \equiv \frac{p^{\alpha+1}-1}{p-1}\bar{p}(5pn) - \frac{p(p^{\alpha}-1)}{p-1}\bar{p}\left(\frac{5n}{p}\right) \pmod{5}. \tag{3.17}$$

If $p \equiv 4 \pmod{5}$, then

$$\frac{p^{2\alpha+2}-1}{p-1} = \sum_{j=0}^{2\alpha+1} p^j \equiv \sum_{j=0}^{2\alpha+1} (-1)^j \equiv 0 \pmod{5}. \tag{3.18}$$

Replacing α by $2\alpha+1$ in (3.17) and employing (3.18), we see that

$$\bar{p}(5p^{4\alpha+3}n) \equiv -\frac{p(p^{2\alpha+1}-1)}{p-1}\bar{p}\left(\frac{5n}{p}\right) \pmod{5}. \tag{3.19}$$

By (3.19) and the fact that $\bar{p}\big(\frac{5n}{p}\big) = 0$ when $p \nmid n$, we get

$$\bar{p}(5p^{4\alpha+3}n) \equiv 0 \pmod{5}. \tag{3.20}$$

It follows from (3.16) and (3.20) that (1.9) holds. ∎

4. Proof of Theorem 1.4. Assume that $p \geq 3$ is a prime and α is a nonnegative integer in this section. If $p \equiv 1 \pmod{5}$, then

$$\begin{aligned} \frac{p-p^{5\alpha+1}}{p-1} &= \frac{p-1+1-p^{5\alpha+1}}{p-1} = 1 - \frac{p^{5\alpha+1}-1}{p-1} \\ &= 1 - \sum_{j=0}^{5\alpha} p^j \equiv 1-(5\alpha+1) \equiv 0 \pmod{5}. \end{aligned} \tag{4.1}$$

If $p \equiv 2 \pmod{5}$, then

$$\begin{aligned} \frac{p-p^{4\alpha+1}}{p-1} &= \frac{p-1+1-p^{4\alpha+1}}{p-1} = 1 - \frac{p^{4\alpha+1}-1}{p-1} \\ &= 1 - \sum_{j=0}^{4\alpha} p^j \equiv 1 - \sum_{j=0}^{4\alpha} 2^j \equiv 1-(2^{4\alpha+1}-1) \equiv 0 \pmod{5}. \end{aligned} \tag{4.2}$$

If $p \equiv 3 \pmod 5$, then

$$\begin{aligned} \frac{p-p^{4\alpha+1}}{p-1} &= \frac{p-1+1-p^{4\alpha+1}}{p-1} = 1 - \frac{p^{4\alpha+1}-1}{p-1} \\ &= 1 - \sum_{j=0}^{4\alpha} p^j \equiv 1 - \sum_{j=0}^{4\alpha} 3^j \equiv 1 - \frac{3^{4\alpha+1}-1}{2} \equiv 0 \pmod 5. \end{aligned} \tag{4.3}$$

If $p \equiv 4 \pmod 5$, then

$$\begin{aligned} \frac{p-p^{2\alpha+1}}{p-1} &= \frac{p-1+1-p^{2\alpha+1}}{p-1} = 1 - \frac{p^{2\alpha+1}-1}{p-1} \\ &= 1 - \sum_{j=0}^{2\alpha} p^j \equiv 1 - \sum_{j=0}^{2\alpha} (-1)^j \equiv 0 \pmod 5. \end{aligned} \tag{4.4}$$

If $p = 5$, then

$$\frac{p-p^{\alpha+1}}{p-1} \equiv 0 \pmod 5. \tag{4.5}$$

Based on (4.1)–(4.5), we get

$$\frac{p-p^{\frac{\nu(p)\alpha}{2}+1}}{p-1} \equiv 0 \pmod 5, \tag{4.6}$$

where $\nu(p)$ is defined by

$$\nu(p) := \begin{cases} 10 & \text{if } p \equiv 1 \pmod 5, \\ 8 & \text{if } p \equiv \pm 2 \pmod 5, \\ 4 & \text{if } p \equiv 4 \pmod 5, \\ 2 & \text{if } p = 5. \end{cases}$$

Replacing α by $\nu(p)\alpha/2$ in (3.6) and using (4.6) yields

$$\bar{p}(5p^{\nu(p)\alpha}n) \equiv \lambda(p, \nu(p)\alpha/2, n)\bar{p}(5n) \pmod 5. \tag{4.7}$$

Replacing n by $4^k(8n+7)$ in (4.7) and using (1.6), we arrive at the congruences stated in Theorem 1.4. ∎

5. Proof of Theorem 1.5.

We begin with the following lemma.

LEMMA 5.1. *Let p be an odd prime and let $r_5(n)$ denote the number of representations of n as the sum of five squares. For $n, \alpha \geq 0$,*

$$r_5(p^{2\alpha}n) = \frac{p^{3\alpha}-1}{p^3-1} r_5(p^2 n) + \frac{p^3-p^{3\alpha}}{p^3-1} r_5(n). \tag{5.1}$$

Proof. We use induction on α. It is easy to check that (5.1) is true when $\alpha = 0$ and $\alpha = 1$. Assume that (5.1) is true when $\alpha = m$ and $\alpha = m + 1$

$(m \geq 0)$, that is,

$$r_5(p^{2m}n) = \frac{p^{3m}-1}{3^5-1} r_5(p^2n) + \frac{p^3-p^{3m}}{p^5-1} r_5(n), \tag{5.2}$$

$$r_5(p^{2m+2}n) = \frac{p^{3m+3}-1}{p^3-1} r_5(p^2n) + \frac{p^3-p^{3m+3}}{p^3-1} r_5(n), \tag{5.3}$$

where p is an odd prime. Cooper [6] proved that

$$r_5(p^2n) = \left(p^3 - p\left(\frac{n}{p}\right) + 1\right) r_5(n) - p^3 r_5\left(\frac{n}{p^2}\right). \tag{5.4}$$

Replacing n by p^2n in (5.4) yields

$$r_5(p^4n) = (p^3+1) r_5(p^2n) - p^3 r_5(n). \tag{5.5}$$

Replacing n by $p^{2m}n$ in (5.5) and employing (5.2) and (5.3) yields

$$\begin{aligned} r_5(p^{2m+4}n) &= (p^3+1) r_5(p^{2m+2}n) - p^3 r_5(p^{2m}n) \\ &= (p^3+1)\left(\frac{p^{3m+3}-1}{p^3-1} r_5(p^2n) + \frac{p^3-p^{3m+3}}{p^3-1} r_5(n)\right) \\ &\quad - p^3\left(\frac{p^{3m}-1}{p^3-1} r_5(p^2n) + \frac{p^3-p^{3m}}{p^3-1} r_5(n)\right) \\ &= \frac{p^{3m+6}-1}{p^3-1} r_5(p^2n) + \frac{p^3-p^{3m+6}}{p^6-1} r_5(n), \end{aligned}$$

which implies (5.1) when $\alpha = m+2$. ∎

Now, we are ready to prove Theorem 1.5. By (5.1) and (5.4),

$$\begin{aligned} r_5(p^{2\alpha}n) &= \frac{p^{3\alpha}-1}{p^3-1}\left(\left(p^3 - p\left(\frac{n}{p}\right) + 1\right) r_5(n) - p^3 r_5\left(\frac{n}{p^2}\right)\right) \\ &\quad + \frac{p^3-p^{3\alpha}}{p^3-1} r_5(n) \\ &= \lambda_2(p,\alpha,n) r_5(n) - \frac{p^3(p^{3\alpha}-1)}{p^3-1} r_5\left(\frac{n}{p^2}\right), \end{aligned} \tag{5.6}$$

where

$$\lambda_2(p,\alpha,n) = \frac{p^{3\alpha+3} - p^{3\alpha+1}\left(\frac{n}{p}\right) + p\left(\frac{n}{p}\right) - 1}{p^3-1}. \tag{5.7}$$

If $\left(\frac{n}{p}\right) = -1$ and $p \equiv k \pmod 9$ with $k \in \{1,4,7\}$, then $k^3 \equiv 1 \pmod 9$ and for $\alpha \geq 0$,

$$\begin{aligned}
(5.8)\quad \lambda_2\left(p, 9\alpha+\frac{c(p)}{2}, n\right) &= \frac{p^{27\alpha+3c(p)/2+3}+p^{27\alpha+3c(p)/2+1}-p-1}{p^3-1} \\
&= \frac{p^{27\alpha+3c(p)/2+3}-1}{p^3-1}+p\frac{p^{27\alpha+3c(p)/2}-1}{p^3-1} \\
&= \sum_{i=0}^{9\alpha+c(p)/2} p^{3i}+p\sum_{i=0}^{9\alpha+c(p)/2-1} p^{3i} \\
&\equiv \sum_{i=0}^{9\alpha+c(p)/2} k^{3i}+k\sum_{i=0}^{9\alpha+c(p)/2-1} k^{3i} \\
&\equiv 9\alpha+\frac{c(p)}{2}+1+k\left(9\alpha+\frac{c(p)}{2}\right) \equiv 0 \pmod 9,
\end{aligned}$$

where $c(p)$ is defined by

$$c(p) := \begin{cases} 8 & \text{if } p \equiv 1 \pmod 9, \\ 14 & \text{if } p \equiv 4 \pmod 9, \\ 2 & \text{if } p \equiv 7 \pmod 9. \end{cases}$$

Replacing α by $9\alpha + c(p)/2$ in (5.6) and using (5.8), we get

$$(5.9)\qquad r_5(p^{18\alpha+c(p)}n) \equiv -\frac{p^3(p^{27\alpha+3c(p)/2}-1)}{p^3-1} r_5\left(\frac{n}{p^2}\right) \pmod 9.$$

Moreover, $\left(\frac{n}{p}\right) = -1$ implies that n/p^2 is not an integer and

$$(5.10)\qquad r_5\left(\frac{n}{p^2}\right) = 0.$$

Combining (5.9) and (5.10), we see that if $p \equiv k \pmod 9$ with $k \in \{1, 4, 7\}$ and $\left(\frac{n}{p}\right) = -1$, then

$$(5.11)\qquad r_5(p^{18\alpha+c(p)}n) \equiv 0 \pmod 9.$$

Fortin, Jacob and Mathieu [9] and Hirschhorn and Sellers [12] also proved that

$$(5.12)\qquad \sum_{n=0}^{\infty} \bar{p}(3n)q^n = \frac{(q^2;q^2)_\infty^4 (q^3;q^3)_\infty^6}{(q;q)_\infty^8 (q^6;q^6)_\infty^3}.$$

By the binomial theorem,

$$(5.13)\qquad (q;q)_\infty^9 \equiv (q^3;q^3)_\infty^3 \pmod 9.$$

Based on (5.12) and (5.13), we deduce

$$(5.14)\qquad \sum_{n=0}^{\infty} \bar{p}(3n)q^n \equiv \frac{(q;q)_\infty^{10}}{(q^2;q^2)_\infty^5} \pmod 9.$$

It should be noted that the generating function of $r_5(n)$ is

$$\sum_{n=0}^{\infty} r_5(n)q^n = \frac{(q^2;q^2)_\infty^{25}}{(q;q)_\infty^{10}(q^4;q^4)_\infty^{10}}. \tag{5.15}$$

Replacing q by $-q$ in (5.15) and using (2.1), we get

$$\sum_{n=0}^{\infty} (-1)^n r_5(n)q^n = \frac{(q;q)_\infty^{10}}{(q^2;q^2)_\infty^{5}}. \tag{5.16}$$

It follows from (5.14) and (5.16) that for $n \geq 0$,

$$\bar{p}(3n) \equiv (-1)^n r_5(n) \pmod{9}. \tag{5.17}$$

As observed by Lovejoy and Osburn [16], the second and third authors [13] proved that for $n \geq 0$,

$$\bar{p}(3n) \equiv (-1)^n r_5(n) \pmod{3}.$$

Replacing n by $p^{18\alpha+c(p)}n$ in (5.17) and using (5.11), we get the congruences stated in Theorem 1.5. ■

6. Proofs of Theorems 1.6 and 1.7. In this section, we assume that p is a prime with $p \equiv 1 \pmod 3$. By (1.2) and (5.17), we see that for $n, \beta \geq 0$,

$$r_5(9^\beta(9n+6)) \equiv 0 \pmod{3}. \tag{6.1}$$

Replacing n by $9^\beta(9n+6)$ in (5.6) yields

$$\begin{aligned} r_5(9^\beta p^{2\alpha}(9n+6)) &= \lambda_2(p,\alpha,9^\beta(9n+6))r_5(9^\beta(9n+6)) \\ &\quad - \frac{p^3(p^{3\alpha}-1)}{p^3-1} r_5\left(\frac{9^\beta(9n+6)}{p^2}\right), \end{aligned} \tag{6.2}$$

where $\lambda_2(p,\alpha,n)$ is defined by (5.7). If $p \mid (3n+2)$, then

$$\lambda_2(p,3\alpha+2,9^\beta(9n+6)) = \frac{p^{9\alpha+9}-1}{p^3-1} = \sum_{i=0}^{3\alpha+2} p^{3i} \equiv 3\alpha+3 \equiv 0 \pmod{3}. \tag{6.3}$$

If $p^2 \nmid (3n+2)$, then $9^\beta(9n+6)/p^2$ is not an integer and

$$r_5\left(\frac{9^\beta(9n+6)}{p^2}\right) = 0. \tag{6.4}$$

Replacing α by $3\alpha+2$ in (6.2) and utilizing (6.1), (6.3) and (6.4), we find that if $p \mid (3n+2)$ and $p^2 \nmid (3n+2)$, then

$$r_5(9^\beta p^{6\alpha+4}(9n+6)) \equiv 0 \pmod{9}. \tag{6.5}$$

Replacing n by $9^\beta p^{6\alpha+4}(9n+6)$ in (5.17) and employing (6.5), we obtain (1.10). This completes the proof of Theorem 1.6. ■

We now turn to the proof of Theorem 1.7. It is easy to check that if

$$\left(\frac{9n+6}{p}\right) = -1, \tag{6.6}$$

then $\left(\frac{9^\beta(9n+6)}{p}\right) = -1$ and

$$\begin{aligned} \lambda_2(p, 3\alpha+1, 9^\beta(9n+6)) &= \frac{p^{9\alpha+6} + p^{9\alpha+4} - p - 1}{p^3 - 1} \\ &= \frac{p^{9\alpha+6} - 1}{p^3 - 1} + p\frac{p^{9\alpha+3} - 1}{p^3 - 1} \\ &= \sum_{i=0}^{3\alpha+1} p^{3i} + p\sum_{i=0}^{3\alpha} p^{3i} \\ &\equiv 3\alpha + 2 + 3\alpha + 1 \equiv 0 \pmod 3. \end{aligned} \tag{6.7}$$

Note that (6.6) also implies that $9^\beta(9n+6)/p^2$ is not an integer and

$$r_5\left(\frac{9^\beta(9n+6)}{p^2}\right) = 0. \tag{6.8}$$

Replacing α by $3\alpha+1$ in (6.2) and employing (6.1), (6.7) and (6.8), we deduce that if (6.6) holds, then

$$r_5(9^\beta p^{6\alpha+2}(9n+6)) \equiv 0 \pmod 9. \tag{6.9}$$

Replacing n by $9^\beta p^{6\alpha+2}(9n+6)$ in (5.17) and utilizing (6.9), we get (1.11). ∎

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