$m\text{-}\mathbf{ARY}$ PARTITIONS WITH NO GAPS: A CHARACTERIZATION MODULO m

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ABSTRACT. In a recent work, the authors provided the first-ever characterization of the values $b_m(n)$ modulo m where $b_m(n)$ is the number of (unrestricted) m-ary partitions of the integer n and $m \geq 2$ is a fixed integer. That characterization proved to be quite elegant and relied only on the base m representation of n. Since then, the authors have been motivated to consider a specific restricted m-ary partition function, namely $c_m(n)$, the number of m-ary partitions of n where there are no "gaps" in the parts. (That is to say, if m^i is a part in a partition counted by $c_m(n)$, and i is a positive integer, then m^{i-1} must also be a part in the partition.) Using tools similar to those utilized in the aforementioned work on $b_m(n)$, we prove the first-ever characterization of $c_m(n)$ modulo m. As with the work related to $b_m(n)$ modulo m, this characterization of $c_m(n)$ modulo m is also based solely on the base m representation of n.

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1. INTRODUCTION

In this note, we will focus our attention on congruence properties for the partition functions which enumerate restricted integer partitions known as *m*-ary partitions. These are partitions of an integer *n* wherein each part is a power of a fixed integer $m \ge 2$. Throughout this note, we will let $b_m(n)$ denote the number of *m*-ary partitions of *n*.

As an example, note that there are five 3-ary partitions of n = 9:

9, 3+3+3, 3+3+1+1+1, 3+1+1+1+1+1+1, 1+1+1+1+1+1+1+1) = 5.

Thus, $b_3(9) = 5$.

In the late 1960s, Churchhouse [5, 6] initiated the study of congruence properties of binary partitions (*m*-ary partitions with m = 2). Within months, other mathematicians proved Churchhouse's conjectures and proved natural extensions of his results. These included Rødseth [9] who extended Churchhouse's results to include the functions $b_p(n)$ where p is any prime as well as Andrews [2] and Gupta [7, 8] who proved that corresponding results also held for $b_m(n)$ where m could be any integer greater than 1. As part of an infinite family of results, these authors proved that, for any $m \ge 2$ and any nonnegative integer n, $b_m(m(mn-1)) \equiv 0 \pmod{m}$.

Quite recently, the authors [3] provided the following mod m characterization of $b_m(mn)$ relying solely on the base m representation of n:

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Theorem 1.1. If $m \ge 2$ is a fixed integer and

$$n = \alpha_0 + \alpha_1 m + \dots + \alpha_j m^j$$

is the base m representation of n (so that $0 \le \alpha_i \le m-1$ for each i), then

$$b_m(mn) \equiv \prod_{i=0}^j (\alpha_i + 1) \pmod{m}.$$

In this note, we provide a similar mod m result for the values $c_m(mn)$, where $c_m(n)$ is the number of m-ary partitions of n with "no gaps" in the parts. More specifically, $c_m(n)$ counts the number of partitions of n into powers of m such that, if m^i is a part in a partition counted by $c_m(n)$, and i is a positive integer, then m^{i-1} must also be a part in the partition. For example, there are six such partitions counted by $c_3(15)$:

Note, in particular, that 9 + 1 + 1 + 1 + 1 + 1 + 1 does not appear in the above list because it does not contain the part 3, and 3 + 3 + 3 + 3 + 3 + 3 is missing from the list because it does not contain the part 1.

This family of functions $c_m(n)$ is motivated by a recent work of Bessenrodt, Olsson, and Sellers [4] in which the function $c_2(n)$ plays a critical role.

2. The Main Result

The following theorem provides a complete characterization of $c_m(mn)$ modulo m:

Theorem 2.1. Let $m \ge 2$ be a fixed integer and let

$$n = \sum_{i=j}^{\infty} \alpha_i m^i$$

be the base m representation of n where $1 \le \alpha_j < m$ and $0 \le \alpha_i < m$ for i > j.

(1) If j is even, then

$$c_m(mn) \equiv \alpha_j + (\alpha_j - 1) \sum_{i=j+1}^{\infty} \alpha_{j+1} \dots \alpha_i \pmod{m}.$$

(2) If j is odd, then

$$c_m(mn) \equiv 1 - \alpha_j - (\alpha_j - 1) \sum_{i=j+1}^{\infty} \alpha_{j+1} \dots \alpha_i \pmod{m}.$$

Remark 2.2. Note that Lemma 2.7 (which appears below) implies that Theorem 2.1 tells us the congruence class of $c_m(n)$ modulo m for all n, not just those values of n which are divisible by m.

In order to prove Theorem 2.1, we need a few elementary tools. We describe these tools here.

First, it is important to note the generating function for $c_m(n)$.

Lemma 2.3.

$$C_m(q) := 1 + \sum_{n=0}^{\infty} \frac{q^{1+m+m^2+\dots+m^n}}{(1-q)(1-q^m)\dots(1-q^{m^n})}.$$

Proof. The proof follows from a standard argument from [1, Chapter 1].

Next, we wish to find the generating function for $c_m(mn)$.

Lemma 2.4.

(1)
$$\sum_{n=0}^{\infty} c_m(mn)q^n = 1 + \frac{q}{1-q}C_m(q)$$

Proof. Note that $C_m(q)$ can be rewritten as

$$C_m(q) = 1 + \sum_{n=0}^{\infty} \frac{q^{m+m^2+\dots+m^n}}{(1-q^m)\dots(1-q^{m^n})} \frac{q}{1-q}$$

= $1 + \frac{q}{1-q} + \sum_{n=1}^{\infty} \frac{q^{m+m^2+\dots+m^n}}{(1-q^m)\dots(1-q^{m^n})} \cdot \sum_{j=1}^{\infty} q^j.$

Hence,

$$\sum_{n=0}^{\infty} c_m(mn)q^{mn} = \frac{1}{1-q^m} + \sum_{n=1}^{\infty} \frac{q^{m+m^2+\dots+m^n}}{(1-q^m)\dots(1-q^{m^n})} \cdot \sum_{j=1}^{\infty} q^{jm}$$
$$= \frac{1}{1-q^m} + \frac{q^m}{1-q^m} \cdot \sum_{n=1}^{\infty} \frac{q^{m+m^2+\dots+m^n}}{(1-q^m)\dots(1-q^{m^n})}$$
$$= \frac{1}{1-q^m} + \frac{q^m}{1-q^m} (C_m(q^m) - 1)$$
$$= 1 + \frac{q^m}{1-q^m} + \frac{q^m}{1-q^m} C_m(q^m).$$

The proof follows by replacing q^m by q.

From Lemma 2.4, we have the following recurrence satisfied by $c_m(mn)$. Lemma 2.5. For $n \ge 1$,

$$c_m(mn) = c_m(0) + c_m(1) + \dots + c_m(n-1).$$

Proof. Compare coefficients of q^n on both sides of the identity in Lemma 2.4.

Lemma 2.6.

$$C_m(q) = -q^{-1} - q^{-2} - \dots - q^{-(m-1)} + (1 + q^{-1} + \dots + q^{-(m-1)}) \sum_{n=0}^{\infty} c_m(mn) q^{mn}$$

Proof. By Lemma 2.4,

$$\sum_{n=0}^{\infty} c_m(mn)q^{mn} = 1 + \frac{q^m}{1-q^m}C_m(q^m).$$

On the other hand,

$$C_m(q) = 1 + \frac{q}{1-q} + \sum_{n=1}^{\infty} \frac{q^{m+\dots+m^n}}{(1-q^m)\dots(1-q^{mn})} \cdot \frac{q}{1-q}$$

= $\frac{1}{1-q} + \frac{q}{1-q} \sum_{n=0}^{\infty} \frac{q^{m(1+m+\dots+m^n)}}{(1-q^m)\dots(1-q^{m\cdot m^n})}$
= $\frac{1}{1-q} + \frac{q}{1-q} C_m(q^m).$

Therefore,

$$C_m(q^m) = q^{-1}(C_m(q)(1-q) - 1)$$

and so

$$\sum_{n=0}^{\infty} c_m(mn)q^{mn} = 1 + \frac{q^{m-1}}{1-q^m}(C_m(q)(1-q)-1).$$

Solving for $C_m(q)$ gives the desired result.

Lemma 2.6 can now be used to prove that the values of the function $c_m(n)$ come in *m*-tuples as described in the next lemma.

Lemma 2.7. For all $n \ge 1$,

 $c_m(mn) = c_m(mn-1) = c_m(mn-2) = \dots = c_m(mn-(m-1)).$

Proof. Compare coefficients of q^n on both sides of the identity in Lemma 2.6.

We now begin the consideration of $c_m(mn)$ modulo m by proving the following lemma:

Lemma 2.8. If $n \equiv k \pmod{m}$ where $1 \leq k \leq m$, then for all $n \geq 1$,

$$c_m(mn) \equiv 1 + (k-1)c_m(n) \pmod{m}.$$

Proof. By Lemma 2.5,

$$c_m(mn) = c_m(0) + c_m(1) \dots + c_m(n-1)$$

Next, we write n = jm + k for some integer j. Then

$$c_{m}(mn) = c_{m}(0) + c_{m}(1) + \dots + c_{m}(m) + c_{m}(m+1) + \dots + c_{m}(2m)$$

$$\vdots + c_{m}((j-1)m+1) + \dots + c_{m}((j-1)m+m) + c_{m}(jm+1) + \dots + c_{m}(jm+k-1)$$

$$\equiv 1 + c_{m}(jm+1) + \dots + c_{m}(jm+k-1) \pmod{m}$$
by Lemma 2.7
$$\equiv 1 + (k-1)c_{m}(jm+k) \pmod{m}$$
by Lemma 2.7
$$= 1 + (k-1)c_{m}(n).$$

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Next, we prove an additional lemma involving an "internal" congruence satisfied by c_m modulo m. It is interesting to note that a similar result holds for $b_m(n)$, the unrestricted m-ary partition function studied in [3, 5, 6].

Lemma 2.9. For all $n \ge 0$,

$$c_m(m^3n) \equiv c_m(mn) \pmod{m}.$$

Proof. By Lemma 2.8, we know

$$c_m(m^3n) = c_m(m(m^2n))$$

$$\equiv 1 + (m-1)c_m(m^2n) \pmod{m}$$

$$= 1 + (m-1)c_m(m(mn))$$

$$\equiv 1 + (m-1)(1 + (m-1)c_m(mn)) \pmod{m}$$

$$\equiv c_m(mn) \pmod{m}.$$

Lemma 2.9 enables a significant reduction in the number of cases which will need to be checked when we prove Theorem 2.1. This is because of the following. Given n written in m-ary notation as

$$n = \alpha m^j + \beta m^k + \dots + \gamma m^r,$$

we see immediately that

$$mn = \alpha m^{j+1} + \beta m^{k+1} + \dots + \gamma m^{r+1},$$

where $\alpha, \beta, \ldots, \gamma \in \{1, 2, \ldots, m-1\}$ and $j < k < \cdots < r$. Thus, we can divide by m^2 for as many times as we wish if $j \ge 2$ (because $j + 1 \ge 3$). Therefore, we only need to consider the cases j = 0 and j = 1 in what follows.

We are now in a position to prove Theorem 2.1 which provides a characterization of $c_m(mn)$ modulo m simply based on the m-ary representation of n.

Proof. By Lemma 2.9, we see that if $j \ge 2$, then $m^3 \mid mn$. This means $c_m(mn) \equiv c_m\left(\frac{n}{m}\right) \pmod{m}$. Thus, we may assume j = 0 or j = 1 without loss of generality. Now we consider two cases (based on the parity of j).

• Case 1: j is even, so we can assume j = 0. Hence,

$$c_m(mn) \equiv 1 + (\alpha_0 - 1)c_m(n) \pmod{m}$$

= $1 + (\alpha_0 - 1)c_m(\alpha_0 + \alpha_1 m + \alpha_2 m^2 + ...).$

Now since $m > \alpha_0 \ge 1$, we may replace α_0 by m (thanks to Lemma 2.7). Then the above becomes

$$c_m(mn) \equiv 1 + (\alpha_0 - 1)c_m((\alpha_1 + 1)m + \alpha_2m^2 + \dots) \pmod{m}$$

= 1 + (\alpha_0 - 1)c_m(m((\alpha_1 + 1) + \alpha_2m + \alpha_3m^2 + \dots))
\equiv 1 + (\alpha_0 - 1)(1 + \alpha_1c_m((\alpha_1 + 1) + \alpha_2m + \alpha_3m^2 + \dots)) \quad (mod m).

Now $1 \le \alpha_1 + 1 \le m$, so by Lemma 2.7 we may replace $\alpha_1 + 1$ by m in the above to obtain

$$c_m(mn) \equiv 1 + (\alpha_0 - 1)(1 + \alpha_1 c_m(m(\alpha_2 + 1) + \alpha_3 m + \dots)) \pmod{m}.$$

Now $1 \le \alpha_2 + 1 \le m$, so we may apply Lemma 2.7 again, and the process continues until we hit some $\alpha_i = 0$ at which time the process terminates. The result is

$$c_m(mn) \equiv 1 + (\alpha_0 - 1)(1 + \alpha_1(1 + \alpha_2(1 + \alpha_3 + \dots))) \pmod{m}$$

= $\alpha_0 + (\alpha_0 - 1) \sum_{i=1}^{\infty} \alpha_1 \alpha_2 \dots \alpha_i$

which is equivalent to the first case of Theorem 2.1.

• Case 2: j is odd, so we can assume j = 1. Hence, $n \equiv m \pmod{m}$, and by Lemma 2.8,

$$c_m(mn) \equiv 1 - c_m(n) \pmod{m}$$
$$= 1 - c_m\left(m\sum_{j=0}^{\infty} \alpha_{j+1}m^j\right)$$

Now Case 1 above is applicable to $n' = \sum_{j=0}^{\infty} \alpha_{j+1} m^j$ because $1 \le \alpha_1 < m$. Hence, the desired result follows.

With the goal of demonstrating the applicability of Theorem 2.1, we compute a few examples.

• Let m = 4, $n = 123 = 3 + 2 \cdot 4 + 3 \cdot 4^2 + 1 \cdot 4^3$. Then

$$c_4(4 \cdot 123) = c_4(492) = 5843 \equiv 3 \pmod{4}.$$

This is an example of the case j = 0. Theorem 2.1 asserts that

$$c_4(4 \cdot 123) \equiv 3 + (3 - 1)(2 + 2 \cdot 3 + 2 \cdot 3 \cdot 1) \pmod{4}$$

= 3 + 2 \cdot 14
= 3 \quad (mod 4)

as computed above.

• Let m = 5, $n = 485 = 2 \cdot 5 + 4 \cdot 5^2 + 3 \cdot 5^3$. Then

$$c_5(5 \cdot 485) = c_5(2425) = 230358 \equiv 3 \pmod{5}.$$

This is an example of the case j = 1. Theorem 2.1 asserts that

$$c_{5}(5 \cdot 485) \equiv 1 - 2 - (2 - 1)(4 + 4 \cdot 3) \pmod{5}$$

= 1 - 2 - 16
= -17
= 3 (mod 5)

as computed above.

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