On Sloane's Generalization of Non–Squashing Stacks of Boxes

George E. Andrews and James A. Sellers

Department of Mathematics, Penn State University, University Park, PA, 16802 USA

Abstract

Recently, Sloane and Sellers solved a certain box stacking problem related to *non–squashing* partitions. These are defined as partitions $n = p_1 + p_2 + \cdots + p_k$ with $1 \le p_1 \le p_2 \le \cdots \le p_k$ wherein $p_1 + \cdots + p_j \le p_{j+1}$ for $1 \le j \le k-1$. Sloane has also hinted at a generalized box stacking problem which is closely related to generalized non–squashing partitions. We solve this generalized box stacking problem by obtaining a generating function for the number of such stacks and discuss partition functions which arise via this generating function.

Key words: partitions, non–squashing partitions, m–ary partitions, stacking boxes, partition analysis
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1 Introduction

In a recent paper, Sloane and Sellers [8] considered the following combinatorial problem and provided its solution:

> We are given n boxes, labeled $1, 2, \ldots, n$. For $i = 1, \ldots, n$, box i weighs i grams and can support a total weight of i grams. What is the number $a_2(n)$ of different ways to build a single stack of boxes in which no box will be squashed by the weight of the boxes above it?

For example, $a_2(4) = 14$ where the allowable stacks of boxes are as follows:

Email address: andrews@math.psu.edu and sellersj@math.psu.edu (George E. Andrews and James A. Sellers).

$$\emptyset,\ \boxed{1},\ \boxed{2},\ \boxed{3},\ \boxed{4},\ \begin{array}{c}\boxed{1}\\\boxed{2}\end{array},\ \begin{array}{c}\boxed{1}\\\boxed{3}\end{array},\ \begin{array}{c}\boxed{1}\\\boxed{4}\end{array},\ \begin{array}{c}\boxed{2}\\\boxed{3}\end{array},\ \begin{array}{c}\boxed{2}\\\boxed{4}\end{array},\ \begin{array}{c}\boxed{3}\\\boxed{4}\end{array},\ \begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array},\ \begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{4}\end{array},\ \begin{array}{c}\boxed{1}\\\boxed{3}\\\boxed{4}\end{array}$$

The other two possible stacks,

$$\begin{array}{c}\boxed{2}\\\boxed{3}\\\boxed{4}\end{array}\ \text{and}\ \begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\\\boxed{4}\end{array},$$

are excluded since $2 + 3 > 4$ and the box labeled 4 would collapse in both cases. More recently, Sloane suggested a generalization to this problem:

> We are given n boxes, labeled $1, 2, \ldots, n$. For $i = 1, \ldots, n$, box i weighs $(m - 1)i$ grams (where $m \geq 2$ is a fixed integer) and box i can support a total weight of i grams. What is the number $a_m(n)$ of different ways to build a single stack of boxes in which no box will be squashed by the weight of the boxes above it?

See, for example, sequences A090631 and A090632 in Sloane's Online Encyclopedia of Integer Sequences [7] which give the first several values of $a_3(n)$ and $a_4(n)$ respectively. Our goal in this work is to answer the general question above by proving the following generating function identity:

Theorem 1.1 *For $m \geq 3$,*

$$\sum_{n=0}^{\infty} a_m(n)q^n = \frac{1}{(1-q)^2 \prod_{i=0}^{\infty}(1 - q^{(m-1)m^i})}.$$

We note in passing that the $m = 2$ case of the above problem, which Sloane and Sellers considered in detail, is fundamentally different from the cases when $m \geq 3$. In particular, we note that the right–hand side of the identity in Theorem 1.1 is representable as one infinite product. The generating function found by Sloane and Sellers for $a_2(n)$ [8, Corollary 9] does not seem to be representable as one infinite product.

The reader may also wish to see a recent work by Rødseth and Sellers [5]. Although the problem considered in [5] is related, it is nevertheless different and the proof technique used is quite dissimilar from that given below.

Before closing this introduction, a comment is in order regarding a corollary of Theorem 1.1 above.

Corollary 1.2 *For fixed $m \geq 3$ and $n \geq 0$, let $c_m(n)$ be the number of partitions of n whose parts are taken from the set*

$$\{1, 2, 2(m-1), 2(m-1)m, 2(m-1)m^2, 2(m-1)m^3, \dots\}.$$

Then $a_m(n) = c_m(2n)$.

The proof of this corollary follows from straightforward generating function dissection. While Corollary 1.2 may seem a bit artificial, it is the case that the original motivation for considering this problem arose from a conjecture that the values of $a_4(n)$ equal $f(2n)$ where $f(n)$ is the number of partitions of n whose parts are factorial numbers [7, A064986]. While the conjecture ultimately proved to be false, it did lead us to pursue the problem and so served as an excellent motivation.

In Section 2, we provide a brief amount of necessary mathematical background, especially as it relates to partition analysis, a proof technique initially developed by MacMahon [4] and more recently utilized by a variety of authors. (See, for example, [1], [2], and [6].) In Section 3, we prove Theorem 1.1. We then close the paper in Section 4 by proving an infinite family of congruences modulo powers of 2 satisfied by $a_m(n)$ for certain values of m.

2 Mathematical Background

In order to make our work below more precise, we say that a partition

$$n = p_1 + p_2 + \cdots + p_j, \qquad 1 \leq p_1 \leq p_2 \leq \cdots \leq p_j \tag{1}$$

of a natural number n into j parts is *m–non–squashing* if

$$(m-1)(p_1 + \cdots + p_r) \leq p_{r+1} \text{ for } 1 \leq r \leq j-1 \tag{2}$$

where m is some fixed integer greater than 1. In the context of our box stacking problem, we see that the boxes in the stack will not collapse if and only if the corresponding partition is m–non–squashing. We note in passing that Hirschhorn and Sellers [3] as well as Sloane and Sellers [8] have proven that the number of m–non–squashing partitions of n equals the number of m–ary partitions of n, the number of partitions of n wherein each part is a power of m.

In the box stacking problem as stated above, the boxes must have distinct labels and the sum of their weights cannot exceed $\binom{n+1}{2}$. Therefore $a_m(n)$ is equal to the total number of m–non–squashing partitions of numbers from 0 to $\binom{n+1}{2}$ which have distinct parts each of which must be less than or equal to n.

Before turning to the proof of Theorem 1.1, we briefly mention a few key items from MacMahon's partition analysis. First, we define the Omega operator $\underset{\geq}{\Omega}$.

Definition 2.1 *The operator* $\underset{\geq}{\Omega}$ *is given by*

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_j=-\infty}^{\infty} A_{s_1,\ldots,s_j} \lambda_1^{s_1} \ldots \lambda_j^{s_j} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_j=0}^{\infty} A_{s_1,\ldots,s_j},$$

where the domain of the $A_{s_1,\ldots,s_j}$ *is the field of rational functions over* $\mathbb{C}$ *in several complex variables restricted to a small neighborhood of the origin and the* λ_i *are restricted to annuli of the form* $1-\varepsilon < |\lambda_i| < 1+\varepsilon$.

Finally, we need the following two lemmas involving the Omega operator.

Lemma 2.2 *For nonnegative integers* $s_1, s_2, \ldots, s_r$,

$$\begin{aligned} &\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y_1}{\lambda^{s_1}}\right)\left(1-\frac{y_2}{\lambda^{s_2}}\right)\cdots\left(1-\frac{y_r}{\lambda^{s_r}}\right)} \\ &= \frac{1}{(1-x)(1-x^{s_1}y_1)(1-x^{s_2}y_2)\cdots(1-x^{s_r}y_r)}. \end{aligned}$$

Remark: Note that MacMahon [4, pp. 1147–1148] proves Lemma 2.2 in the cases $r=1$, $s_1=1,2,3$.

Proof:

$$\begin{aligned} &\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y_1}{\lambda^{s_1}}\right)\left(1-\frac{y_2}{\lambda^{s_2}}\right)\cdots\left(1-\frac{y_r}{\lambda^{s_r}}\right)} \\ &= \underset{\geq}{\Omega} \sum_{n_0,n_1,\ldots n_r \geq 0} x^{n_0} y_1^{n_1} y_2^{n_2} \cdots y_r^{n_r} \lambda^{n_0-n_1s_1-n_2s_2-\cdots-n_rs_r} \\ &= \sum_{\substack{n_0,n_1,\cdots n_r \geq 0 \\ n_0 \geq n_1s_1+n_2s_2+\cdots n_rs_r}} x^{n_0} y_1^{n_1} y_2^{n_2} \cdots y_r^{n_r} \\ &= \sum_{n_0,n_1,\cdots n_r \geq 0} x^{n_0+n_1s_1+n_2s_2+\cdots n_rs_r} y_1^{n_1} y_2^{n_2} \cdots y_r^{n_r} \\ &= \frac{1}{(1-x)(1-x^{s_1}y_1)(1-x^{s_2}y_r)\cdots(1-x^{s_r}y_r)}. \end{aligned}$$

■

Lemma 2.3

$$\Omega_{\geq} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)} = \frac{1}{(1-x)(1-xy)}.$$

Proof: This is the case $r = s = 1$ of Lemma 2.2 above. ■

3 A Proof Of Theorem 1.1 Via Partition Analysis

We are now prepared to prove Theorem 1.1.

Proof: We fix an integer $m \geq 3$. Note that the generating function for $a_m(n)$ can be written as

$$\sum_{n=0}^{\infty} a_m(n) q^n = \lim_{j\to\infty} F_{m,j}(q)$$

where

$$F_{m,j}(q) = \Omega_{\geq} \sum_{n=0}^{\infty} q^n \sum_{p_1,p_2,\ldots,p_j \geq 0} \lambda_{j+1}^{n-p_j} \prod_{k=2}^{j} \lambda_k^{p_k-(m-1)(p_1+p_2+\cdots+p_{k-1})}.$$

The exponents on the λ–parameters "encode" the inequalities inherent in the problem (namely, that the largest part in any partition is to be at most n and that the partitions are to be m–non–squashing). Next, we rewrite $F_{m,j}(q)$ in terms of products rather than sums via geometric series:

$$\begin{aligned}
&F_{m,j}(q)\\
&= \Omega_{\geq} \frac{1}{(1-q\lambda_{j+1})\left(1-\frac{\lambda_j}{\lambda_{j+1}}\right)\left[\prod_{k=2}^{j-1}\left(1-\frac{\lambda_k}{\prod_{\ell=k+1}^{j}\lambda_\ell^{m-1}}\right)\right]\left(1-\frac{1}{\prod_{k=2}^{j}\lambda_k^{m-1}}\right)}
\end{aligned}$$

We now proceed with the annihilation of the λ–parameters, beginning with λ_{j+1}.

$$\begin{aligned}
&F_{m,j}(q)\\
&= \Omega_{\geq} \frac{1}{(1-q\lambda_{j+1})\left(1-\frac{\lambda_j}{\lambda_{j+1}}\right)\left[\prod_{k=2}^{j-1}\left(1-\frac{\lambda_k}{\prod_{\ell=k+1}^{j}\lambda_\ell^{m-1}}\right)\right]\left(1-\frac{1}{\prod_{k=2}^{j}\lambda_k^{m-1}}\right)}\\
&= \frac{1}{(1-q)}\Omega_{\geq} \frac{1}{(1-q\lambda_j)\left[\prod_{k=2}^{j-1}\left(1-\frac{\lambda_k}{\prod_{\ell=k+1}^{j}\lambda_\ell^{m-1}}\right)\right]\left(1-\frac{1}{\prod_{k=2}^{j}\lambda_k^{m-1}}\right)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{using Lemma 2.3}
\end{aligned}$$

$$= \frac{1}{(1-q)^2} \underset{\geq}{\Omega} \frac{1}{(1-q^{m-1}\lambda_{j-1}) \left[\prod_{k=2}^{j-2}\left(1-\frac{q^{m-1}\lambda_k}{\prod_{\ell=k+1}^{j-1}\lambda_\ell^{m-1}}\right)\right]\left(1-\frac{q^{m-1}}{\prod_{k=2}^{j-1}\lambda_k^{m-1}}\right)}$$

using Lemma 2.2

$$= \frac{1}{(1-q)^2(1-q^{m-1})} \times$$

$$\times \underset{\geq}{\Omega} \frac{1}{(1-q^{(m-1)m}\lambda_{j-2}) \left[\prod_{k=2}^{j-3}\left(1-\frac{q^{(m-1)m}\lambda_k}{\prod_{\ell=k+1}^{j-2}\lambda_\ell^{m-1}}\right)\right]\left(1-\frac{q^{(m-1)m}}{\prod_{k=2}^{j-2}\lambda_k^{m-1}}\right)}$$

$$= \frac{1}{(1-q)^2(1-q^{m-1})(1-q^{(m-1)m})} \times$$

$$\times \underset{\geq}{\Omega} \frac{1}{(1-q^{(m-1)m^2}\lambda_{j-3}) \left[\prod_{k=2}^{j-4}\left(1-\frac{q^{(m-1)m^2}\lambda_k}{\prod_{\ell=k+1}^{j-3}\lambda_\ell^{m-1}}\right)\right]\left(1-\frac{q^{(m-1)m^2}}{\prod_{k=2}^{j-3}\lambda_k^{m-1}}\right)}$$

$$= \frac{1}{(1-q)^2(1-q^{m-1})(1-q^{(m-1)m})(1-q^{(m-1)m^2})} \times$$

$$\times \underset{\geq}{\Omega} \frac{1}{(1-q^{(m-1)m^3}\lambda_{j-4}) \left[\prod_{k=2}^{j-5}\left(1-\frac{q^{(m-1)m^3}\lambda_k}{\prod_{\ell=k+1}^{j-4}\lambda_\ell^{m-1}}\right)\right]\left(1-\frac{q^{(m-1)m^3}}{\prod_{k=2}^{j-4}\lambda_k^{m-1}}\right)}$$

$$\vdots$$

$$= \frac{1}{(1-q)^2(1-q^{m-1})(1-q^{(m-1)m})(1-q^{(m-1)m^2})\dots(1-q^{(m-1)m^{j-2}})}$$

after annihilation of all λ–parameters. Therefore,

$$\sum_{n=0}^{\infty} a_m(n)q^n = \lim_{j\to\infty} F_{m,j}(q) = \lim_{j\to\infty} \frac{1}{(1-q)^2\prod_{i=0}^{j-2}(1-q^{(m-1)m^i})}$$

and the result of the theorem follows. ∎

4 Closing Thoughts

As was the case in [5] and [8], a natural question to ask is whether $a_m(n)$ satisfies any special arithmetic properties (due to its clear relationship to m–ary partition functions and the many properties that such functions satisfy). We answer this question affirmatively by proving an infinite family of congruence properties satisfied by $a_m(n)$.

Theorem 4.1 *Assume t is the largest positive integer such that $m \equiv 1 \pmod{2^t}$. Then, for each j, $1 \leq j \leq t$, and for all $n \geq 0$,*

$$a_m(2^j n + 2^j - 1) \equiv 0 \pmod{2^j}.$$

Proof: Let

$$A(c,q) = \frac{1}{(1-q)^2 \prod_{i=0}^{\infty}(1-q^{cm^i})}$$

for some positive integer c. Then we see that

$$\sum_{n=0}^{\infty} a_m(n)q^n = A(m-1,q).$$

Next, assume t is the largest positive integer such that $m \equiv 1 \pmod{2^t}$ as in the statement of the theorem. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} a_m(2n+1)q^{2n+1} &= \frac{1}{2}\frac{1}{\prod_{i=0}^{\infty} 1-q^{(m-1)m^i}}\left[\frac{1}{(1-q)^2} - \frac{1}{(1+q)^2}\right] \\
&= \frac{1}{2}\frac{1}{\prod_{i=0}^{\infty} 1-q^{(m-1)m^i}}\left[\frac{4q}{(1-q^2)^2}\right] \\
&= \frac{2q}{(1-q^2)^2}\frac{1}{\prod_{i=0}^{\infty} 1-q^{(m-1)m^i}}.
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} a_m(2n+1)q^n = \frac{2}{(1-q)^2 \prod_{i=0}^{\infty} 1-q^{((m-1)/2)m^i}} = 2A((m-1)/2,q).$$

This process of generating function dissection, which in effect involves replacing n by $2n+1$, can then be iterated to yield

$$\sum_{n=0}^{\infty} a_m(2(2n+1)+1)q^n = 2(2A((m-1)/4,q))$$

or

$$\sum_{n=0}^{\infty} a_m(4n+3)q^n = 4A((m-1)/4,q)$$

followed by

$$\sum_{n=0}^{\infty} a_m(8n+7)q^n = 8A((m-1)/8,q)$$

and so on. This iterative process terminates once the power $((m-1)/2^j)m^k$ becomes odd. Since m is known to be odd, this power becomes odd exactly when $j=t$. The result follows.

$\blacksquare$

Based on computational experimentation, it appears that $a_m(n)$ satisfies many other congruence properties. A fuller treatment of these properties will be the subject of a future work.

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