CONGRUENCES FOR THE FISHBURN NUMBERS

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Dedicated to our good friend and colleague Wen-Ching Winnie Li

ABSTRACT. The Fishburn numbers, $\xi(n),$ are defined by a formal power series expansion

$$\sum_{n=0}^{\infty} \xi(n)q^n = 1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} (1 - (1-q)^j)$$

For half of the primes p, there is a non-empty set of numbers T(p) lying in [0, p-1] such that if $j \in T(p)$, then for all $n \ge 0$,

 $\xi(pn+j) \equiv 0 \pmod{p}.$

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1. INTRODUCTION

The Fishburn numbers $\xi(n)$ are defined by the formal power series

(1)
$$\sum_{n=0}^{\infty} \xi(n)q^n = \sum_{n=0}^{\infty} (1-q;1-q)_n$$

where

(2)
$$(A;q)_n = (1-A)(1-Aq)\dots(1-Aq^{n-1}).$$

The Fishburn numbers have arisen in a wide variety of combinatorial settings. One can gain some sense of the extent of their applications in [9, Sequence A022493]. Namely, these numbers arise in such combinatorial settings as linearized chord diagrams, Stoimenow diagrams, nonisomorphic interval orders, unlabeled (2 + 2)-free posets, and ascent sequences. They were first defined in the work of Fishburn (cf. [6, 7, 8]), and have recently found a connection with mock modular forms [4].

It turns out that the Fishburn numbers satisfy congruences reminiscent of those for the partition function p(n) [2, Chapter 1]. Surprisingly, in contrast to p(n), we shall see in Section 4 that there are congruences of the form $\xi(pn+b) \equiv 0 \pmod{p}$

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for half of all the primes p. For example, for all $n \ge 0$,

(3)
$$\xi(5n+3) \equiv \xi(5n+4) \equiv 0 \pmod{5},$$

(4) $\xi(7n+6) \equiv 0 \pmod{7},$

(5)
$$\xi(11n+8) \equiv \xi(11n+9) \equiv \xi(11n+10) \equiv 0 \pmod{11},$$

- (6) $\xi(17n+16) \equiv 0 \pmod{17}$, and
- (7) $\xi(19n+17) \equiv \xi(19n+18) \equiv 0 \pmod{19}.$

These results all follow from a general result stated as Theorem 3.1 in Section 3. The next section is devoted to background lemmas. Theorem 3.1 is then proved in Section 3. In Section 4 we discuss an infinite family of primes p for which these congruences hold. We conclude with some open problems.

2. Background Lemmas

The sequence of pentagonal numbers is given by

(8)
$$\{n(3n-1)/2\}_{n=-\infty}^{\infty} = \{0, 1, 2, 5, 7, 12, 15, 22, \dots\}.$$

Throughout this work the symbol λ will be used to designate a pentagonal number.

In our first lemma, f(q) will denote an arbitrary polynomial in $\mathbb{Z}[q]$, and p will be a fixed prime. Then we separate the terms in f(q) according to the residue of the exponent modulo p. Thus,

(9)
$$f(q) = \sum_{i=0}^{p-1} q^i \phi_i(q^p)$$

We also suppose that for every p^{th} root of unity ζ (including $\zeta = 1$),

$$f(\zeta) = \sum_{\lambda} c_{\lambda} \zeta^{\lambda}$$

where the λ 's sum over some set of pentagonal numbers that includes 0. The *c*'s are thus defined to be 0 outside this prescribed set of pentagonal numbers, and the *c*'s are independent of the choice of ζ .

Lemma 2.1. Under the above conditions, $\phi_j(1) = 0$ if j is not a pentagonal number.

Proof. The assertion is not immediate because the p^{th} roots of unity are not linearly independent. In particular, if ζ is a primitive p^{th} root of unity, then

$$1 + \zeta + \zeta^2 + \dots + \zeta^{p-1} = 0.$$

However, we know that the ring of integers in $\mathbb{Q}(\zeta)$ has $1, \zeta, \zeta^2, \ldots, \zeta^{p-2}$ as a basis [1, page 187]. Hence,

$$\phi_0(1)(-\zeta - \zeta^2 - \dots - \zeta^{p-1}) + \sum_{j=1}^{p-1} \zeta^j \phi_j(1) = c_0(-\zeta - \zeta^2 - \dots - \zeta^{p-1}) + \sum_{\lambda \neq 0} c_\lambda \zeta^\lambda.$$

Therefore, if $1 \le j \le p - 1$,

$$\phi_j(1) - \phi_0(1) = \begin{cases} c_\lambda - c_0 & \text{if } j \text{ is one of the designated pentagonal numbers,} \\ -c_0 & \text{otherwise} \end{cases}$$

is a linear system of p-1 equations in p variables $\phi_j(1), 0 \leq j \leq p-1$. However, the $\zeta = 1$ case adds one further equation

$$\phi_0(1) + \phi_1(1) + \dots + \phi_{p-1}(1) = \sum_{\lambda} c_{\lambda}.$$

We now have a linear system of p equations in p variables, and the determinant of the system is p. Hence, there is a unique solution which is the obvious solution

$$\phi_j(1) = \begin{cases} c_\lambda & \text{if } j \text{ is one of the designated pentagonal numbers,} \\ 0 & \text{otherwise.} \end{cases}$$

In the next three lemmas, we require some variations on Leibniz's rule for taking the n^{th} derivative of a product. Each is probably in the literature, but is included here for completeness.

Lemma 2.2.

$$\left(q\frac{d}{dq}\right)^n \left(A(q)B(q)\right) = \sum_{j=1}^n q^j c_{n,j} \left(\frac{d}{dq}\right)^j \left(A(q)B(q)\right),$$

where the $c_{n,j}$ are the Stirling numbers of the second kind given by $c_{n,0} = c_{n,n+1} = 0$, $c_{1,1} = 1$, and $c_{n+1,j} = jc_{n,j} + c_{n,j-1}$ for $1 \le j \le n+1$.

Proof. The result is a tautology when n = 1. To pass from n to n + 1, we note

$$\begin{aligned} q\frac{d}{dq} \right)^{n+1} (A(q)B(q)) &= q\frac{d}{dq} \left(\left(q\frac{d}{dq} \right)^n (A(q)B(q)) \right) \\ &= q\frac{d}{dq} \sum_{j=1}^n q^j c_{n,j} \left(\frac{d}{dq} \right)^j (A(q)B(q)) \\ &= \sum_{j=1}^n jq^j c_{n,j} \left(\frac{d}{dq} \right)^j (A(q)B(q)) \\ &+ \sum_{j=1}^n q^{j+1} c_{n,j} \left(\frac{d}{dq} \right)^{j+1} (A(q)B(q)) \\ &= \sum_{j=1}^{n+1} q^j (jc_{n,j} + c_{n,j-1}) \left(\frac{d}{dq} \right)^j (A(q)B(q)) \\ &= \sum_{j=1}^{n+1} q^j c_{n+1,j} \left(\frac{d}{dq} \right)^j (A(q)B(q)). \end{aligned}$$

Lemma 2.3.

$$\left(\frac{d}{dt}\right)^n f(qe^t)\Big|_{t=0} = \left(q\frac{d}{dq}\right)^n f(q).$$

Proof. By Lemma 2.2 with A(q) = f(q) and B(q) = 1, we see that

(10)
$$\left(q\frac{d}{dq}\right)^{n} f(q) = \sum_{j=1}^{n} q^{j} c_{n,i} f^{(j)}(q).$$

On the other hand, we claim

(11)
$$\left(\frac{d}{dt}\right)^{n} f(qe^{t}) = \sum_{j=1}^{n} q^{j} e^{jt} c_{n,j} f^{(j)}(qe^{t}).$$

When n = 1, this is just the chain rule applied to $f(qe^t)$. To pass from n to n + 1, we note

$$\left(\frac{d}{dt}\right)^{n+1} f(qe^t) = \frac{d}{dt} \left(\frac{d}{dt}\right)^n f(qe^t)$$

$$= \frac{d}{dt} \sum_{j=1}^n q^j e^{jt} c_{n,j} f^{(j)}(qe^t)$$

$$= \sum_{j=1}^n jq^j e^{jt} c_{n,j} f^{(j)}(qe^t)$$

$$+ \sum_{j=1}^n q^{j+1} e^{(j+1)t} c_{n,j} f^{(j+1)}(qe^t)$$

$$= \sum_{j=1}^{n+1} (jc_{n,j} + c_{n,j-1}) q^j e^{jt} f^{(j)}(qe^t)$$

$$= \sum_{j=1}^{n+1} c_{n+1,j} q^j e^{jt} f^{(j)}(qe^t).$$

Comparing (11) with t = 0 to (10), we see that our lemma is established.

We now turn to the generating function for the Fishburn numbers as given by Zagier [10, page 946]. Namely,

(12)
$$F(1-q) = \sum_{n=0}^{\infty} \xi(n)q^n = \sum_{n=0}^{\infty} (1-q; 1-q)_n.$$

To facilitate the study, we concentrate on

(13)
$$F(q) = \sum_{n=0}^{\infty} (q;q)_n$$

and

(14)
$$F(q,N) = \sum_{n=0}^{N} (q;q)_n = \sum_{i=0}^{p-1} q^i A_p(N,i,q^p),$$

where $A_p(N, i, q^p)$ is a polynomial in q^p . We note that if ζ is a p^{th} root of unity (15) $F(\zeta) = F(\zeta, m) = F(\zeta, p-1)$

for all $m \ge p$. Furthermore,

(16)
$$\left(q\frac{d}{dq}\right)^r F(q)\Big|_{q=\zeta} = \left(q\frac{d}{dq}\right)^r F(q,m)\Big|_{q=\zeta} = \left(q\frac{d}{dq}\right)^r F(q,(r+1)p-1)\Big|_{q=\zeta}$$

for all $m \ge (r+1)p$ because $(1-q^p)^{r+1}$ divides $(q;q)_j$ for all $j \ge (r+1)p$.

Similarly, for all $m \ge (r+1)p$,

(17)
$$F^{(r)}(q)\Big|_{q=\zeta} = F^{(r)}(q,m)\Big|_{q=\zeta} = F^{(r)}(q,(r+1)p-1)\Big|_{q=\zeta}$$

In the next lemma, we require a Stirling–like array of numbers $C_{N,i,j}(p)$ given by $C_{N,i,0}(p) = i^N (C_{0,0,0}(p) = 1), C_{N,i,N+1}(p) = 0$, and for $1 \le j \le N$, (18) $C_{N+1,i,j}(p) = (i+jp)C_{N,i,j}(p) + pC_{N,i,j-1}(p)$.

Lemma 2.4.

$$\left(q\frac{d}{dq}\right)^{N}F(q,n) = \sum_{j=0}^{N}\sum_{i=0}^{p-1}C_{N,i,j}(p)q^{i+jp}A_{p}^{(j)}(n,i,q^{p}).$$

Proof. In light of the fact that $C_{0,i,0}(p) = 1$ for all *i*, the N = 0 assertion is

$$F(q,n) = \sum_{i=0}^{p-1} q^i A_p(n,i,q^p),$$

which is just the definition of the A's given in (14). To pass from N to N + 1, we note

$$\begin{pmatrix} q \frac{d}{dq} \end{pmatrix}^{N+1} F(q,n) = q \frac{d}{dq} \sum_{j=0}^{N} \sum_{i=0}^{p-1} C_{N,i,j}(p) q^{i+jp} A_p^{(j)}(n,i,q^p)$$

$$= \sum_{j=0}^{N} \sum_{i=0}^{p-1} C_{N,i,j}(p) (i+jp) q^{i+jp} A_p^{(j)}(n,i,q^p)$$

$$+ \sum_{j=0}^{N} \sum_{i=0}^{p-1} C_{N,i,j}(p) q^{i+jp} pq^p A_p^{(j+1)}(n,i,q^p)$$

$$= \sum_{j=0}^{N+1} \sum_{i=0}^{p-1} ((i+jp) C_{N,i,j}(p) + p C_{N,i,j-1}(p)) q^{i+jp} A_p^{(j)}(n,i,q^p)$$

$$= \sum_{j=0}^{N+1} \sum_{i=0}^{p-1} C_{N+1,i,j}(p) q^{i+jp} A_p^{(j)}(n,i,q^p).$$

We now define, for any positive integer p, two special sets of integers:

(19) $S(p) = \{j \mid 0 \le j \le p-1 \text{ such that } n(3n-1)/2 \equiv j \pmod{p} \text{ for some } n\}$ and

(20) $T(p) = \{k \mid 0 \le k \le p - 1 \text{ such that } k \text{ is larger than every element of } S(p)\}.$ For example, for p = 11, we have

 $S(11) = \{0, 1, 2, 4, 5, 7\} \quad \text{ and } \quad T(11) = \{8, 9, 10\} \,.$

Lemma 2.5. If $i \notin S(p)$, then

 $A_p(pn-1, i, q) = (1-q)^n \alpha_p(n, i, q)$

where the $\alpha_p(n, i, q)$ are polynomials in $\mathbb{Z}[q]$.

Proof. This result is equivalent to the assertion that for $0 \le j < n$,

$$A_p^{(j)}(pn-1, i, 1) = 0$$

and by (17) and Lemma 2.1 we need only prove for $j \ge 0$,

(21)
$$A_p^{(j)}((j+1)p-1,i,1) = 0$$

because $n \ge (j+1)$.

We proceed to prove (21) by induction on j. When j = 0, we only need show that if $i \notin S(p)$,

$$A_p(p-1,i,1) = 0$$

Following [10, Section 5], we define (where ζ is now an N^{th} root of unity)

(22)
$$F(\zeta e^{t}) = \sum_{n=0}^{\infty} \frac{b_{n}(\zeta)(-t)^{n}}{n!}$$
$$= e^{-t/24} \sum_{n=0}^{\infty} \frac{c_{n}(\zeta)(-t)^{n}}{24^{n}n!}$$
$$= \sum_{M=0}^{\infty} \frac{(-t)^{M}}{24^{M}M!} \sum_{n=0}^{M} \binom{M}{n} c_{n}(\zeta),$$

where we have replaced Zagier's ξ with ζ to avoid confusion with $\xi(n)$. In [10, Section 5], we see that

(23)
$$c_n(\zeta) = \frac{(-1)^n N^{2n+1}}{2n+2} \sum_{m=1}^{N/2} \chi(m) \zeta^{(m^2-1)/24} B_{2n+2}\left(\frac{m}{N}\right),$$

where the *B*'s are Bernoulli polynomials and $\chi(m) = \left(\frac{12}{m}\right)$. Note that the only non-zero terms in the sum in (23) have

(24)
$$\zeta^{((6m\pm 1)^2 - 1)/24} \chi(6m\pm 1) = (-1)^m \zeta^{m(3m\pm 1)/2},$$

i.e., $c_n(\zeta)$ is a linear combination of powers of ζ where each exponent is a pentagonal number. Hence, by (22) we see that $b_n(\zeta)$ is a linear combination of powers of ζ where each exponent is a pentagonal number.

Hence, if ζ is now a p^{th} root of unity,

$$F(\zeta) = F(\zeta, p-1)$$

= $b_0(\zeta)$
= $\sum_{\lambda} c_{\lambda} \zeta^{\lambda}$,

where the sum over λ is restricted to a subset of the pentagonal numbers. On the other hand,

$$F(\zeta) = F(\zeta, p-1) \\ = \sum_{i=0}^{p-1} \zeta^{i} A_{p}(p-1, i, 1).$$

Hence, by Lemma 2.1, for $i \notin S(p)$,

$$A_p(p-1,i,1) = 0$$

which is (21) when j = 0. Now let us assume that

(25)
$$A_p^{(j)}(p(j+1) - 1, i, 1) = 0$$

for $0 \le j < \nu < n$. By Lemma 2.4,

(26)
$$\left(q\frac{d}{dq}\right)^{\nu} F(q, p(\nu+1)-1)\Big|_{q=\zeta} = \sum_{j=0}^{\nu} \sum_{i=0}^{p-1} C_{\nu,i,j}(p)\zeta^{i} A_{p}^{(j)}(p(\nu+1)-1, i, 1).$$

But for $j < \nu$,

$$A_p^{(j)}(p(\nu+1), i, 1) = A_p^{(j)}(p(j+1) - 1, i, 1) = 0.$$

Hence the only terms in the sum in (26) where ζ is raised to a non-pentagonal power, *i*, arise from the terms with $j = \nu$, namely

(27)
$$C_{\nu,i,\nu}(p)\zeta^{i}A_{p}^{(\nu)}(p(\nu+1)-1,i,1),$$

and we note that $C_{\nu,i,\nu}(p) \neq 0$.

Applying Lemma 2.3 to the left side of (26), we see that by (22)

(28)
$$(-1)^{\nu} b_{\nu}(\zeta) = \left(q \frac{d}{dq}\right)^{\nu} F(q) \Big|_{q=\zeta}$$
$$= \left(q \frac{d}{dq}\right)^{\nu} F(q, (\nu+1)p-1) \Big|_{q=\zeta}$$
$$= \sum_{j=0}^{\nu} \sum_{i=0}^{p-1} C_{\nu,i,j}(p) \zeta^{i} A_{p}^{(j)}(p(\nu+1)-1, i, 1).$$

Recall that $b_{\nu}(\zeta)$ is a linear combination of powers of ζ where the exponents are pentagonal numbers. Hence the expression given in (27) must be zero by Lemma 2.1. Therefore,

$$A_{p}^{(\nu)}(p(\nu+1)-1,i,1) = 0,$$

and this proves (21) and thus proves Lemma 2.5.

3. The Main Theorem

We recall from (12) that

$$\sum_{n=0}^{\infty} \xi(n)q^n = \sum_{j=0}^{\infty} (1-q;1-q)_j$$
$$= 1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \sum_{h=1}^{i} (-1)^{h-1}q^h \binom{i}{h}$$
$$= 1 + \sum_{j=1}^{\infty} (j!q^j + O(q^{j+1})).$$

Hence,

(29)
$$\sum_{n=0}^{\infty} \xi(n)q^n = F(1-q,N) + O(q^{N+1}).$$

We are now in a position to state and prove the main theorem of this paper.

Theorem 3.1. If p is a prime and $i \in T(p)$ (as defined in (20)), then for all $n \ge 0$,

$$\xi(pn+i) \equiv 0 \pmod{p}.$$

Remark 3.2. Congruences (3)–(7) are the cases p = 5, 7, 11, 17 and 19 of Theorem 3.1.

Proof. We begin with a simple observation derived from Lucas's theorem for the congruence class of binomial coefficients modulo p [5, page 271]. Namely if π is any integer congruent to a pentagonal number modulo p, and $i \in T(p)$, then

(30)
$$\binom{\pi}{i} \equiv 0 \pmod{p},$$

because the final digit in the p-ary expansion of π is smaller than i because i is in T(p).

Now by Lemma 2.5, we may write

$$F(q, pn-1) = \sum_{i=0}^{p-1} q^i A_p(pn-1, i, q^p)$$

=
$$\sum_{\substack{i=0\\i\in S(p)}}^{p-1} q^i A_p(pn-1, i, q^p) + \sum_{\substack{i=0\\i\notin S(p)}}^{p-1} q^i (1-q^p)^n \alpha_p(n, i, q^p)$$

 So

$$F(1-q, pn-1) = \sum_{\substack{i=0\\i\in S(p)}}^{p-1} (1-q)^i A_p(pn-1, i, (1-q)^p) + \sum_{\substack{i=0\\i\notin S(p)}}^{p-1} (1-q)^i (1-(1-q)^p)^n \alpha_p(n, i, (1-q)^p) := \Sigma_1 + \Sigma_2.$$

Now modulo p,

$$\Sigma_2 \equiv \sum_{\substack{i=0\\i\notin S(p)}}^{p-1} (1-q)^i q^{pn} \alpha_p(n,i,1-q^p)$$
$$= O(q^{pn}).$$

Therefore, modulo p,

$$F(1-q, pn-1) \equiv \sum_{\substack{i=0\\i \in S(p)}}^{p-1} (1-q)^i A_p(pn-1, i, 1-q^p) + O(q^{pn}) \pmod{p}.$$

Let us look at the terms in this sum where q is raised to a power that is congruent to an element of T(p). Such a term must arise from the expansion of some $(1-q)^i$ where $i \in S(p)$ because $A_p(pn-1, 1, 1-q^p)$ is a polynomial in q^p .

By (30) all such terms have a coefficient congruent to 0 modulo p. Therefore, every term q^j in F(1-q, pn-1) where j < pn and j is congruent to an element of T(p) must have a coefficient congruent to 0 modulo p.

To conclude the proof, we let $n \to \infty$.

4. An Infinite Set of Primes With Congruences

At this stage, one might ask whether one can identify an infinite set of primes p for which congruences such as those described in Theorem 3.1 are found. The answer to this question can be answered affirmatively.

Theorem 4.1. Let $p \neq 23$ be a prime. T(p) is not empty, i.e., at least one congruence such as those described in Theorem 3.1 must hold modulo p, if and only if p = 23k + r for some nonnegative integer k and some r, 0 < r < 23, such that $\left(\frac{r}{23}\right) = -1$.

Remark 4.2. From the Prime Number Theorem for primes in arithmetic progression, we see that, asymptotically, T(p) is not empty for half of the primes and T(p) equals the empty set for half of the primes.

Proof. In light of the fact that T(p) is the set of all least nonnegative residues modulo p that are larger than every element of S(p), we see that T(p) is empty if and only if p-1 is in S(p). That is, T(p) is empty if and only if there is a solution to the congruence $n(3n-1)/2 \equiv p-1 \pmod{p}$.

Now if $n(3n-1)/2 \equiv -1 \pmod{p}$ for some integer *n*, then by completing the square we obtain $(6n-1)^2 \equiv -23 \pmod{p}$. Thus, our original congruence holds if and only if $\left(\frac{-23}{p}\right) = -1$. Therefore, $\left(\frac{-23}{p}\right) = -1$ if and only if T(p) is not empty. But thanks to properties of the Legendre symbol, we know

$$\begin{pmatrix} -23\\ p \end{pmatrix} = \begin{pmatrix} -1\\ p \end{pmatrix} \begin{pmatrix} 23\\ p \end{pmatrix}$$

$$= (-1)^{\frac{p-1}{2}} (-1)^{\frac{23-1}{2}\frac{p-1}{2}} \begin{pmatrix} p\\ 23 \end{pmatrix}$$
 by quadratic reciprocity
$$= (-1)^{\frac{12(p-1)}{2}} \begin{pmatrix} r\\ 23 \end{pmatrix}$$
 since $p = 23k + r$

$$= \begin{pmatrix} r\\ 23 \end{pmatrix}$$

and we want this value to be -1. The theorem then follows since $\left(\frac{r}{23}\right) = -1$.

Thus, we clearly have infinitely many primes p for which the Fishburn numbers will exhibit at least one congruence modulo p.

5. CONCLUSION

There are many natural open questions that could be posed at this point.

- First, we believe that Theorem 3.1 lists all the congruences of the form $\xi(pn+b) \equiv 0 \pmod{p}$, but we have not proved this at this time.
- Numerical evidence seems to indicate that Theorem 3.1 can be strengthened. Namely, for certain values of j > 1 and certain primes p, it appears that

$$\xi(p^j n + b) \equiv 0 \pmod{p^j}$$

for certain values b and all n.

• Numerical evidence suggests that Lemma 2.5 could be strengthened as follows: If $i \notin S(p)$, then

$$A_p(pn-1, i, q) = (q; q)_n \beta_p(n, i, q)$$

for some polynomial $\beta_p(n, i, q)$. That is to say, in Lemma 5, it was proved that $(1-q)^n$ divides $A_p(pn-1, i, q)$; it appears that the factor $(1-q)^n$ can be strengthened to $(q;q)_n$.

• With an eye towards the recent work of Andrews and Jelínek [3], consider the power series given by

$$\sum_{n=0}^{\infty} a(n)q^n := \sum_{n=0}^{\infty} \left(\frac{1}{1-q}, \frac{1}{1-q}\right)_n$$

which begins

$$1 - q + q^2 - 2q^3 + 5q^4 - 16q^5 + 61q^6 - 271q^7 + 1372q^8 - 7795q^9 + \dots$$

We conjecture that, for all $n \ge 0$, $a(5n + 4) \equiv 0 \pmod{5}$.

6. Acknowledgements

Since the writing of this manuscript, the authors have learned of work of F. G. Garvan which provides a generalization of the results above. In particular, Garvan considers what he calls the r-Fishburn numbers whose generating function, for a fixed integer r, is given by

$$\sum_{n=0}^{\infty} \xi_r(n) q^n = F((1-q)^r).$$

In doing so, Garvan extends the set of congruences provided in Theorem 3.1. (In particular, he proves that $\xi(n)$ satisfies certain congruence properties modulo 23.) Garvan has also proven the last conjecture shared above; namely, that $a(5n+4) \equiv 0 \pmod{5}$ for all $n \geq 0$ where

$$\sum_{n=0}^{\infty} a(n)q^n := \sum_{n=0}^{\infty} \left(\frac{1}{1-q}, \frac{1}{1-q}\right)_n$$

This is done by considering the case r = -1 in Garvan's generalization.

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