ELEMENTARY PROOFS OF PARITY RESULTS FOR 5–REGULAR PARTITIONS

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Abstract

In a recent paper, Calkin, Drake, James, Law, Lee, Penniston and Radder use the theory of modular forms to examine 5–regular partitions modulo 2 and 13–regular partitions modulo 2 and 3. They obtain and conjecture various results. In this note, we use nothing more than Jacobi's triple product identity to obtain results for 5–regular partitions stronger than those obtained by Calkin and his collaborators. We find infinitely many Ramanujan–type congruences for $b_5(n)$, and we prove the striking result that the number of 5–regular partitions of the number n is even for at least 75% of the positive integers n.

Introduction

In the seven–author paper [1], Calkin et al. examine the parity of 5–regular partitions which are defined by

$$\sum_{n\geq 0} b_5(n)q^n = \frac{(q^5;q^5)_\infty}{(q;q)_\infty}.$$

Using the theory of modular forms, they prove results equivalent to the following:

$b_5(2n)$ is odd if and only if $12n+1$ is a square, and, for all $n \geq 0$,

$$b_5(20n+5) \text{ is even and } b_5(20n+13) \text{ is even.}$$

Combining these two results, one can deduce that $b_5(n)$ is even for at least 60% of the positive integers n.

In this note we use nothing more than Jacobi's triple product identity to prove their two results above.

We also prove infinitely many new Ramanujan–type congruences for $b_5(n)$, of which the "smallest" is

$$b_5(1156n+65) \equiv 0 \pmod 2,$$

and we prove that

$$b_5(n) \text{ is even for at least 75\% of the positive integers } n.$$

This theorem is striking, because it is believed that the unrestricted partition function, $p(n)$, is even for half of the positive integers n.

2. The proofs

We begin with a fundamental theorem which provides the 2–dissection of the generating function of $b_5(n)$.

Theorem 1:

$$\sum_{n\geq 0} b_5(n)q^n = \frac{(q^8;q^8)_\infty (q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^2 (q^{40};q^{40})_\infty} + q\frac{(q^4;q^4)_\infty^3 (q^{10};q^{10})_\infty (q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^3 (q^8;q^8)_\infty (q^{20};q^{20})_\infty}$$

where $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)(1-aq^3)\dots$

Proof: We start by noting that

$$\sum_{n\geq 0} b_5(n)q^n = \frac{(q^5;q^5)_\infty}{(q;q)_\infty}$$

$$
\begin{aligned}
&= \frac{1}{(q,q^2,q^3,q^4;q^5)_\infty} \\
&= \frac{1}{(q,q^2,q^3,q^4,q^6,q^7,q^8,q^9;q^{10})_\infty} \\
&= \frac{(-q,-q^3,-q^7,-q^9,q^{10},q^{10};q^{10})_\infty}{(q^2,q^2,q^4,q^6,q^6,q^8,q^{10},q^{10},q^{12},q^{14},q^{14},q^{16},q^{18},q^{18},q^{20},q^{20};q^{20})_\infty} \\
&= \frac{(q^4;q^4)_\infty(-q,-q^3,-q^7,-q^9,q^{10},q^{10};q^{10})_\infty}{(q^2;q^2)_\infty^2(q^{20};q^{20})_\infty}
\end{aligned}
$$

where $(a_1,a_2,\ldots,a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\ldots(a_k;q)_\infty$.
Now,

$$
\begin{aligned}
&(-q,-q^3,-q^7,-q^9,q^{10},q^{10};q^{10})_\infty \\
&= \sum_{m,n=-\infty}^{\infty} q^{5m^2-4m+5n^2-2n} \quad \text{by Jacobi's triple product identity} \\
&= \sum_{r,s=-\infty}^{\infty} q^{5(r+s)^2-4(r+s)+5(r-s)^2-2(r-s)} + \sum_{r,s=-\infty}^{\infty} q^{5(r+s+1)^2-4(r+s+1)+5(r-s)^2-2(r-s)} \\
&= \sum_{r,s=-\infty}^{\infty} q^{10r^2-6r+10s^2-2s} + q\sum_{r,s=-\infty}^{\infty} q^{10r^2+4r+10s^2+8s} \\
&= (-q^4,-q^8,-q^{12},-q^{16},q^{20},q^{20};q^{20})_\infty + q(-q^2,-q^6,-q^{14},-q^{18},q^{20},q^{20};q^{20})_\infty \\
&= \frac{(-q^4;q^4)_\infty(q^{20};q^{20})_\infty^2}{(-q^{20};q^{20})_\infty} + q\frac{(-q^2;q^2)_\infty(-q^{20};q^{20})_\infty(q^{20};q^{20})_\infty^2}{(-q^4;q^4)_\infty(-q^{10};q^{10})_\infty} \\
&= \frac{(q^8;q^8)_\infty(q^{20};q^{20})_\infty^3}{(q^4;q^4)_\infty(q^{40};q^{40})_\infty} + q\frac{(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty(q^8;q^8)_\infty}.
\end{aligned}
$$

Therefore,

$$
\sum_{n\geq 0} b_5(n)q^n = \frac{(q^8;q^8)_\infty(q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^2(q^{40};q^{40})_\infty} + q\frac{(q^4;q^4)_\infty^3(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^3(q^8;q^8)_\infty(q^{20};q^{20})_\infty}
$$

as claimed. $\square$

Theorem 2 ([1, Theorem 1]): For all $n \geq 0$, $b_5(2n)$ is odd if and only if $12n+1$ is a perfect square.

Proof: Thanks to Theorem 1 above, we know

$$\begin{aligned}
\sum_{n\geq 0} b_5(2n)q^n &= \frac{(q^4;q^4)_\infty (q^{10};q^{10})_\infty^2}{(q;q)_\infty^2 (q^{20};q^{20})_\infty} \\
&\equiv \frac{(q^4;q^4)_\infty (q^{20};q^{20})_\infty}{(q^2;q^2)_\infty (q^{20};q^{20})_\infty} \pmod 2 \\
&= (-q^2;q^2)_\infty \\
&\equiv (q^2;q^2)_\infty \pmod 2 \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+n} \\
&\equiv \sum_{n=-\infty}^{\infty} q^{3n^2+n} \pmod 2.
\end{aligned}$$

Thus,

$$\sum_{n\geq 0} b_5(2n)q^{12n+1} \equiv \sum_{n=-\infty}^{\infty} q^{(6n+1)^2} \pmod 2$$

from which the result follows. $\square$

Theorem 3: For all $n \geq 0$, $b(4n+1)$ is even unless $24n+7 = 2x^2+5y^2$ for some integers x, y.

Proof: From Theorem 1, we know

$$\sum_{n\geq 0} b_5(2n+1)q^n = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^3} \cdot \frac{(q^5;q^5)_\infty (q^{20};q^{20})_\infty}{(q^4;q^4)_\infty (q^{10};q^{10})_\infty}.$$

Now,

$$\begin{aligned}\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^3} &= \prod_{n\geq 1}\left(\frac{1-q^{2n}}{1-q^n}\right)^3\\ &= \prod_{n\geq 1}(1+q^n)^3\\ &\equiv \prod_{n\geq 1}(1+q^n+q^{2n}+q^{3n}) \pmod 2\\ &= \prod_{n\geq 1}\frac{1-q^{4n}}{1-q^n}\\ &= \frac{(q^4;q^4)_\infty}{(q;q)_\infty}.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n\geq 0} b_5(2n+1)q^n &\equiv \frac{(q^5;q^5)_\infty(q^{20};q^{20})_\infty}{(q;q)_\infty(q^{10};q^{10})_\infty} \pmod 2\\ &= \prod_{n\geq 1}(1+q^{10n})\sum_{n\geq 0} b_5(n)q^n\\ &\equiv \sum_{n=-\infty}^{\infty} q^{10(3n^2+n)/2}\left(\sum_{n=-\infty}^{\infty} q^{2(3n^2+n)} + \sum_{n\geq 0} b_5(2n+1)q^{2n+1}\right) \pmod 2.\end{aligned}$$

This means

$$\sum_{n\geq 0} b_5(4n+1)q^n \equiv \sum_{m,n=-\infty}^{\infty} q^{(3m^2+m)+5(3n^2+n)/2} \pmod 2$$

which implies

$$\sum_{n\geq 0} b_5(4n+1)q^{24n+7} \equiv \sum_{m,n=-\infty}^{\infty} q^{2(6m+1)^2+5(6n+1)^2} \pmod 2.$$

The result follows. $\square$

Theorem 4 ([1, Theorem 3]): For all $n \geq 0$,

$$b_5(20n+5) \equiv 0 \pmod 2 \text{ and}$$
$$b_5(20n+13) \equiv 0 \pmod 2.$$

Proof: From Theorem 3, we know $b(20n+5)$ is even unless $24(5n+1)+7 = 2x^2+5y^2$ for some integers x, y. Consideration of this equation modulo 5 yields $x^2 \equiv 3 \pmod 5$. Since 3 is a quadratic non–residue modulo 5, we know that there can be no such solutions. This proves the first congruence. A proof of the second congruence is obtained based on the fact that 2 is the other quadratic non–residue modulo 5. $\square$

Theorem 5: If p is any prime greater than 3 such that -10 is a quadratic non–residue modulo p, if u is the reciprocal of 24 modulo p^2 and $r \not\equiv 0 \pmod p$, then for all m,

$$b_5(4p^2m + 4u(pr-7)+1) \equiv 0 \pmod 2.$$

Proof: If we set $n = p^2m + u(pr-7)$ then

$$24n + 7 \equiv 24p^2m + pr = p(24pm + r) \pmod{p^2}$$

is divisible by p but not by p^2. If $24n+7 = 2x^2+5y^2$, then $2x^2+5y^2 \equiv 0 \pmod p$ but $2x^2+5y^2 \not\equiv 0 \pmod{p^2}$. This is impossible, so by Theorem 3, $b_5(4n+1) \equiv 0 \pmod 2$. $\square$

Examples: With $p = 17$, we find that for $r \not\equiv 0 \pmod{17}$ and all m,

$$b_5(1156m + 340r + 337) \equiv 0 \pmod 2.$$

In particular, with $r = 6$ (and m replaced by $m-2$),

$$b_5(1156m + 65) \equiv 0 \pmod 2.$$

We close with one last observation about the parity of $b_5(n)$.

Theorem 6: $b_5(n)$ is even for at least 75% of the positive integers n.

Proof: By Theorem 2, $b_5(2n)$ is almost always even, and by Theorem 3, $b_5(4n+1)$ is almost always even.

The latter statement is true because in the prime factorisation of $24n+7 = 2x^2 + 5y^2$, primes congruent to

$$3,\ 17,\ 21,\ 27,\ 29,\ 31,\ 33 \text{ or } 39 \pmod{40},$$

those for which -10 is a quadratic non–residue, necessarily occur to an even power (3 itself does not occur). The density of such numbers is

$$\frac{1}{\prod_{\text{such } p>3}\left(1+\frac{1}{p}\right)} = 0. \quad \square$$

Bibliography

[1] N. Calkin, N. Drake, K. James, S. Law, P. Lee, D. Penniston and J. Radder, Divisibility properties of the 5–regular and 13–regular partition functions, Integers: Electronic Journal of Combinatorial Number Theory 8 (2008), #A60.

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