

ARITHMETIC PROPERTIES OF ANDREWS' SINGULAR OVERPARTITIONS

SHI-CHAO CHEN, MICHAEL D. HIRSCHHORN, AND JAMES A. SELLERS

ABSTRACT. In a very recent work, G. E. Andrews defined the combinatorial objects which he called *singular overpartitions* with the goal of presenting a general theorem for overpartitions which is analogous to theorems of Rogers–Ramanujan type for ordinary partitions with restricted successive ranks. As a small part of his work, Andrews noted two congruences modulo 3 which followed from elementary generating function manipulations. In this work, we show that Andrews' results modulo 3 are two examples of an infinite family of congruences modulo 3 which hold for that particular function. We also expand the consideration of such arithmetic properties to other functions which are part of Andrews' framework for singular overpartitions.

2010 Mathematics Subject Classification: 05A17, 11P83

Keywords: singular overpartition, congruence, generating function, sums of squares

1. INTRODUCTION

In a very recent work, Andrews [1] defined the combinatorial objects which he called *singular overpartitions* with the goal of presenting a general theorem for overpartitions which is analogous to theorems of Rogers–Ramanujan type for ordinary partitions with restricted successive ranks. In the process, Andrews proves that these singular overpartitions, which depend on two parameters k and i , can be enumerated by the function $\overline{C}_{k,i}(n)$ which gives the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. Andrews also notes that, for all $n \geq 0$, $\overline{C}_{3,1}(n) = \overline{A}_3(n)$ where $\overline{A}_3(n)$ is the number of overpartitions of n into parts not divisible by 3. The function $\overline{A}_k(n)$, which counts the number of overpartitions of n into parts not divisible by k , plays a key role in the work of Lovejoy [5].

As part of his work, Andrews [1] uses elementary generating function manipulations to prove that, for all $n \geq 0$,

$$(1) \quad \overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}.$$

In Section 2, we prove (1) as part of an infinite family of mod 3 congruences satisfied by $\overline{C}_{3,1}(n)$. We also prove a number of arithmetic properties modulo powers of 2 satisfied by $\overline{C}_{3,1}(n)$. In Section 3, we prove similar results for $\overline{C}_{4,1}(n)$ while in Section 4, we prove a wide variety of results for $\overline{C}_{6,1}(n)$ and $\overline{C}_{6,2}(n)$, respectively. All of the proofs will follow from elementary generating function considerations and q -series manipulations.

Before we transition to our proofs, we note that, for $k \geq 3$ and $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, the generating function for $\overline{C}_{k,i}(n)$ is given by

$$(2) \quad \sum_{n=0}^{\infty} \overline{C}_{k,i}(n)q^n = \frac{(q^k; q^k)_{\infty}(-q^i; q^k)_{\infty}(-q^{k-i}; q^k)_{\infty}}{(q; q)_{\infty}}$$

where

$$(A; q)_n = (1 - A)(1 - Aq) \dots (1 - Aq^{n-1})$$

and

$$(A; q)_{\infty} = \lim_{n \rightarrow \infty} (A; q)_n.$$

For certain values of k and i , (2) can be manipulated in elementary ways to generate the Ramanujan-like congruences which appear in this paper.

2. RESULTS FOR $\overline{C}_{3,1}(n)$

Motivated by Andrews, we first focus on the generating function for $\overline{C}_{3,1}(n)$ which, according to (2), is given by

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{3,1}(n)q^n &= \frac{(q^3; q^3)_{\infty}(-q; q^3)_{\infty}(-q^2; q^3)_{\infty}}{(q; q)_{\infty}} \\ &= \frac{(q^3; q^3)_{\infty}(-q; q)_{\infty}}{(q; q)_{\infty}(-q^3; q^3)_{\infty}} \\ &= \frac{(q^3; q^3)_{\infty}^2 (q^2; q^2)_{\infty}}{(q; q)_{\infty}^2 (q^6; q^6)_{\infty}} \\ &= \frac{(q^3; q^3)_{\infty}^2}{(q^6; q^6)_{\infty}} \bigg/ \frac{(q; q)_{\infty}^2}{(q^2; q^2)_{\infty}} \\ &= \frac{\varphi(-q^3)}{\varphi(-q)}, \end{aligned}$$

where $\varphi(q)$ is Ramanujan's theta function given by

$$(3) \quad \varphi(q) = 1 + 2 \sum_{n \geq 1} q^{n^2}.$$

Given that

$$(4) \quad \sum_{n=0}^{\infty} \overline{C}_{3,1}(n)q^n = \frac{\varphi(-q^3)}{\varphi(-q)},$$

we can see rather quickly how one might develop congruences modulo 3 which are satisfied by $\overline{C}_{3,1}(n)$.

Theorem 2.1. *Let N be a positive integer which is not expressible as the sum of two nonnegative squares. Then*

$$\overline{C}_{3,1}(N) \equiv 0 \pmod{3}.$$

Proof. Note that

$$\begin{aligned}
\sum_{n=0}^{\infty} (-1)^n \overline{C}_{3,1}(n) q^n &= \frac{\varphi(q^3)}{\varphi(q)} \\
&\equiv \frac{\varphi(q)^3}{\varphi(q)} \pmod{3} \\
&= \varphi(q)^2 \\
&= \left(1 + 2 \sum_{n=1}^{\infty} q^{n^2}\right)^2 \\
&\equiv 1 + \sum_{n=1}^{\infty} q^{n^2} + \sum_{m,n=1}^{\infty} q^{m^2+n^2} \pmod{3}.
\end{aligned}$$

The result follows. ■

Let $r_2(n)$ be the number of representations of n as the sum of two squares. From the proof of Theorem 2.1, we know

$$\overline{C}_{3,1}(n) \equiv (-1)^n r_2(n) \pmod{3}.$$

Recall the well-known formula for $r_2(n)$, as noted in [3], which states

$$r_2(n) = 4 \prod_{\substack{p|n \\ p \equiv 1 \pmod{4}}} (1 + \nu_p(n)) \prod_{\substack{p|n \\ p \equiv 3 \pmod{4}}} \frac{1 + (-1)^{\nu_p(n)}}{2},$$

where p is prime and $\nu_p(n)$ is the exponent of p dividing n . In light of this formula for $r_2(n)$, we have the following corollaries of Theorem 2.1.

Corollary 2.2. *For all $k, m \geq 0$,*

$$\overline{C}_{3,1}(2^k(4m+3)) \equiv 0 \pmod{3}.$$

Corollary 2.3. *Let $p \equiv 1 \pmod{4}$ be prime. Then for all $k, m \geq 0$ with $p \nmid m$,*

$$\overline{C}_{3,1}(p^{3k+2}m) \equiv 0 \pmod{3}.$$

Corollary 2.4. *Let $p \equiv 3 \pmod{4}$ be prime. Then for all $k, m \geq 0$ with $p \nmid m$,*

$$\overline{C}_{3,1}(p^{2k+1}m) \equiv 0 \pmod{3}.$$

Remark 2.5. *Note that Andrews' original congruences modulo 3, as given in (1), are the $p = 3, k = 0$ and $m \equiv 1, 2 \pmod{3}$ cases of Corollary 2.4.*

We now transition to a consideration of congruence results satisfied by $\overline{C}_{3,1}(n)$ modulo small powers of 2. We begin with a lemma which will allow us to obtain an alternate form of the generating function for $\overline{C}_{3,1}(n)$ from which we can obtain such results.

Lemma 2.6. $\varphi(-q^2)^2 = \varphi(q)\varphi(-q)$ where $\varphi(q)$ is defined in (3).

Proof.

$$\varphi(q)\varphi(-q) = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2} \cdot \frac{(q; q)_{\infty}^2}{(q^2; q^2)_{\infty}} = \frac{(q^2; q^2)_{\infty}^4}{(q^4; q^4)_{\infty}^2} = \varphi(-q^2)^2.$$

■

Corollary 2.7.

$$\frac{1}{\varphi(-q)} = \frac{\varphi(q)}{\varphi(-q^2)^2}.$$

Proof. This result is obvious based on the previous lemma. ■

Corollary 2.8.

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(n)q^n = \varphi(-q^3) \prod_{i=0}^{\infty} \varphi(q^{2^i})^{2^i}.$$

Proof. We simply use (4) and iterate Corollary 2.7 ad infinitum. ■

We can now state a few characterization theorems for $\overline{C}_{3,1}(n)$ modulo small powers of 2.

Theorem 2.9. *For all $n \geq 1$, $\overline{C}_{3,1}(n) \equiv 0 \pmod{2}$.*

Proof. This follows from (4) and (3). ■

Theorem 2.10. *For all $n \geq 1$,*

$$\overline{C}_{3,1}(n) \equiv \begin{cases} 2 \pmod{4} & \text{if } n = k^2 \text{ or } n = 3k^2, \\ 0 \pmod{4} & \text{otherwise.} \end{cases}$$

Proof. Thanks to Corollary 2.8 and (3) we know

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{3,1}(n)q^n &= \varphi(-q^3) \prod_{i=0}^{\infty} \varphi(q^{2^i})^{2^i} \\ &\equiv \varphi(-q^3)\varphi(q) \pmod{4} \\ &= \left(1 + 2 \sum_{k \geq 1} (-1)^k q^{3k^2}\right) \left(1 + 2 \sum_{k \geq 1} q^{k^2}\right) \\ &\equiv 1 + 2 \sum_{k \geq 1} q^{k^2} + 2 \sum_{k \geq 1} q^{3k^2} \pmod{4}. \end{aligned}$$

The result follows. ■

It is clear that one can write down numerous Ramanujan-like congruences modulo 4 satisfied by $\overline{C}_{3,1}(n)$ thanks to Theorem 2.10. We refrain from doing so here.

Note that it is also possible to write a relatively clean characterization modulo 8 for $\overline{C}_{3,1}(n)$ using the same strategy as that employed above. This is because

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{3,1}(n)q^n &= \varphi(-q^3) \prod_{i=0}^{\infty} \varphi(q^{2^i})^{2^i} \\ &\equiv \varphi(-q^3)\varphi(q)\varphi(q^2)^2 \pmod{8}. \end{aligned}$$

With that said, we consider a slight variant, namely obtaining a clean characterization modulo 8 for $(-1)^n \overline{C}_{3,1}(n)$, which can then be used rather quickly to prove numerous Ramanujan-like congruences modulo 8.

Theorem 2.11.

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^n \overline{C}_{3,1}(n) q^n \\ \equiv & 1 + 6 \sum_{k \geq 1} q^{k^2} + 4 \sum_{k \geq 1} q^{2k^2} + 2 \sum_{k \geq 1} q^{3k^2} + 4 \sum_{k, \ell \geq 1} q^{k^2+3\ell^2} \pmod{8}. \end{aligned}$$

Proof. Since $\varphi(q) = 1 + 2 \sum_{n \geq 1} q^{n^2}$, it is clear that $\varphi(q)^4 \equiv 1 \pmod{8}$. It follows that

$$\begin{aligned} & \sum_{n=0}^{\infty} (-1)^n \overline{C}_{3,1}(n) q^n \\ = & \frac{\varphi(q^3)}{\varphi(q)} \\ \equiv & \varphi(q^3) \varphi(q)^3 \pmod{8} \\ = & \left(1 + 2 \sum_{k \geq 1} q^{k^2} \right)^3 \left(1 + 2 \sum_{k \geq 1} q^{3k^2} \right) \\ \equiv & \left(1 + 6 \sum_{k \geq 1} q^{k^2} + 4 \sum_{k, \ell \geq 1} q^{k^2+\ell^2} \right) \left(1 + 2 \sum_{k \geq 1} q^{3k^2} \right) \pmod{8} \\ \equiv & 1 + 6 \sum_{k \geq 1} q^{k^2} + 2 \sum_{k \geq 1} q^{3k^2} + 4 \sum_{k, \ell \geq 1} q^{k^2+\ell^2} + 4 \sum_{k, \ell \geq 1} q^{k^2+3\ell^2} \pmod{8} \\ \equiv & 1 + 6 \sum_{k \geq 1} q^{k^2} + 4 \sum_{k \geq 1} q^{2k^2} + 2 \sum_{k \geq 1} q^{3k^2} + 4 \sum_{k, \ell \geq 1} q^{k^2+3\ell^2} \pmod{8} \end{aligned}$$

since solutions of $n = k^2 + \ell^2$ with $k \neq \ell$ come in pairs. ■

We close this section by briefly noting a few corollaries of Theorem 2.11.

Corollary 2.12. *For all $k, m \geq 0$,*

$$\begin{aligned} \overline{C}_{3,1}(4^k(16m+6)) & \equiv 0 \pmod{8}, \\ \overline{C}_{3,1}(4^k(16m+10)) & \equiv 0 \pmod{8}, \text{ and} \\ \overline{C}_{3,1}(4^k(16m+14)) & \equiv 0 \pmod{8}. \end{aligned}$$

Corollary 2.13. *For all $k, m \geq 0$,*

$$\overline{C}_{3,1}(2^k(6m+5)) \equiv 0 \pmod{8}.$$

Corollary 2.14. *Let p be prime, $p \equiv 5, 11 \pmod{12}$. For all $k, m \geq 0$ with $p \nmid m$,*

$$\overline{C}_{3,1}(p^{2k+1}m) \equiv 0 \pmod{8}.$$

Proof. Note that $n = p^{2k+1}m$ with $p \nmid m$ is neither a square, twice a square, three times a square, nor of the form $x^2 + 3y^2$ (since $\left(\frac{-3}{p}\right) = -1$ and $\nu_p(n)$ is odd). ■

3. RESULTS FOR $\overline{C}_{4,1}(n)$

We now wish to consider other examples of the functions $\overline{C}_{k,i}(n)$ where arithmetic properties can be proven using elementary means. In this section, we concentrate on the function $\overline{C}_{4,1}(n)$.

Theorem 3.1.

$$\sum_{n=0}^{\infty} \overline{C}_{4,1}(n)q^n = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2}.$$

Proof. Beginning with (2), we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{4,1}(n)q^n &= \frac{(q^4; q^4)_{\infty}(-q; q^4)_{\infty}(-q^3; q^4)_{\infty}}{(q; q)_{\infty}} \\ &= \frac{(q^4; q^4)_{\infty}(-q; q^2)_{\infty}}{(q; q)_{\infty}} \\ &= \frac{(q^4; q^4)_{\infty}(q^2; q^4)_{\infty}}{(q; q)_{\infty}(q; q^2)_{\infty}} \\ &= \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2}. \end{aligned}$$

■

Theorem 3.1 provides the following characterization of $\overline{C}_{4,1}(n)$ modulo 2.

Theorem 3.2. *For all $n \geq 1$,*

$$\overline{C}_{4,1}(n) \equiv \begin{cases} 1 \pmod{2} & \text{if } n = k(3k-1), \\ 0 \pmod{2} & \text{otherwise.} \end{cases}$$

Proof. Thanks to Theorem 3.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{4,1}(n)q^n &= \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} \\ &\equiv \frac{(q^2; q^2)_{\infty}^2}{(q^2; q^2)_{\infty}} \pmod{2} \\ &= (q^2; q^2)_{\infty} \\ &\equiv \sum_{k=-\infty}^{\infty} q^{k(3k-1)} \pmod{2} \end{aligned}$$

thanks to Euler's Pentagonal Number Theorem [2, Corollary 1.7].

■

Two corollaries follow immediately from the above.

Corollary 3.3. *For all $n \geq 0$,*

$$\overline{C}_{4,1}(2n+1) \equiv 0 \pmod{2}.$$

Proof. Note that $k(3k-1)$ is even for all integers k .

■

Corollary 3.4. *Let p be prime and let $1 \leq r \leq p-1$ with $12r+1$ a quadratic nonresidue modulo p . Then, for all $m \geq 0$,*

$$\overline{C}_{4,1}(pm+r) \equiv 0 \pmod{2}.$$

Proof. We have

$$\sum_{n=0}^{\infty} \overline{C}_{4,1}(n) q^{12n+1} \equiv \sum_{k=-\infty}^{\infty} q^{(6k-1)^2} \pmod{2}.$$

Here $n = pm + r$, so $12n + 1 = 12pm + 12r + 1 \equiv 12r + 1 \pmod{p}$ is not a square modulo p . Thus, $12n + 1$ is not a square, and $\overline{C}_{4,1}(n) \equiv 0 \pmod{2}$. ■

From Theorem 3.1, we can also obtain results modulo 4 satisfied by $\overline{C}_{4,1}(n)$.

Theorem 3.5. *If n cannot be represented as the sum of two pentagonal numbers, or if n cannot be represented as the sum of a square and twice a pentagonal number, then $\overline{C}_{4,1}(n) \equiv 0 \pmod{4}$.*

Proof. From the fact that $(1 - q^2)^2 \equiv (1 - q)^4 \pmod{4}$, we find

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{4,1}(n) q^n &= \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} \\ &\equiv \frac{(q; q)_{\infty}^4}{(q; q)_{\infty}^2} \pmod{4} \\ &= (q; q)_{\infty}^2 \\ &= \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{\frac{k(3k-1)}{2} + \frac{l(3l-1)}{2}}, \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n \overline{C}_{4,1}(n) q^n &\equiv (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty}^2 \pmod{4} \\ &= \frac{(q^2; q^4)_{\infty}^2 (q^2; q^2)_{\infty}^2}{(q; q^2)_{\infty}^2} \\ &= \frac{(q^2; q^2)_{\infty}^6}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2} \\ &= (q^2; q^2)_{\infty} \varphi(q) \\ &= \sum_{k,l=-\infty}^{\infty} (-1)^k q^{k(3k-1) + \ell^2}. \end{aligned}$$

The result follows. ■

Using Theorem 3.5, we can explicitly write infinitely many Ramanujan-like congruences modulo 4 satisfied by $\overline{C}_{4,1}(n)$.

Corollary 3.6. *Let $p \geq 5$ be a prime and $p \not\equiv 1 \pmod{12}$. Then for all $k, m \geq 0$ with $p \nmid m$,*

$$\overline{C}_{4,1} \left(p^{2k+1}m + \frac{p^{2k+2} - 1}{12} \right) \equiv 0 \pmod{4}.$$

Proof. First suppose p is prime, $p \equiv 7$ or $11 \pmod{12}$. We have

$$\sum_{n=0}^{\infty} \overline{C}_{4,1}(n) q^{24n+2} \equiv \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{(6k-1)^2 + (6l-1)^2} \pmod{4}.$$

Thus, if $24n + 2$ is not the sum of two squares, $\overline{C}_{4,1}(n) \equiv 0 \pmod{4}$.

We have $n = p^{2k+1}m + \frac{p^{2k+2} - 1}{12}$, so

$$24n + 2 = 24p^{2k+1}m + 2p^{2k+2} = p^{2k+1}(24m + 2p)$$

and $\nu_p(24n + 2)$ is odd. By Fermat's two-squares theorem, $24n + 2$ is not the sum of two squares, so $\overline{C}_{4,1}(n) \equiv 0 \pmod{4}$.

Now suppose p is prime, $p \equiv 5 \pmod{12}$. We have

$$\sum_{n=0}^{\infty} (-1)^n \overline{C}_{4,1}(n) q^{12n+1} \equiv \sum_{k,l=-\infty}^{\infty} (-1)^k q^{(6k-1)^2 + 3(2l)^2} \pmod{4}.$$

If N is of the form $x^2 + 3y^2$, then it follows by a standard argument that $\nu_p(N)$ is even since $\left(\frac{-3}{p}\right) = -1$.

However, here $n = p^{2k+1}m + \frac{p^{2k+2} - 1}{12}$ and $\nu_p(12n + 1)$ is odd. So $12n + 1$ is not of the form $x^2 + 3y^2$, and $\overline{C}_{4,1}(n) \equiv 0 \pmod{4}$. \blacksquare

As we close this section, we note that the generating function for $\overline{C}_{4,1}(n)$ is a modular function on $\Gamma_0(2)$. As such, we can slightly modify the proof of Theorem 1 of [4] to obtain the following:

Theorem 3.7. *Let $p \geq 5$ be prime and let δ_p be the least positive residue of p modulo 12. Then, for all $m \geq 0$ with $p \nmid m$,*

$$\overline{C}_{4,1}\left(pm + \frac{p^2 - 1}{12}\right) \equiv 0 \pmod{2^{\delta_p - 1}}.$$

Hence, for example, we have the following:

Corollary 3.8. *For all $m \geq 0$,*

$$\begin{aligned} \overline{C}_{4,1}(5m + 2) &\equiv 0 \pmod{2^4} \text{ if } 5 \nmid m, \\ \overline{C}_{4,1}(7m + 4) &\equiv 0 \pmod{2^6} \text{ if } 7 \nmid m, \\ \overline{C}_{4,1}(11m + 10) &\equiv 0 \pmod{2^{10}} \text{ if } 11 \nmid m. \end{aligned}$$

4. RESULTS FOR $\overline{C}_{6,1}(n)$ AND $\overline{C}_{6,2}(n)$

Next, we consider the two functions $\overline{C}_{6,1}(n)$ and $\overline{C}_{6,2}(n)$. We begin by proving an elementary result for $\overline{C}_{6,1}(n)$ modulo 3.

Theorem 4.1. *If n cannot be represented as the sum of a pentagonal number and twice a triangular number, or if n cannot be represented as the sum of a triangular number and four times a pentagonal number, then $\overline{C}_{6,1}(n) \equiv 0 \pmod{3}$.*

Proof. Beginning with (2), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{C}_{6,1}(n)q^n &= \frac{(q^6; q^6)_{\infty}(-q; q^6)_{\infty}(-q^5; q^6)_{\infty}}{(q; q)_{\infty}} \\
&= \frac{(q^6; q^6)_{\infty}(-q; q^2)_{\infty}}{(q; q)_{\infty}(-q^3; q^6)_{\infty}} \\
&= \frac{(q^6; q^6)_{\infty}(q^2; q^4)_{\infty}(q^3; q^6)_{\infty}}{(q; q)_{\infty}(q; q^2)_{\infty}(q^6; q^{12})_{\infty}} \\
&= \frac{(q^2; q^2)_{\infty}^2 (q^3; q^3)_{\infty} (q^{12}; q^{12})_{\infty}}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty} (q^6; q^6)_{\infty}} \\
&\equiv \frac{(q^2; q^2)_{\infty}^2 (q; q)_{\infty}^3 (q^4; q^4)_{\infty}^3}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty} (q^2; q^2)_{\infty}^3} \pmod{3} \\
&= \frac{(q; q)_{\infty} (q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}} \\
&= (q; q)_{\infty} \psi(q^2) \\
&= \sum_{k=-\infty}^{\infty} \sum_{\ell=1}^{\infty} (-1)^k q^{\frac{k(3k-1)}{2} + \ell(\ell-1)},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} (-1)^n \overline{C}_{6,1}(n)q^n &\equiv \frac{(-q; q^2)_{\infty} (q^2; q^2)_{\infty} (q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}} \pmod{3} \\
&= \frac{(q^2; q^4)_{\infty} (q^4; q^4)_{\infty}^2}{(q; q^2)_{\infty}} \\
&= \frac{(q^2; q^2)_{\infty}^2 (q^4; q^4)_{\infty}}{(q; q)_{\infty}} \\
&= (q^4; q^4)_{\infty} \psi(q) \\
&= \sum_{k=-\infty}^{\infty} \sum_{\ell=1}^{\infty} (-1)^k q^{2k(3k-1) + \frac{\ell(\ell-1)}{2}}.
\end{aligned}$$

Here we have used Ramanujan's theta function $\psi(q)$ which satisfies

$$\psi(q) = \sum_{\ell=1}^{\infty} q^{\frac{\ell(\ell-1)}{2}} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}}.$$

The result follows. ■

From Theorem 4.1, we can obtain the following set of Ramanujan-like congruences modulo 3 for $\overline{C}_{6,1}(n)$.

Corollary 4.2. *Let $p \geq 5$ be prime and $p \not\equiv 1, 7 \pmod{24}$. Then for $k, m \geq 0$ with $p \nmid m$,*

$$\overline{C}_{6,1} \left(p^{2k+1}m + 7 \times \frac{p^{2k+2} - 1}{24} \right) \equiv 0 \pmod{3}.$$

Proof. First suppose p is prime, $p \equiv 13, 17, 19$ or $23 \pmod{24}$. We have

$$\sum_{n=0}^{\infty} \overline{C}_{6,1}(n) q^{24n+7} \equiv \sum_{k=-\infty}^{\infty} \sum_{\ell=1}^{\infty} (-1)^k q^{(6k-1)^2+6(2\ell-1)^2} \pmod{3}.$$

So, if $24n+7$ is not of the form $(6k-1)^2+6(2\ell-1)^2$, then $\overline{C}_{6,1}(n) \equiv 0 \pmod{3}$.

If N is of the form x^2+6y^2 , then $\nu_p(N)$ is even since $\left(\frac{-6}{p}\right) = -1$. However, here $n = p^{2k+1}m + 7 \times \frac{p^{2k+2}-1}{24}$, and $\nu_p(24n+7)$ is odd. So $24n+7$ is not of the form x^2+6y^2 , and $\overline{C}_{6,1}(n) \equiv 0 \pmod{3}$.

Now suppose p is prime, $p \equiv 5$ or $11 \pmod{24}$. We have

$$\sum_{n \geq 0} (-1)^n \overline{C}_{6,1}(n) q^{24n+7} \equiv \sum_{k=-\infty}^{\infty} \sum_{\ell=1}^{\infty} (-1)^k q^{(12k-2)^2+3(2\ell-1)^2} \pmod{3}.$$

If N is of the form x^2+3y^2 , then $\nu_p(N)$ is even since $\left(\frac{-3}{p}\right) = -1$. However, $\nu_p(24n+7)$ is odd, so $24n+7$ is not of the form x^2+3y^2 . Therefore, we can conclude that $\overline{C}_{6,1}(n) \equiv 0 \pmod{3}$. \blacksquare

We now transition to a similar analysis of $\overline{C}_{6,2}(n)$.

Theorem 4.3. *For all $n \geq 1$,*

$$\overline{C}_{6,2}(n) \equiv \begin{cases} 1 \pmod{2} & \text{if } n \text{ is a pentagonal number,} \\ 0 \pmod{2} & \text{otherwise.} \end{cases}$$

Proof. Beginning with (2), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{6,2}(n) q^n &= \frac{(q^6; q^6)_{\infty} (-q^2; q^6)_{\infty} (-q^4; q^6)_{\infty}}{(q; q)_{\infty}} \\ &= \frac{(q^6; q^6)_{\infty} (-q^2; q^2)_{\infty}}{(q; q)_{\infty} (-q^6; q^6)_{\infty}} \\ &= \frac{(q^6; q^6)_{\infty} (q^4; q^4)_{\infty} (q^6; q^6)_{\infty}}{(q; q)_{\infty} (q^2; q^2)_{\infty} (q^{12}; q^{12})_{\infty}} \\ &= \frac{(q^4; q^4)_{\infty} (q^6; q^6)_{\infty}^2}{(q; q)_{\infty} (q^2; q^2)_{\infty} (q^{12}; q^{12})_{\infty}} \\ &\equiv \frac{(q; q)_{\infty}^4 (q^{12}; q^{12})_{\infty}}{(q; q)_{\infty} (q; q)_{\infty}^2 (q^{12}; q^{12})_{\infty}} \pmod{2} \\ &= (q; q)_{\infty} \\ &\equiv \sum_{k=-\infty}^{\infty} q^{\frac{k(3k-1)}{2}} \pmod{2}. \end{aligned}$$

Corollary 4.4. *Let $p \geq 5$ be prime and let $1 \leq r \leq p-1$ with $24r+1$ a quadratic nonresidue modulo p . Then, for all $m \geq 0$,*

$$\overline{C}_{6,2}(pm+r) \equiv 0 \pmod{2}.$$

Proof. We have

$$\sum_{n=0}^{\infty} \overline{C}_{6,2}(n) q^{24n+1} \equiv \sum_{k=-\infty}^{\infty} q^{(6k-1)^2} \pmod{2}.$$

Here, $n = pm + r$, so $24n + 1 = 24pm + 24r + 1 \equiv 24r + 1 \pmod{p}$ is not a square modulo p . Thus, $24n + 1$ is not a square, and $\overline{C}_{6,2}(n) \equiv 0 \pmod{2}$. ■

We close this section by considering $\overline{C}_{6,2}(n)$ modulo 3.

Corollary 4.5. *If n cannot be represented as the sum of a pentagonal number and a square, or if n cannot be written as the sum of a pentagonal number and twice a square, then*

$$\overline{C}_{6,2}(n) \equiv 0 \pmod{3}.$$

Proof. We have

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{6,2}(n) q^n &= \frac{(q^4; q^4)_{\infty} (q^6; q^6)_{\infty}^2}{(q; q)_{\infty} (q^2; q^2)_{\infty} (q^{12}; q^{12})_{\infty}} \\ &\equiv \frac{(q^4; q^4)_{\infty} (q^2; q^2)_{\infty}^6}{(q; q)_{\infty} (q^2; q^2)_{\infty} (q^4; q^4)_{\infty}^3} \pmod{3} \\ &= \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty} (q^4; q^4)_{\infty}^2} \\ &= (q; q)_{\infty} \varphi(q) \\ &= \sum_{k, \ell=-\infty}^{\infty} (-1)^k q^{\frac{k(3k-1)}{2} + \ell^2} \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n \overline{C}_{6,2}(n) q^n &\equiv (-q; q^2)_{\infty} (q^2; q^2)_{\infty} \varphi(-q) \pmod{3} \\ &= (-q; q^2)_{\infty} (q^2; q^2)_{\infty} \frac{(q; q)_{\infty}^2}{(q^2; q^2)_{\infty}} \\ &= \frac{(q; q)_{\infty}^2 (q^2; q^4)_{\infty}}{(q; q^2)_{\infty}} \\ &= \frac{(q; q)_{\infty}^2 (q^2; q^2)_{\infty}^2}{(q; q)_{\infty} (q^4; q^4)_{\infty}} \\ &= \frac{(q; q)_{\infty} (q^2; q^2)_{\infty}^2}{(q^4; q^4)_{\infty}} \\ &= (q; q)_{\infty} \varphi(-q^2) \\ &= \sum_{k, \ell=-\infty}^{\infty} (-1)^{k+\ell} q^{\frac{k(3k-1)}{2} + 2\ell^2}. \end{aligned}$$

The result follows. ■

We close our paper by demonstrating an infinite family of Ramanujan-like congruences satisfied by $\overline{C}_{6,2}(n)$ modulo 3.

Corollary 4.6. *Let $p \geq 5$ be prime, $p \not\equiv 1$ or $7 \pmod{24}$, then for all $k, m \geq 0$ with $p \nmid m$,*

$$\overline{C}_{6,2} \left(p^{2k+1}m + \frac{p^{2k+2} - 1}{24} \right) \equiv 0 \pmod{3}.$$

Proof. First suppose that p is prime, $p \equiv 13, 17, 19$ or $23 \pmod{24}$. This means $\left(\frac{-6}{p}\right) = -1$. We have

$$\sum_{n=0}^{\infty} \overline{C}_{6,2}(n)q^{24n+1} \equiv \sum_{k,\ell=-\infty}^{\infty} (-1)^k q^{(6k-1)^2+6(2\ell)^2} \pmod{3}.$$

The proof now goes through just as the proof of the first half of Corollary 4.2.

Next, suppose p is prime, $p \equiv 5$ or $11 \pmod{24}$. Then $\left(\frac{-3}{p}\right) = -1$. We have

$$\sum_{n=0}^{\infty} (-1)^n \overline{C}_{6,2}(n)q^{24n+1} \equiv \sum_{k,\ell=-\infty}^{\infty} (-1)^{k+\ell} q^{(6k-1)^2+3(4\ell)^2} \pmod{3}.$$

The proof now goes through just as the proof of the second half of Corollary 4.2. ■

REFERENCES

1. G. E. Andrews, Singular overpartitions, in progress
2. G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading 1976; reprinted, Cambridge University Press, Cambridge, 1984, 1998
3. A. Berkovich and H. Yesilyurt, Ramanujan's identities and representation of integers by certain binary and quaternary quadratic forms, *Ramanujan J.* **20** (2009), 375–408
4. S.-C. Chen, Arithmetic properties of a partition pair function, *Int. J. Num. Thy.*, to appear
5. J. Lovejoy, Gordon's theorem for overpartitions, *J. Comb. Thy. A* **103** (2003), 393–401

INSTITUTE OF CONTEMPORARY MATHEMATICS, DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES, HENAN UNIVERSITY, KAIFENG, 475001, CHINA,

SCHOOL OF MATHEMATICS AND STATISTICS, UNSW, SYDNEY 2052, AUSTRALIA,

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY PARK, PA 16802, USA,