CONGRUENCE PROPERTIES OF PARTITIONS INTO FOUR SQUARES

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Abstract

The number of partitions of a number into four squares has recently become an object of serious study. It has been discovered that the number of partitions into four squares of a number three less than a multiple of 72 is even, as are also the number of partitions into four positive squares, four distinct squares and four distinct positive squares. Here we find the generating functions of the relevant sequences, from which the results follow directly.

Keywords: partitions, four squares

§1 Introduction

Ramanujan discovered that the partition function $p(n)$ possesses many congruence properties, including

$$p(5n+4) \equiv 0 \pmod 5.$$

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Many proofs have been given of this, but perhaps the most strikingly simple and elegant is Ramanujan's

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{(q^5)_\infty^5}{(q)_\infty^6}.$$

In two earlier papers [3,5] we made a study of $p_{4\square}(n)$, the number of partitions of n into four squares, and discovered (and proved combinatorially) that

$$p_{4\square}(72n+69) \equiv 0 \pmod 2.$$

The object of this note is to give a proof of this result of the same sort as that mentioned above of Ramanujan's result, by showing that

$$\sum_{n\geq 0} p_{4\square}(72n+69)q^n = 2(\Pi_1 + \Pi_2 + \Pi_4)$$

where Π_1, Π_2 and Π_4 are all infinite products.

Indeed, we shall give the same type of proof for each of the results

$$p_{4\square}^{+}(72n+69) \equiv 0 \pmod 2,$$
$$p_{4\square}^{d}(72n+69) \equiv 0 \pmod 2,$$
$$p_{4\square}^{d+}(72n+69) \equiv 0 \pmod 2$$

where $p_{4\square}^{+}(n)$, $p_{4\square}^{d}(n)$ and $p_{4\square}^{d+}(n)$, are, respectively, the number of partitions of n into four positive squares, four distinct squares and four distinct positive squares.

With the standard notation

$$(a;q)_\infty = \prod_{n\geq 1}(1-aq^{n-1}),$$

$$(a_1, a_2, \cdots, a_k; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_k;q)_\infty,$$
$$\left(\begin{matrix} a_1, \cdots, a_k \\ b_1, \cdots, b_k \end{matrix}; q\right)_\infty = \frac{(a_1, \cdots, a_k; q)_\infty}{(b_1, \cdots, b_k; q)_\infty}$$

we prove

Theorem 1. Let

$$\Pi_1 = \left(\begin{matrix} q^6, q^6, q^{10}, q^{12}, q^{12}, q^{14}, q^{18}, q^{18}, q^{24}, q^{24}, q^{24}, q^{24} \\ q, q, q^5, q^7, q^{11}, q^{11}, q^{13}, q^{13}, q^{17}, q^{19}, q^{23}, q^{23} \end{matrix}; q^{24}\right)_\infty,$$

$$\Pi_2 = \left(\begin{matrix} q^4, q^8, q^{12}, q^{16}, q^{20}, q^{24}, q^{24}, q^{24} \\ q, q^2, \ q^7, q^{10}, q^{14}, q^{17}, q^{22}, q^{23} \end{matrix} ; q^{24} \right)_\infty ,$$

$$\Pi_3 = q^2 \left(\begin{matrix} q^4, q^8, q^{12}, q^{16}, q^{20}, q^{24}, q^{24}, q^{24} \\ q^2, q^5, q^{10}, q^{11}, q^{13}, q^{14}, q^{19}, q^{22} \end{matrix} ; q^{24} \right)_\infty ,$$

$$\Pi_4 = q^3 \left(\begin{matrix} q^2, q^6, q^6, q^{12}, q^{12}, q^{18}, q^{18}, q^{22} q^{24}, q^{24}, q^{24}, q^{24} \\ q, q^5, q^5, \ q^7, \ q^7, q^{11}, q^{13}, q^{17}, q^{17}, q^{19}, q^{19}, q^{23} \end{matrix} ; q^{24} \right)_\infty .$$

Then

$$\sum_{n\geq 0} p_{4\square}(72n+69)q^n = 2\left(\Pi_1 + \Pi_2 + \Pi_4\right),$$

$$\sum_{n\geq 0} p^{+}_{4\square}(72n+69)q^n = 2\left(\Pi_1 + \Pi_3 + \Pi_4\right),$$

$$\sum_{n\geq 0} p^{d}_{4\square}(72n+69)q^n = 2\left(\Pi_1 - \Pi_3 + \Pi_4\right),$$

$$\sum_{n\geq 0} p^{d+}_{4\square}(72n+69)q^n = 2\left(\Pi_1 - \Pi_2 + \Pi_4\right).$$

§2 Some preliminary facts

Let $\phi(q) = \sum_{-\infty}^{\infty} q^{n^2}$ and

$$\sum_{n\geq 0} a(n)q^n = \phi(q)^4, \quad \sum_{n\geq 0} b(n)q^n = \phi(q)^3, \quad \sum_{n\geq 0} c(n)q^n = \phi(q)^2\phi(q^2),$$

$$\sum_{n\geq 0} d(n)q^n = \phi(q)^2, \quad \sum_{n\geq 0} e(n)q^n = \phi(q)\phi(q^2), \quad \sum_{n\geq 0} f(n)q^n = \phi(q)\phi(q^3),$$

$$\sum_{n\geq 0} g(n)q^n = \phi(q^2)^2, \quad \sum_{n\geq 0} h(n)q^n = \phi(q), \quad \sum_{n\geq 0} j(n)q^n = \phi(q^2),$$

$$\sum_{n\geq 0} k(n)q^n = \phi(q^3), \quad \sum_{n\geq 0} l(n)q^n = \phi(q^4).$$

Then, as was shown in [5], for $n \geq 1$,

$$\begin{aligned}
p_{4\square}(n) &= \frac{1}{384}((a(n) + 4b(n) + 12c(n) + 18d(n) + 24e(n) + 32f(n) + 12g(n) \\
&\qquad + 60h(n) + 36j(n) + 32k(n) + 48l(n)), \\
p^{+}_{4\square}(n) &= \frac{1}{384}((a(n) - 4b(n) + 12c(n) - 6d(n) - 24e(n) + 32f(n) + 12g(n) \\
&\qquad - 12h(n) - 12j(n) - 32k(n) + 48l(n)), \\
p^{d}_{4\square}(n) &= \frac{1}{384}((a(n) + 4b(n) - 12c(n) - 6d(n) - 24e(n) + 32f(n) + 12g(n) \\
&\qquad + 12h(n) + 12j(n) + 32k(n) - 48l(n)), \\
p^{d+}_{4\square}(n) &= \frac{1}{384}((a(n) - 4b(n) - 12c(n) + 18d(n) + 24e(n) + 32f(n) + 12g(n) \\
&\qquad - 60h(n) - 36j(n) - 32k(n) - 48l(n)).
\end{aligned}$$

We now consider the subsequence given by $72n + 69$. It is easy to show using classical results of Jacobi, Dirichlet and Lorenz (see [4]) and elementary arguments that

$$\begin{aligned}
&d(72n+69) = e(72n+69) = f(72n+69) = g(72n+69) \\
&= h(72n+69) = j(72n+69) = k(72n+69) = l(72n+69) = 0.
\end{aligned}$$

Thus we have
Theorem 2.

$$\begin{aligned}
p_{4\square}(72n+69) &= \frac{1}{384}\left(a(72n+69) + 4b(72n+69) + 12c(72n+69)\right), \\
p^{+}_{4\square}(72n+69) &= \frac{1}{384}\left(a(72n+69) - 4b(72n+69) + 12c(72n+69)\right), \\
p^{d}_{4\square}(72n+69) &= \frac{1}{384}\left(a(72n+69) + 4b(72n+69) - 12c(72n+69)\right), \\
p^{d+}_{4\square}(72n+69) &= \frac{1}{384}\left(a(72n+69) - 4b(72n+69) - 12c(72n+69)\right).
\end{aligned}$$

§3 The generating function for $a(72n + 69)$

Theorem 3.

$$\sum_{n\geq 0} a(72n+69)q^n = 768\left(\Pi_1 + \Pi_4\right).$$

Proof:
It is a famous result of Jacobi (see [4]) that

$$a(n) = 8 \sum_{\substack{d|n \\ 4\nmid d}} d.$$

It follows that

$$a(72n+69) = 8 \sum_{d|72n+69} d = 32 \sum_{d|24n+23} d.$$

Now observe that if $d|24n+23$ then $d = 24k+r$ for some $k \geq 0$ and $r \in R = \{1, 5, 7, 11, 13, 17, 19, 23\}$, and then the codivisor d' of d is $d' = 24l + r'$ for some $l \geq 0$ and $r' = 24 - r$. Also, if $r \in R$, $r' = 24 - r$ and $k,\ l \geq 0$ then $(24k+r)(24l+r') \equiv 23 \pmod{24}$.
It follows that

$$\begin{aligned}
\sum_{n\geq 0} a(72n+69)q^{24n+23} &= 32 \sum_{\substack{k,l\geq 0 \\ r\in R}} (24k+r)q^{(24k+r)(24l+r')} \\
&= 32 \sum_{\substack{k,l\geq 0 \\ r\in\{1,5,7,11\}}} (24k+r+24l+r')q^{(24k+r)(24l+r')} \\
&= 768 \sum_{\substack{k,l\geq 0 \\ r\in\{1,5,7,11\}}} (k+l+1)q^{576kl+24r'k+24rl+rr'} \\
&= 768 \sum_{r\in\{1,5,7,11\}} q^{rr'} \sum_{k,l\geq 0} (k+l+1)q^{576kl+24r'k+24rl}
\end{aligned}$$

and so

$$\begin{aligned}
\sum_{n\geq 0} a(72n+69)q^n = 768\Big\{ &\sum_{k,l\geq 0} (k+l+1)q^{24kl+23k+l} \\
&+ q^3 \sum_{k,l\geq 0} (k+l+1)q^{24kl+19k+5l} \\
&+ q^4 \sum_{k,l\geq 0} (k+l+1)q^{24kl+17k+7l} \\
&+ q^5 \sum_{k,l\geq 0} (k+l+1)q^{24kl+13k+11l}\Big\}.
\end{aligned}$$

Now we have

Lemma 1.

$$\sum_{k,l\geq 0}(k+l+1)q^{(a+b)kl+ak+bl} = \frac{(q^{a+b})_\infty^4}{(q^a,q^b;q^{a+b})_\infty^2}.$$

Proof.

We start with Ramanujan's ${}_1\psi_1$ summation formula [1, Theorem 10.5.1],

$$\sum_{k=-\infty}^{\infty}\frac{(a;q)_k}{(b;q)_k}x^k = \frac{(ax,qa^{-1}x^{-1},q,ba^{-1};q)_\infty}{(x,ba^{-1}x^{-1},b,qa^{-1};q)_\infty},$$

valid for $|ba^{-1}|<|x|<1,\ |q|<1$. Let $b=aq$ and divide by $1-a$ to get

$$\sum_{k=-\infty}^{\infty}\frac{x^k}{1-aq^k} = \frac{(ax,qa^{-1}x^{-1},q,q;q)_\infty}{(x,qx^{-1},a,qa^{-1};q)_\infty},$$

valid for $|q|<|x|<1$.

We manipulate the series as follows:

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\frac{x^k}{1-aq^k} &= \frac{1}{1-a}+\sum_{k\geq 1}\frac{x^k}{1-aq^k}+\sum_{k\geq 1}\frac{x^{-k}}{1-aq^{-k}}\\ &= \frac{1}{1-a}+\sum_{k\geq 1}x^k\left(1+\frac{aq^k}{1-aq^k}\right)-\sum_{k\geq 1}\frac{q^ka^{-1}x^{-k}}{1-a^{-1}q^k}\\ &= \frac{1}{1-a}+\frac{x}{1-x}+\sum_{k\geq 1}\frac{q^kax^k}{1-aq^k}-\sum_{k\geq 1}\frac{q^ka^{-1}x^{-k}}{1-a^{-1}q^k}\\ &= \frac{1-ax}{(1-a)(1-x)}+\sum_{k,l\geq 1}q^{kl}a^lx^k-\sum_{k,l\geq 1}q^{kl}a^{-l}x^{-k}\\ &= \frac{1-ax}{(1-a)(1-x)}+\sum_{k,l\geq 1}q^{kl}a^lx^k-\sum_{k,l\geq 1}q^{kl}a^{-k}x^{-l}.\end{aligned}$$

Therefore

$$\frac{(ax,qa^{-1}x^{-1},q,q;q)_\infty}{(x,qx^{-1},a,qa^{-1};q)_\infty} = \frac{1-ax}{(1-a)(1-x)}-\sum_{k,l\geq 1}q^{kl}a^{-k}x^{-l}\left(1-(ax)^{k+l}\right),$$

and the series on the right converges for $|q|<|a|,|x|<|q|^{-1}$.

It follows that

$$\frac{(qax, qa^{-1}x^{-1}, q, q; q)_\infty}{(x, qx^{-1}, qa, a^{-1}; q)_\infty} = \frac{1}{(1-a^{-1})(1-x)} + a\sum_{k,l\geq 1} q^{kl}a^{-k}x^{-l}\left(1+(ax)+\ \cdots\ +(ax)^{k+l-1}\right).$$

In particular, for $a = x^{-1}$,

$$\begin{aligned}\frac{(q;q)_\infty^4}{(x, qx^{-1}; q)_\infty^2} &= \frac{1}{(1-x)^2} + \sum_{k,l\geq 1}(k+l)q^{kl}x^{k-l-1}\\ &= \sum_{k\geq 0}(k+1)x^k + \sum_{k\geq 0, l\geq 1}(k+l+1)q^{(k+1)l}x^{k-l}\\ &= \sum_{k,l\geq 0}(k+l+1)q^{kl}x^k(qx^{-1})^l.\end{aligned}$$

Now replace q by q^{a+b}, x by q^a.

It follows that

$$\begin{aligned}\sum_{n\geq 0} a(72n+69)q^n = 768\Big\{&\frac{(q^{24})_\infty^4}{(q, q^{23}; q^{24})_\infty^2} + q^3\frac{(q^{24})_\infty^4}{(q^5, q^{19}; q^{24})_\infty^2}\\ &+ q^4\frac{(q^{24})_\infty^4}{(q^7, q^{17}; q^{24})_\infty^2} + q^5\frac{(q^{24})_\infty^4}{(q^{11}, q^{13}; q^{24})_\infty^2}\Big\}.\end{aligned}$$

Now we have

Lemma 2.

$$\frac{(q^{24})_\infty^4}{(q, q^{23}; q^{24})_\infty^2} + q^5\frac{(q^{24})_\infty^4}{(q^{11}, q^{13}; q^{24})_\infty^2} = \Pi_1,$$

$$q^3\frac{(q^{24})_\infty^4}{(q^5, q^{19}; q^{24})_\infty^2} + q^4\frac{(q^{24})_\infty^4}{(q^7, q^{17}; q^{24})_\infty^2} = \Pi_4.$$

Proof:

These identities are equivalent to

$$(q^{11}, q^{13}, q^{24}; q^{24})_\infty^2 + q^5(q, q^{23}, q^{24}; q^{24})_\infty^2 = \prod_{n\geq 1}\frac{1-q^{6n}}{1+q^{6n}}\prod_{n\geq 1}(1+q^{12n-7})(1+q^{12n-5})(1-q^{12n}),$$

$$(q^7, q^{17}, q^{24}; q^{24})_\infty^2 + q(q^5, q^{19}, q^{24}; q^{24})_\infty^2 = \prod_{n\geq 1}\frac{1-q^{6n}}{1+q^{6n}}\prod_{n\geq 1}(1+q^{12n-11})(1+q^{12n-1})(1-q^{12n}).$$

In the first case, both sides are easily shown to be

$$\sum(-1)^{k+l}q^{12k^2+12l^2+k+l}+q^5\sum(-1)^{k+l}q^{12k^2+12l^2+11k+11l}$$

and in the second

$$\sum(-1)^{k+l}q^{12k^2+12l^2+5k+5l}+q\sum(-1)^{k+l}q^{12k^2+12l^2+7k+7l}.$$

Theorem 3 follows.

§4 Some preliminary results

Lemma 3.

$$\begin{aligned}\phi(q)=\phi(q^{144})+2qH(q^{72})+2q^4H(q^{288})+2q^9\psi(q^{72})+2q^{16}A(q^{144})+2q^{25}I(q^{72}) \quad (1)\\ +2q^{100}I(q^{288})+2q^{36}\psi(q^{288})+2q^{256}C(q^{144})+2q^{49}J(q^{72})+2q^{196}J(q^{288})+2q^{64}B(q^{144})\end{aligned}$$

where, as in [2],

$$\psi(q)=\sum_{n\geq 0}q^{(n^2+n)/2},\quad A(q)=\sum_{-\infty}^{\infty}q^{9n^2+2n},\quad B(q)=\sum_{-\infty}^{\infty}q^{9n^2+4n},\quad C(q)=\sum_{-\infty}^{\infty}q^{9n^2+8n},$$

$$H(q)=\sum_{-\infty}^{\infty}q^{(9n^2+n)/2},\quad I(q)=\sum_{-\infty}^{\infty}q^{(9n^2+5n)/2},\quad J(q)=\sum_{-\infty}^{\infty}q^{(9n^2+7n)/2}.$$

Proof: It is easy to check that, modulo 72,

$$n^2\equiv 0,\ 1,\ 4,\ 9,\ 16,\ 25,\ 28,\ 36,\ 40,\ 49,\ 52 \text{ or } 64,$$

and that

$$\begin{aligned}
&n^2 \equiv 0 \text{ when } n \equiv 0 \pmod{12},\\
&n^2 \equiv 1 \text{ when } n \equiv \pm 1 \pmod{18},\\
&n^2 \equiv 4 \text{ when } n \equiv \pm 2 \pmod{36},\\
&n^2 \equiv 9 \text{ when } n \equiv 3 \pmod{6},\\
&n^2 \equiv 16 \text{ when } n \equiv \pm 4 \pmod{36},\\
&n^2 \equiv 25 \text{ when } n \equiv \pm 5 \pmod{18},\\
&n^2 \equiv 28 \text{ when } n \equiv \pm 10 \pmod{36},\\
&n^2 \equiv 36 \text{ when } n \equiv 6 \pmod{12},\\
&n^2 \equiv 40 \text{ when } n \equiv \pm 16 \pmod{36},\\
&n^2 \equiv 49 \text{ when } n \equiv \pm 7 \pmod{18},\\
&n^2 \equiv 52 \text{ when } n \equiv \pm 14 \pmod{36},\\
&n^2 \equiv 64 \text{ when } n \equiv \pm 8 \pmod{36}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\phi(q) = &\sum_{-\infty}^{\infty} q^{(12n)^2} + \left(\sum_{-\infty}^{\infty} q^{(18n+1)^2} + \sum_{-\infty}^{\infty} q^{(18n-1)^2}\right) + \left(\sum_{-\infty}^{\infty} q^{(36n+2)^2} + \sum_{-\infty}^{\infty} q^{(36n-2)^2}\right)\\
&+ \sum_{-\infty}^{\infty} q^{(6n+3)^2} + \left(\sum_{-\infty}^{\infty} q^{(36n+4)^2} + \sum_{-\infty}^{\infty} q^{(36n-4)^2}\right) + \left(\sum_{-\infty}^{\infty} q^{(18n+5)^2} + \sum_{-\infty}^{\infty} q^{(18n-5)^2}\right)\\
&+ \left(\sum_{-\infty}^{\infty} q^{(36n+10)^2} + \sum_{-\infty}^{\infty} q^{(36n-10)^2}\right) + \sum_{-\infty}^{\infty} q^{(12n+6)^2} + \left(\sum_{-\infty}^{\infty} q^{(36n+16)^2} + \sum_{-\infty}^{\infty} q^{(36n-16)^2}\right)\\
&+ \left(\sum_{-\infty}^{\infty} q^{(18n+7)^2} + \sum_{-\infty}^{\infty} q^{(18n-7)^2}\right) + \left(\sum_{-\infty}^{\infty} q^{(36n+14)^2} + \sum_{-\infty}^{\infty} q^{(36n-14)^2}\right)\\
&+ \left(\sum_{-\infty}^{\infty} q^{(36n+8)^2} + \sum_{-\infty}^{\infty} q^{(36n-8)^2}\right)\\
&\\
= &\ \phi(q^{144}) + 2qH(q^{72}) + 2q^4H(q^{288}) + 2q^9\psi(q^{72}) + 2q^{16}A(q^{144}) + 2q^{25}I(q^{72})\\
&+ 2q^{100}I(q^{288}) + 2q^{36}\psi(q^{288}) + 2q^{256}C(q^{144}) + 2q^{49}J(q^{72}) + 2q^{196}J(q^{288}) + 2q^{64}B(q^{144}).
\end{aligned}$$

Lemma 4.

$$\phi(q^2) = \phi(q^{72}) + 2q^2 H(q^{144}) + 2q^8 A(q^{72}) + 2q^{18}\psi(q^{144}) + 2q^{98} J(q^{144}) + 2q^{32} B(q^{72}) + 2q^{50} I(q^{144}) + 2q^{128} C(q^{72}). \tag{2}$$

Proof: We find that, modulo 72,

$$2n^2 \equiv 0,\ 2,\ 8,\ 18,\ 26,\ 32,\ 50 \text{ or } 56,$$

and

$$\begin{aligned}
&2n^2 \equiv 0 \text{ when } n \equiv 0 \pmod 6,\\
&2n^2 \equiv 2 \text{ when } n \equiv \pm 1 \pmod{18},\\
&2n^2 \equiv 8 \text{ when } n \equiv \pm 2 \pmod{18},\\
&2n^2 \equiv 18 \text{ when } n \equiv 3 \pmod 6,\\
&2n^2 \equiv 26 \text{ when } n \equiv \pm 7 \pmod{18},\\
&2n^2 \equiv 32 \text{ when } n \equiv \pm 4 \pmod{18},\\
&2n^2 \equiv 50 \text{ when } n \equiv \pm 5 \pmod{18},\\
&2n^2 \equiv 56 \text{ when } n \equiv \pm 8 \pmod{18}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\phi(q^2) = &\sum_{-\infty}^{\infty} q^{2(6n)^2} + \left(\sum_{-\infty}^{\infty} q^{2(18n+1)^2} + \sum_{-\infty}^{\infty} q^{2(18n-1)^2}\right) + \left(\sum_{-\infty}^{\infty} q^{2(18n+2)^2} + \sum_{-\infty}^{\infty} q^{2(18n-2)^2}\right)\\
&+ \sum_{-\infty}^{\infty} q^{2(6n+3)^2} + \left(\sum_{-\infty}^{\infty} q^{2(18n+7)^2} + \sum_{-\infty}^{\infty} q^{2(18n-7)^2}\right) + \left(\sum_{-\infty}^{\infty} q^{2(18n+4)^2} + \sum_{-\infty}^{\infty} q^{2(18n-4)^2}\right)\\
&+ \left(\sum_{-\infty}^{\infty} q^{2(18n+5)^2} + \sum_{-\infty}^{\infty} q^{2(18n-5)^2}\right) + \left(\sum_{-\infty}^{\infty} q^{2(18n+8)^2} + \sum_{-\infty}^{\infty} q^{2(18n-8)^2}\right)\\
= &\ \phi(q^{72}) + 2q^2 H(q^{144}) + 2q^8 A(q^{72}) + 2q^{18}\psi(q^{144}) + 2q^{98} J(q^{144})\\
&+ 2q^{32} B(q^{72}) + 2q^{50} I(q^{144}) + 2q^{128} C(q^{72}).
\end{aligned}$$

Alternative proof: Put q^2 for q in (1), and use the (easily proven) results

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \quad A(q) = H(q^8) + q^7 C(q^4),$$
$$B(q) = A(q^4) + q^5 J(q^8), \quad C(q) = B(q^4) + qI(q^8).$$

§5 The generating function for $b(72n+69)$

Theorem 4.

$$\sum_{n\geq 0} b(72n+69)q^n = 96\frac{(q^2)_\infty(q^4)_\infty(q^6)_\infty(q^{12})_\infty}{(q)_\infty}.$$

Proof: If we cube (1) and select the appropriate terms, we obtain

$$\sum_{n\geq 0} b(72n+69)q^n = 48H(q)B(q^2)H(q^4) + 48J(q)A(q^2)H(q^4) + 48qI(q)A(q^2)I(q^4)$$
$$+ 48q^2J(q)B(q^2)I(q^4) + 48q^2H(q)A(q^2)J(q^4) + 48q^3I(q)C(q^2)H(q^4)$$
$$+ 48q^3I(q)B(q^2)J(q^4) + 48q^4H(q)C(q^2)I(q^4) + 48q^6J(q)C(q^2)J(q^4)$$

$$= 48\Big(\sum q^{(9r^2+r)/2+18s^2+8s+18t^2+2t} + \sum q^{(9r^2+7r)/2+18s^2+4s+18t^2+2t}$$
$$+ q\sum q^{(9r^2+5r)/2+18s^2+4s+18t^2+10t} + q^2\sum q^{(9r^2+7r)/2+18s^2+8s+18t^2+10t}$$
$$+ q^2\sum q^{(9r^2+r)/2+18s^2+4s+18t^2+14t} + q^3\sum q^{(9r^2+5r)/2+18s^2+16s+18t^2+2t}$$
$$+ q^3\sum q^{(9r^2+5r)/2+18s^2+8s+18t^2+14t} + q^4\sum q^{(9r^2+r)/2+18s^2+16s+18t^2+10t}$$
$$+ q^6\sum q^{(9r^2+7r)/2+18s^2+16s+18t^2+14t}\Big).$$

That is,

$$\sum_{n\geq 0} b(72n+69)q^{72n+69} = 48\Big(\sum q^{(18r+1)^2+(36s-8)^2+(36t+2)^2} + \sum q^{(18r+7)^2+(36s+4)^2+(36t+2)^2}$$
$$+ \sum q^{(18r-5)^2+(36s+4)^2+(36t-10)^2} + \sum q^{(18r+7)^2+(36s-8)^2+(36t-10)^2}$$
$$+ \sum q^{(18r+1)^2+(36s+4)^2+(36t+14)^2} + \sum q^{(18r-5)^2+(36s+16)^2+(36t+2)^2}$$
$$+ \sum q^{(18r-5)^2+(36s-8)^2+(36t+14)^2} + \sum q^{(18r+1)^2+(36s+16)^2+(36t-10)^2}$$
$$+ \sum q^{(18r+7)^2+(36s+16)^2+(36t+14)^2}\Big)$$

$$= 48\sum q^{(6a+1)^2+16(3b+1)^2+4(6c+1)^2}$$

where the sum is taken over all triples (a,b,c) for which, modulo 3,
$(a,b,c) \equiv (0,-1,0),\ (1,0,0),\ (-1,0,-1),\ (1,-1,-1),\ (0,0,1),\ (-1,1,0).\ (-1,-1,1),\ (0,1,-1)$ or $(1,1,1)$.

That is,

$$\begin{aligned}\sum_{n\geq 0} b(72n+69)q^{72n+69} &= 48 \sum_{a-b+c\equiv 1 \pmod 3} q^{(6a+1)^2+(12b+4)^2+(12c+2)^2} \\ &= 48 \sum_{a+b+c\equiv 1 \pmod 3} q^{(6a+1)^2+(12b-4)^2+(12c+2)^2}.\end{aligned}$$

We now split this sum according as a is even or odd.

$$\begin{aligned}\sum_{n\geq 0} b(72n+69)q^{72n+69} &= 48 \sum_{2a+b+c\equiv 1 \pmod 3} q^{(12a+1)^2+(12b-4)^2+(12c+2)^2} \\ &\quad + 48 \sum_{2a-1+b+c\equiv 1 \pmod 3} q^{(12a-5)^2+(12b-4)^2+(12c+2)^2}.\end{aligned}$$

Now, if $2a+b+c \equiv 1 \pmod 3$ then $a-b-c \equiv -1 \pmod 3$, while if $2a-1+b+c \equiv 1 \pmod 3$ then $a-b-c \equiv 1 \pmod 3$. So

$$\begin{aligned}\sum_{n\geq 0} b(72n+69)q^{72n+69} &= 48 \sum_{a+b+c\equiv -1 \pmod 3} q^{(12a+1)^2+(12b+4)^2+(12c-2)^2} \\ &\quad + 48 \sum_{a+b+c\equiv 1 \pmod 3} q^{(12a-5)^2+(12b+4)^2+(12c-2)^2}.\end{aligned}$$

In the first sum put $(a, b, c) = (r+s, r+t-1, r+u)$, and in the second sum put $(a, b, c) = (-r+s+1, -r+u, -r+t)$, where in each case $s+t+u=0$. We obtain

$$\begin{aligned}\sum_{n\geq 0} b(72n+69)q^{72n+69} &= 48 \sum_{-\infty}^{\infty} q^{432r^2-216r} \sum_{s+t+u=0} q^{144(s^2+t^2+u^2)+24s-192t-48u+69} \\ &\quad + 48 \sum_{-\infty}^{\infty} q^{432r^2-216r} \sum_{s+t+u=0} q^{144(s^2+t^2+u^2)+168s-48t+96u+69} \\ &= 96q^{69}\psi(q^{216}) \sum_{s+t+u=0} q^{144(s^2+t^2+u^2)+144s-72t+72u},\end{aligned}$$

and therefore

$$\sum_{n\geq 0} b(72n+69)q^{n} = 96\psi(q^3) \sum_{s+t+u=0} q^{2s^2+2t^2+2u^2+2s-t+u}.$$

So, with the notation

$$\mathrm{CT}_a\left\{\sum_{-\infty}^{\infty} c_n a^n\right\} = c_0,$$

the Constant Term, we have

$$\begin{aligned}
\sum_{n\geq 0} b(72n+69)q^n &= 96\psi(q^3)\mathrm{CT}_a\left\{\sum a^s q^{2s^2+2s}\sum a^t q^{2t^2-t}\sum a^u q^{2u^2+u}\right\}\\
&= 96\psi(q^3)\mathrm{CT}_a\left\{\prod_{n\geq 1}(1+aq^{4n})(1+a^{-1}q^{4n-4})(1-q^{4n})\right.\\
&\qquad\times\prod_{n\geq 1}(1+aq^{4n-3})(1+a^{-1}q^{4n-1})(1-q^{4n})\\
&\qquad\left.\times\prod_{n\geq 1}(1+aq^{4n-1})(1+a^{-1}q^{4n-3})(1-q^{4n})\right\}\\
&= 96\psi(q^3)\prod_{n\geq 1}\frac{(1-q^{4n})^2}{(1-q^{2n})}\mathrm{CT}_a\left\{\prod_{n\geq 1}(1+aq^{4n})(1+a^{-1}q^{4n-4})(1-q^{4n})\right.\\
&\qquad\left.\times\prod_{n\geq 1}(1+aq^{2n-1})(1+a^{-1}q^{2n-1})(1-q^{2n})\right\}\\
&= 96\psi(q^3)\psi(q^2)\mathrm{CT}_a\left\{\sum_{-\infty}^{\infty} a^s q^{2s^2+2s}\sum_{-\infty}^{\infty} a^t q^{t^2}\right\}\\
&= 96\psi(q^2)\psi(q^3)\sum_{-\infty}^{\infty} q^{2s^2+2s+(-s)^2}\\
&= 96\psi(q^2)\psi(q^3)\sum_{-\infty}^{\infty} q^{3s^2+2s}\\
&= 96\psi(q^2)\psi(q^3)X(q)\\
&= 96\frac{(q^4)_\infty^2}{(q^2)_\infty}\frac{(q^6)_\infty^2}{(q^3)_\infty}\frac{(q^2)_\infty^2(q^3)_\infty(q^{12})_\infty}{(q)_\infty(q^4)_\infty(q^6)_\infty}
\end{aligned}$$

$$= 96\frac{(q^2)_\infty(q^4)_\infty(q^6)_\infty(q^{12})_\infty}{(q)_\infty},$$

as claimed.

An alternative proof is given in [2], where it is shown successively that

$$\sum_{n\geq 0} b(n)q^n = \frac{(q^2)_\infty^{15}}{(q)_\infty^3(q^4)_\infty^3},$$

$$\sum_{n\geq 0} b(3n)q^n = \frac{(q^2)_\infty^{20}(q^3)_\infty^2(q^{12})_\infty^2}{(q)_\infty^4(q^4)_\infty^4(q^6)_\infty^5},$$

$$\sum_{n\geq 0} b(9n+6)q^n = 24\frac{(q^6)_\infty^8}{(q^2)_\infty(q^3)_\infty^2(q^{12})_\infty^2},$$

$$\sum_{n\geq 0} b(18n+15)q^n = 48q\frac{(q^3)_\infty^3(q^{24})_\infty^2}{(q)_\infty(q^{12})_\infty},$$

$$\sum_{n\geq 0} b(36n+33)q^n = 48\frac{(q^2)_\infty^3(q^3)_\infty^2(q^{12})_\infty^2}{(q)_\infty^2(q^6)_\infty^2},$$

$$\sum_{n\geq 0} b(72n+69)q^n = 96\frac{(q^2)_\infty(q^4)_\infty(q^6)_\infty(q^{12})_\infty}{(q)_\infty}.$$

§6 The generating function for $c(72n+69)$

Theorem 5.

$$\sum_{n\geq 0} c(72n+69)q^n = 32\frac{(q^3)_\infty^2(q^4)_\infty^4(q^{24})_\infty}{(q)_\infty(q^2)_\infty(q^8)_\infty(q^{12})_\infty}.$$

Proof: If we square (1) and multiply by (2), expand and select the appropriate terms, we find

$$\sum_{n\geq 0} c(72n+69)q^n = 16\psi(q^4)\left(A(q)I(q)+B(q)H(q)+q^2C(q)J(q)\right)$$
$$+ 16q\psi(q)\left(B(q)I(q^4)+C(q)H(q^4)+qA(q)J(q^4)\right)$$

Now we prove

Lemma 5.

$$A(q)I(q)+B(q)H(q)+q^2C(q)J(q) = 2\psi(q^3)P(q^2),$$
$$B(q)I(q^4)+C(q)H(q^4)+qA(q)J(q^4) = 2\psi(q^{12})X(q),$$

where, as in [2],

$$X(q)=\sum_{-\infty}^{\infty}q^{3n^2+2n},\quad P(q)=\sum_{-\infty}^{\infty}q^{(3n^2+n)/2}.$$

Proof. Let

$$\begin{aligned}F(q)&=A(q)I(q)+B(q)H(q)+q^2C(q)J(q)\\&=\sum q^{9n^2+2n+(9m^2+5m)/2}\\&\qquad+\sum q^{9n^2+4n+(9m^2+m)/2}\\&\qquad\qquad+q^2\sum q^{9n^2+8n+(9m^2+7m)/2}.\end{aligned}$$

Then

$$\begin{aligned}q^{33}F(q^{72})&=\sum q^{(18m-5)^2+8(9n+1)^2}\\&\qquad+\sum q^{(18m+1)^2+8(9n-2)^2}\\&\qquad\qquad+\sum q^{(18m+7)^2+8(9n+4)^2}\\&=\sum_{(m,n)\equiv(-1,0),(0,-1)\text{ or }(1,1)\pmod 3}q^{(6m+1)^2+8(3n+1)^2}\\&=\sum_{m+n\equiv-1\pmod 3}q^{(6m+1)^2+8(3n+1)^2}\\&=\sum q^{(6k+12l-5)^2+8(3l-3k+1)^2}\\&=\sum q^{108k^2+216l^2-108k-72l+33}\\&=2q^{33}\psi(q^{216})P(q^{144}).\end{aligned}$$

Next let

$$\begin{aligned}G(q)&=B(q)I(q^4)+C(q)H(q^4)+qA(q)J(q^4)\\&=\sum q^{9n^2+4n+18m^2+10m}\\&\qquad+\sum q^{9n^2+8n+18m^2+2m}\\&\qquad\qquad+\sum q^{9n^2+2n+18m^2+14m+1}.\end{aligned}$$

Then

$$q^{33}G(q^{18})=\sum q^{(18m-5)^2+2(9m-2)^2}$$

$$
\begin{aligned}
&+\sum q^{(18m+1)^2+2(9n+4)^2}\\
&\qquad+\sum q^{(18n+7)^2+2(9m+1)^2}\\
&=\sum_{(m,n)\equiv(-1,-1),(0,1)\text{ or }(1,0)\pmod 3} q^{(6m+1)^2+2(3n+1)^2}\\
&=\sum_{m+n\equiv 1\pmod 3} q^{(6m+1)^2+2(3n+1)^2}\\
&=\sum q^{(6k+6l+1)^2+2(6k-3l+4)^2}\\
&=\sum q^{108k^2+54l^2+108k-36l+33}\\
&=2\psi(q^{216})X(q^{18}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sum_{n\geq 0} c(72n+69)q^n &= 32\psi(q^3)\psi(q^4)P(q^2)+32q\psi(q)\psi(q^{12})X(q)\\
&=32\left(\frac{(q^6)_\infty^2}{(q^3)_\infty}\frac{(q^8)_\infty^2}{(q^4)_\infty}\frac{(q^4)_\infty(q^6)_\infty^2}{(q^2)_\infty(q^{12})_\infty}+q\frac{(q^2)_\infty^2}{(q)_\infty}\frac{(q^{24})_\infty^2}{(q^{12})_\infty}\frac{(q^2)_\infty^2(q^3)_\infty(q^{12})_\infty}{(q)_\infty(q^4)_\infty(q^6)_\infty}\right)\\
&=32\left(\frac{(q^6)_\infty^4(q^8)_\infty^2}{(q^2)_\infty(q^3)_\infty(q^{12})_\infty}+q\frac{(q^2)_\infty^4(q^3)_\infty(q^{24})_\infty^2}{(q)_\infty^2(q^4)_\infty(q^6)_\infty}\right).
\end{aligned}
$$

So we need to show that

$$
\frac{(q^6)_\infty^4(q^8)_\infty^2}{(q^2)_\infty(q^3)_\infty(q^{12})_\infty}+q\frac{(q^2)_\infty^4(q^3)_\infty(q^{24})_\infty^2}{(q)_\infty^2(q^4)_\infty(q^6)_\infty}=\frac{(q^3)_\infty^2(q^4)_\infty^4(q^{24})_\infty}{(q)_\infty(q^2)_\infty(q^8)_\infty(q^{12})_\infty}.
$$

Multiplying by $\dfrac{(q^2)_\infty(q^6)_\infty}{(q^3)_\infty(q^4)_\infty(q^{12})_\infty}$ we see that this is equivalent to

$$
\phi(q^3)\psi(q^4)+q\phi(q)\psi(q^{12})=\frac{(q^3)_\infty(q^4)_\infty^3(q^6)_\infty(q^{24})_\infty}{(q)_\infty(q^8)_\infty(q^{12})_\infty^2}.
$$

We proceed to prove the following results.

Lemma 6.

$$
\frac{(q)_\infty}{(q^3)_\infty}=\frac{(q^2)_\infty(q^{16})_\infty(q^{24})_\infty^2}{(q^6)_\infty^2(q^8)_\infty(q^{48})_\infty}-q\frac{(q^2)_\infty(q^8)_\infty^2(q^{12})_\infty(q^{48})_\infty}{(q^4)_\infty(q^6)_\infty^2(q^{16})_\infty(q^{24})_\infty} \tag{i}
$$

$$
\frac{(q^3)_\infty}{(q)_\infty}=\frac{(q^4)_\infty(q^6)_\infty(q^{16})_\infty(q^{24})_\infty^2}{(q^2)_\infty^2(q^8)_\infty(q^{12})_\infty(q^{48})_\infty}+q\frac{(q^6)_\infty(q^8)_\infty^2(q^{48})_\infty}{(q^2)_\infty^2(q^{16})_\infty(q^{24})_\infty} \tag{ii}
$$

$$\frac{(q)_\infty^2}{(q^3)_\infty^2} = \frac{(q^2)_\infty(q^4)_\infty^2(q^{12})_\infty^4}{(q^6)_\infty^5(q^8)_\infty(q^{24})_\infty} - 2q\frac{(q^2)_\infty^2(q^8)_\infty(q^{12})_\infty(q^{24})_\infty}{(q^4)_\infty(q^6)_\infty^4} \tag{iii}$$

$$\frac{(q^3)_\infty^2}{(q)_\infty^2} = \frac{(q^4)_\infty^4(q^6)_\infty(q^{12})_\infty^2}{(q^2)_\infty^5(q^8)_\infty(q^{24})_\infty} + 2q\frac{(q^4)_\infty(q^6)_\infty^2(q^8)_\infty(q^{24})_\infty}{(q^2)_\infty^4(q^{12})_\infty} \tag{iv}$$

$$\phi(q^6)\psi(q^2) + 2q\psi(q^4)\psi(q^6) = \frac{(q^2)_\infty^4(q^3)_\infty^2(q^8)_\infty(q^{12})_\infty^3}{(q)_\infty^2(q^4)_\infty^2(q^6)_\infty^3(q^{24})_\infty} \tag{v}$$

$$\phi(q^2)\psi(q^6) + 2q\psi(q^2)\psi(q^{12}) = \frac{(q^2)_\infty^3(q^3)_\infty^2(q^4)_\infty(q^{24})_\infty}{(q)_\infty^2(q^6)_\infty^2(q^8)_\infty} \tag{vi}$$

$$\phi(q^3)\psi(q^4) + q\phi(q)\psi(q^{12}) = \frac{(q^3)_\infty(q^4)_\infty^3(q^6)_\infty(q^{24})_\infty}{(q)_\infty(q^8)_\infty(q^{12})_\infty^2}. \tag{vii}$$

Proof of (i).

$$\begin{aligned}
\frac{(q)_\infty}{(q^3)_\infty} &= \prod_{n\ge1}(1-q^{3n-2})(1-q^{3n-1}) \\
&= \prod_{n\ge1}(1-q^{6n-5})(1-q^{6n-4})(1-q^{6n-2})(1-q^{6n-1}) \\
&= \frac{(q^2)_\infty}{(q^6)_\infty^2}\sum_{-\infty}^{\infty}(-1)^n q^{3n^2-2n} \\
&= \frac{(q^2)_\infty}{(q^6)_\infty^2}\left\{\sum_{-\infty}^{\infty}q^{12n^2-4n} - \sum_{-\infty}^{\infty}q^{3(2n+1)^2-2(2n+1)}\right\} \\
&= \frac{(q^2)_\infty}{(q^6)_\infty^2}\left\{\prod_{n\ge1}(1+q^{24n-16})(1+q^{24n-8})(1-q^{24n}) - q\prod_{n\ge1}(1+q^{24n-20})(1+q^{24n-4})(1-q^{24n})\right\} \\
&= \frac{(q^2)_\infty(q^{24})_\infty}{(q^6)_\infty^2}\left\{\prod_{n\ge1}\frac{(1-q^{48n-32})(1-q^{48n-16})}{(1-q^{24n-16})(1-q^{24n-8})} - q\prod_{n\ge1}\frac{(1-q^{48n-40})(1-q^{48n-8})}{(1-q^{24n-20})(1-q^{24n-4})}\right\} \\
&= \frac{(q^2)_\infty(q^{24})_\infty}{(q^6)_\infty^2}\left\{\frac{(q^{16})_\infty(q^{24})_\infty}{(q^8)_\infty(q^{48})_\infty} - q\frac{(q^8)_\infty^2(q^{12})_\infty(q^{48})_\infty}{(q^4)_\infty(q^{16})_\infty(q^{24})_\infty^2}\right\} \\
&= \frac{(q^2)_\infty(q^{16})_\infty(q^{24})_\infty^2}{(q^6)_\infty^2(q^8)_\infty(q^{48})_\infty} - q\frac{(q^2)_\infty(q^8)_\infty^2(q^{12})_\infty(q^{48})_\infty}{(q^4)_\infty(q^6)_\infty^2(q^{16})_\infty(q^{24})_\infty}.
\end{aligned}$$

Proof of (ii).

Start by writing (i)

$$\frac{(q)_\infty}{(q^3)_\infty} = a(q^2) - qb(q^2).$$

It follows that

$$\frac{(q^3)_\infty}{(q)_\infty} = \frac{1}{a(q^2) - qb(q^2)} = \frac{a(q^2) + qb(q^2)}{a(q^2)^2 - q^2 b(q^2)^2}.$$

Now,

$$a(q^2) - qb(q^2) = \frac{(q)_\infty}{(q^3)_\infty} = \frac{(q^2)_\infty}{(q^6)_\infty} \prod_{n\geq 1} \frac{(1-q^{2n-1})}{(1-q^{6n-3})},$$

so

$$\begin{aligned} a(q^2) + qb(q^2) &= \frac{(q^2)_\infty}{(q^6)_\infty} \prod_{n\geq 1} \frac{(1+q^{2n-1})}{(1+q^{6n-3})} \\ &= \frac{(q^2)_\infty}{(q^6)_\infty} \prod_{n\geq 1} \frac{(1-q^{4n-2})}{(1-q^{2n-1})} \frac{(1-q^{6n-3})}{(1-q^{12n-6})} \\ &= \frac{(q^2)_\infty}{(q^6)_\infty} \frac{(q^2)_\infty^2}{(q)_\infty (q^4)_\infty} \frac{(q^3)_\infty (q^{12})_\infty}{(q^6)_\infty^2} \\ &= \frac{(q^2)_\infty^3 (q^3)_\infty (q^{12})_\infty}{(q)_\infty (q^4)_\infty (q^6)_\infty^3}. \end{aligned}$$

It follows that

$$a(q^2)^2 - q^2 b(q^2)^2 = \frac{(q^2)_\infty^3 (q^{12})_\infty}{(q^4)_\infty (q^6)_\infty^3}$$

and

$$\begin{aligned} \frac{(q^3)_\infty}{(q)_\infty} &= \frac{(q^4)_\infty (q^6)_\infty^3}{(q^2)_\infty^3 (q^{12})_\infty} \left\{ \frac{(q^2)_\infty (q^{16})_\infty (q^{24})_\infty^2}{(q^6)_\infty^2 (q^8)_\infty (q^{48})_\infty} + q \frac{(q^2)_\infty (q^8)_\infty^2 (q^{12})_\infty (q^{48})_\infty}{(q^4)_\infty (q^6)_\infty^2 (q^{16})_\infty (q^{24})_\infty} \right\} \\ &= \frac{(q^4)_\infty (q^6)_\infty (q^{16})_\infty (q^{24})_\infty^2}{(q^2)_\infty^2 (q^8)_\infty (q^{12})_\infty (q^{48})_\infty} + q \frac{(q^6)_\infty (q^8)_\infty^2 (q^{48})_\infty}{(q^2)_\infty^2 (q^{16})_\infty (q^{24})_\infty}. \end{aligned}$$

Proof of (iii).

$$\begin{aligned} \frac{(q)_\infty^2}{(q^3)_\infty^2} &= \prod_{n\geq 1} (1-q^{3n-2})^2 (1-q^{3n-1})^2 \\ &= \prod_{n\geq 1} (1-q^{6n-5})^2 (1-q^{6n-4})^2 (1-q^{6n-2})^2 (1-q^{6n-1})^2 \end{aligned}$$

$$= \frac{(q^2)^2_\infty}{(q^6)^4_\infty}\left(\sum_{-\infty}^{\infty}(-1)^n q^{3n^2-2n}\right)^2$$
$$= \frac{(q^2)^2_\infty}{(q^6)^4_\infty}\sum(-1)^{m+n}q^{3m^2-2m+3n^2-2n}$$
$$= \frac{(q^2)^2_\infty}{(q^6)^4_\infty}\left\{\sum q^{3(k+l)^2+3(k-l)^2-2(2k)} - \sum q^{3(k+l+1)^2+3(k-l)^2-2(2k+1)}\right\}$$
$$= \frac{(q^2)^2_\infty}{(q^6)^4_\infty}\left\{\sum q^{6k^2+6l^2-4k} - q\sum q^{6k^2+6l^2+2k+6l}\right\}$$
$$= \frac{(q^2)^2_\infty}{(q^6)^4_\infty}\left\{\prod_{n\geq 1}(1+q^{12n-10})(1+q^{12n-6})^2(1+q^{12n-2})(1-q^{12n})^2 \right.$$
$$\left. -2q\prod_{n\geq 1}(1+q^{12n-8})(1+q^{12n-4})(1+q^{12n})^2(1-q^{12n})^2\right\}$$
$$= \frac{(q^2)^2_\infty(q^{12})^2_\infty}{(q^6)^4_\infty}\left\{\prod_{n\geq 1}(1+q^{4n-2})(1+q^{12n-6}) - 2q\prod_{n\geq 1}(1+q^{4n})(1+q^{12n})\right\}$$
$$= \frac{(q^2)^2_\infty(q^{12})^2_\infty}{(q^6)^4_\infty}\left\{\prod_{n\geq 1}\frac{(1-q^{8n-4})}{(1-q^{4n-2})}\frac{(1-q^{24n-12})}{(1-q^{12n-6})} - 2q\prod_{n\geq 1}\frac{(1-q^{8n})}{(1-q^{4n})}\frac{(1-q^{24n})}{(1-q^{12n})}\right\}$$
$$= \frac{(q^2)^2_\infty(q^{12})^2_\infty}{(q^6)^4_\infty}\left\{\frac{(q^4)^2_\infty}{(q^2)_\infty(q^8)_\infty}\frac{(q^{12})^2_\infty}{(q^6)_\infty(q^{24})_\infty} - 2q\frac{(q^8)_\infty}{(q^4)_\infty}\frac{(q^{24})_\infty}{(q^{12})_\infty}\right\}$$
$$= \frac{(q^2)_\infty(q^4)^2_\infty(q^{12})^4_\infty}{(q^6)^5_\infty(q^8)_\infty(q^{24})_\infty} - 2q\frac{(q^2)^2_\infty(q^8)_\infty(q^{12})_\infty(q^{24})_\infty}{(q^4)_\infty(q^6)^4_\infty}.$$

Proof of (iv).
Start by writing (iii)

$$\frac{(q)^2_\infty}{(q^3)^2_\infty} = c(q^2) - 2qd(q^2).$$

It follows that

$$\frac{(q^3)^2_\infty}{(q)^2_\infty} = \frac{1}{c(q^2)-2qd(q^2)} = \frac{c(q^2)+2qd(q^2)}{c(q^2)^2-4q^2d(q^2)^2}.$$

Now,

$$c(q^2) - 2qd(q^2) = \frac{(q)^2_\infty}{(q^3)^2_\infty} = \frac{(q^2)^2_\infty}{(q^6)^2_\infty}\prod_{n\geq 1}\frac{(1-q^{2n-1})^2}{(1-q^{6n-3})^2},$$

so

$$\begin{aligned}c(q^2)+2qd(q^2)&=\frac{(q^2)_\infty^2}{(q^6)_\infty^2}\prod_{n\geq 1}\frac{(1+q^{2n-1})^2}{(1+q^{6n-3})^2}\\&=\frac{(q^2)_\infty^2}{(q^6)_\infty^2}\prod_{n\geq 1}\frac{(1-q^{4n-2})^2}{(1-q^{2n-1})^2}\frac{(1-q^{6n-3})^2}{(1-q^{12n-6})^2}\\&=\frac{(q^2)_\infty^2}{(q^6)_\infty^2}\frac{(q^2)_\infty^4}{(q)_\infty^2(q^4)_\infty^2}\frac{(q^3)_\infty^2(q^{12})_\infty^2}{(q^6)_\infty^4}\\&=\frac{(q^2)_\infty^6(q^3)_\infty^2(q^{12})_\infty^2}{(q)_\infty^2(q^4)_\infty^2(q^6)_\infty^6}.\end{aligned}$$

It follows that

$$c(q^2)^2-4q^2d(q^2)^2=\frac{(q^2)_\infty^6(q^{12})_\infty^2}{(q^4)_\infty^2(q^6)_\infty^6}$$

and

$$\begin{aligned}\frac{(q^3)_\infty^2}{(q)_\infty^2}&=\frac{(q^4)_\infty^2(q^6)_\infty^6}{(q^2)_\infty^6(q^{12})_\infty^2}\left\{\frac{(q^2)_\infty(q^4)_\infty^2(q^{12})_\infty^4}{(q^6)_\infty^5(q^8)_\infty(q^{24})_\infty}+2q\frac{(q^2)_\infty^2(q^8)_\infty(q^{12})_\infty(q^{24}_\infty}{(q^4)_\infty(q^6)_\infty^4}\right\}\\&=\frac{(q^4)_\infty^4(q^6)_\infty(q^{12})_\infty^2}{(q^2)_\infty^5(q^8)_\infty(q^{24})_\infty}+2q\frac{(q^4)_\infty(q^6)_\infty^2(q^8)_\infty(q^{24})_\infty}{(q^2)_\infty^4(q^{12})_\infty}.\end{aligned}$$

Proof of (v).

If we multiply (iv) by $\frac{(q^2)_\infty^4(q^8)_\infty(q^{12})_\infty^3}{(q^4)_\infty^2(q^6)_\infty^3(q^{24}_\infty}$ we obtain

$$\begin{aligned}\frac{(q^2)_\infty^4(q^3)_\infty^2(q^8)_\infty(q^{12})_\infty^3}{(q)_\infty^2(q^4)_\infty^2(q^6)_\infty^3(q^{24})_\infty}&=\frac{(q^4)_\infty^2(q^{12})_\infty^5}{(q^2)_\infty(q^6)_\infty^2(q^{24})_\infty^2}+2q\frac{(q^8)_\infty^2(q^{12})_\infty^2}{(q^4)_\infty(q^6)_\infty}\\&=\phi(q^6)\psi(q^2)+2q\psi(q^4)\psi(q^6).\end{aligned}$$

Proof of (vi).

If we multiply (iv) by $\frac{(q^2)_\infty^2(q^4)_\infty(q^{24})_\infty}{(q^6)_\infty^2(q^8)_\infty}$ we obtain

$$\begin{aligned}\frac{(q^2)_\infty^3(q^3)_\infty^2(q^4)_\infty(q^{24})_\infty}{(q)_\infty^2(q^6)_\infty^2(q^8)_\infty}&=\frac{(q^4)_\infty^5(q^{12})_\infty^2}{(q^2)_\infty^2(q^6)_\infty(q^8)_\infty^2}+2q\frac{(q^4)_\infty^2(q^{24})_\infty^2}{(q^2)_\infty(q^{12})_\infty}\\&=\phi(q^2)\psi(q^6)+2q\psi(q^2)\psi(q^{12}).\end{aligned}$$

Proof of (vii).

$$\begin{aligned}
&\phi(q^3)\psi(q^4)+q\phi(q)\psi(q^{12})\\
&\qquad=\Big(\phi(q^{12})+2q^3\psi(q^{24})\Big)\psi(q^4)+q\Big(\phi(q^4)+2q\psi(q^8)\Big)\psi(q^{12})\\
&\qquad=\Big(\phi(q^{12})\psi(q^4)+2q^2\psi(q^8)\psi(q^{12})\Big)\\
&\qquad\qquad+q\Big(\phi(q^4)\psi(q^{12})+2q^2\psi(q^4)\psi(q^{24})\Big)\\
&\qquad=\frac{(q^4)^4_\infty(q^6)^2_\infty(q^{16})_\infty(q^{24})^3_\infty}{(q^2)^2_\infty(q^8)^2_\infty(q^{12})^3_\infty(q^{48})_\infty}+q\frac{(q^4)^3_\infty(q^6)^2_\infty(q^8)_\infty(q^{48})_\infty}{(q^2)^2_\infty(q^{12})^2_\infty(q^{16})_\infty}\text{ by (v) and (vi)}\\
&\qquad=\frac{(q^4)^3_\infty(q^6)_\infty(q^{24})_\infty}{(q^8)_\infty(q^{12})^2_\infty}\left\{\frac{(q^4)_\infty(q^6)_\infty(q^{16})_\infty(q^{24})^2_\infty}{(q^2)^2_\infty(q^8)_\infty(q^{12})_\infty(q^{48})_\infty}+q\frac{(q^6)_\infty(q^8)^2_\infty(q^{48})_\infty}{(q^2)^2_\infty(q^{16})_\infty(q^{24})_\infty}\right\}\\
&\qquad=\frac{(q^3)_\infty(q^4)^3_\infty(q^6)_\infty(q^{24})_\infty}{(q)_\infty(q^8)_\infty(q^{12})^2_\infty}\text{ by (ii).}
\end{aligned}$$

Theorem 5 follows.

§7 The final chapter

We require

Theorem 6.

$$12\sum_{n\ge0}c(72n+69)q^n+4\sum_{n\ge0}b(72n+69)q^n=768\Pi_2,$$

$$12\sum_{n\ge0}c(72n+69)q^n-4\sum_{n\ge0}b(72n+69)q^n=768\Pi_3.$$

Proof: These identities are equivalent to

$$\begin{aligned}
&\prod_{n\ge1}(1+q^{12n-10})(1-q^{12n-9})(1-q^{12n-3})(1+q^{12n-2})(1-q^{12n})^2\\
&\quad+\prod_{n\ge1}(1-q^{12n-10})(1+q^{12n-9})(1+q^{12n-3})(1-q^{12n-2})(1-q^{12n})^2\\
&\qquad=2\prod_{n\ge1}(1-q^{24n-19})(1-q^{24n-13})(1-q^{24n-11})(1-q^{24n-5})(1-q^{24n})^2,
\end{aligned}$$

$$\prod_{n\ge1}(1+q^{12n-10})(1-q^{12n-9})(1-q^{12n-3})(1+q^{12n-2})(1-q^{12n})^2$$

$$-\prod_{n\geq 1}(1-q^{12n-10})(1+q^{12n-9})(1+q^{12n-3})(1-q^{12n-2})(1-q^{12n})^2$$
$$=2q^2\prod_{n\geq 1}(1-q^{24n-23})(1-q^{24n-17})(1-q^{24n-7})(1-q^{24n-1})(1-q^{24n})^2.$$

In the first case, both sides are

$$2\Big\{\sum q^{24k^2+24l^2+8k+6l}-q^5\sum q^{24k^2+24l^2+16k+18l}\Big\}$$

and in the second

$$2q^2\Big\{\sum q^{24k^2+24l^2+6k+16l}-q\sum q^{24k^2+24l^2+18k+8l}\Big\}.$$

Theorem 1 now follows from Theorems 2, 3 and 6.

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