Arithmetic Properties of Non-Squashing Partitions into Distinct Parts

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Abstract

A partition $n = p_1+p_2+\cdots+p_k$ with $1 \le p_1 \le p_2 \le \cdots \le p_k$ is non-squashing if $p_1+\cdots+p_j \le p_{j+1}$ for $1 \le j \le k-1$. On their way towards the solution of a certain box-stacking problem, Sloane and Sellers were led to consider the number $b(n)$ of non-squashing partitions of n into distinct parts. Sloane and Sellers did briefly consider congruences for $b(n)$ modulo 2. In this paper we show that 2^{r-2} is the exact power of 2 dividing the difference $b(2^{r+1}n) - b(2^{r-1}n)$ for n odd and $r \ge 2$.

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1 Introduction

We begin by considering the following combinatorial problem. Suppose we have boxes with labels $1, 2, 3, \ldots$ A box labeled i weighs i pounds and can support a total weight of i pounds. We wish to build single stacks of boxes with distinct labels in such a way that no box will be squashed by the weight of the boxes above it. What is the number of different ways to build such a single stack of boxes where the total weight of all the boxes in the stack is exactly n pounds?

For the sake of precision, let us say that a partition of a natural number n is *non-squashing* if, when the parts are arranged in nondecreasing order, say

$$n = p_1 + p_2 + \cdots + p_k \text{ with } 1 \leq p_1 \leq p_2 \leq \cdots \leq p_k,$$

we have

$$p_1 + \cdots + p_j \leq p_{j+1} \text{ for } 1 \leq j \leq k-1.$$

If the boxes in a stack are labeled (from the top) $p_1, p_2, \ldots, p_k$, the stack will not collapse if and only if the corresponding partition is non-squashing.

It was shown by Hirschhorn and Sellers [1] that the number of non-squashing partitions of n is equal to the number of "binary partitions" of n, a much studied partition function. In fact, Hirschhorn and Sellers proved a more general result, and an alternative proof is given in [4].

Throughout this paper, we will denote the number of non-squashing partitions of n into *distinct* parts by $b(n)$. So the question posed in the opening paragraph is: What is $b(n)$ for a given positive integer n?

As an example, we see that $b(10) = 9$ with the following stacks being allowed:

$$\begin{array}{ccccccccc}
 & & & & \boxed{1} & & \boxed{1} & \boxed{1} & \boxed{2} \\
 & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{2} & \boxed{4} & \boxed{3} & \boxed{4} & \boxed{3} \\
\boxed{10} & \boxed{9} & \boxed{8} & \boxed{7} & \boxed{7} & \boxed{6} & \boxed{6} & \boxed{5} & \boxed{5}
\end{array}$$

Note that the stack

$$\begin{array}{c}
\boxed{1} \\
\boxed{2} \\
\boxed{3} \\
\boxed{4}
\end{array}$$

is not allowed even though the numbers 1, 2, 3, 4 are distinct and sum to 10. The bottom box of this stack, which can withstand a combined weight of 4 pounds, will be squashed by the weight of the boxes above it.

The first several values of the sequence $\{b(n)\}_{n\geq 0}$ can be found in Sloane's Online Encyclopedia of Integer Sequences [3, Sequence A088567]. In their recent work, Sloane and Sellers [4] extensively studied $b(n)$. In particular, they showed that the generating function $B(q) = \sum_{n=0}^{\infty} b(n)q^n$ satisfies the functional equation

$$B(q) = \frac{1}{1-q}B(q^2) - \frac{q^2}{1-q^2}, \tag{1}$$

and is given explicitly by

$$B(q) = \frac{1}{1-q} + \sum_{i=1}^{\infty} \frac{q^{3\cdot 2^{i-1}}}{\prod_{j=0}^{i}(1-q^{2^j})}. \tag{2}$$

An immediate consequence of (2) is that $b(n)$, the number of non-squashing partitions of n into distinct parts, is equal to the number of partitions of n into non-decreasing powers of 2 such that either all parts are equal to 1 or, if the largest part has size $2^i > 1$, then there is also at least one part of size 2^{i-1} present in the partition.

Sloane and Sellers [4, Corollary 4] did briefly consider congruences for $b(n)$ modulo 2. Since $b(n)$ can be viewed as a restricted binary partition function (given the interpretation above), we searched for congruence properties of $b(n)$ similar to those satisfied by some other restricted binary partition functions, as studied by Rødseth and Sellers [2], and discovered the following result.

Theorem 1 *For each integer $r \geq 2$, we have*

$$b(2^{r+1}n) - b(2^{r-1}n) \equiv 0 \pmod{2^{r-2}}. \tag{3}$$

Moreover, no higher power of 2 *divides the left hand side of* (3) *if n is odd.*

We prove Theorem 1 using tools developed by Rødseth and Sellers [2] as well as the functional equation (1).

2 Auxiliaries

The power series in this paper will be elements of $\mathbb{Z}[[q]]$, the ring of formal power series in q with coefficients in $\mathbb{Z}$. We define a $\mathbb{Z}$-linear operator

$$U : \mathbb{Z}[[q]] \longrightarrow \mathbb{Z}[[q]]$$

by

$$U\sum_n a(n)q^n = \sum_n a(2n)q^n.$$

Notice that if $f(q), g(q) \in \mathbb{Z}[[q]]$, then

$$U(f(q)g(q^2)) = (Uf(q))g(q). \tag{4}$$

Moreover, if $f(q) = \sum_n a(n)q^n \in \mathbb{Z}[[q]]$, $g(q) = \sum_n c(n)q^n \in \mathbb{Z}[[q]]$, and M is a positive integer, then we have

$$f(q) \equiv g(q) \pmod{M} \qquad (\text{in } \mathbb{Z}[[q]])$$

if and only if, for all n,

$$a(n) \equiv c(n) \pmod{M} \qquad (\text{in } \mathbb{Z}).$$

We shall use below the following identity for binomial coefficients:

$$\binom{2n+r-1}{r} = \sum_{i=\lceil r/2\rceil}^{r} (-1)^{r-i} 2^{2i-r} \binom{i}{r-i}\binom{n+i-1}{i}. \tag{5}$$

The truth of this relation follows by expanding both sides of the identity

$$\frac{1}{(1-q)^{2n}} = \frac{1}{(1-q(2-q))^n}$$

and comparing the coefficient of q^r on each side of the equation.

Let

$$h_i = h_i(q) = \frac{q}{(1-q)^{i+1}}, \qquad i \geq 0.$$

Then

$$h_i = \sum_{n=1}^{\infty} \binom{n+i-1}{i} q^n, \tag{6}$$

so that

$$Uh_r = \sum_{n=1}^{\infty} \binom{2n+r-1}{r} q^n.$$

It follows from (5) and (6) that

$$Uh_r = \sum_{i=\lceil r/2 \rceil}^{r} (-1)^{r-i} 2^{2i-r} \binom{i}{r-i} h_i \tag{7}$$

for $r \geq 0$.

Next, we recursively define $K_r = K_r(q)$ by

$$K_2 = 2^3 h_2 \quad \text{and} \quad K_{i+1} = U\Big(\frac{1}{1-q} K_i\Big) \tag{8}$$

for $i \geq 2$. We have the following lemma regarding K_r.

Lemma 1 *For $1 \leq i \leq r-1$, there exist $\gamma_r(i) \in \mathbb{Z}$ such that*

$$K_r = \sum_{i=1}^{r-1} \gamma_r(i) h_{i+1}, \tag{9}$$

where

$$\gamma_r(i) \equiv 0 \pmod{2^{r+i}}. \tag{10}$$

Proof. This is a weak version of [2, Lemma 1]. ■

Lemma 2 *For $r \geq 2$ and $1 \leq i \leq r$, there exist $\delta_r(i) \in \mathbb{Z}$ such that*

$$UK_r = \sum_{i=1}^{r} \delta_r(i) h_i, \tag{11}$$

where

$$\delta_r(i) \equiv 0 \pmod{2^{r+i}}. \tag{12}$$

Proof. This is a weak version of [2, Lemma 2]. ■

Now we define

$$L_2 = 2^2 h_2 + h_1,$$

and, for $i \geq 2$,

$$L_{i+1} = K_{i+1} - (UK_i)\frac{1}{1-q} + UL_i. \tag{13}$$

Lemma 3 *For $r \geq 2$, there exist $\lambda_r(i) \in \mathbb{Z}$ such that*

$$L_r = \sum_{i=1}^{r} \lambda_r(i) h_i, \tag{14}$$

where

$$\lambda_r(1) \equiv 2^{r-2} \pmod{2^{r-1}} \tag{15}$$

and

$$\lambda_r(i) \equiv 0 \pmod{2^{r+i-2}} \quad \textit{for } 2 \leq i \leq r. \tag{16}$$

Proof. We use induction on r. The lemma is true for $r = 2$. Suppose that for some $r \geq 3$ there are integers $\lambda_{r-1}(j)$ such that

$$L_{r-1} = \sum_{j=1}^{r-1} \lambda_{r-1}(j) h_j, \tag{17}$$

where

$$\lambda_{r-1}(1) \equiv 2^{r-3} \pmod{2^{r-2}} \tag{18}$$

and

$$\lambda_{r-1}(j) \equiv 0 \pmod{2^{r+j-3}} \quad \text{for } 2 \leq j \leq r-1. \tag{19}$$

Then, by (17) and (7),

$$\begin{aligned}
UL_{r-1} &= \sum_{j=1}^{r-1} \lambda_{r-1}(j) U h_j \\
&= \sum_{j=1}^{r-1} \lambda_{r-1}(j) \sum_{i=\lceil j/2 \rceil}^{j} (-1)^{j-i} 2^{2i-j} \binom{i}{j-i} h_i \\
&= \sum_{i=1}^{r-1} \sum_{j=i}^{\min(r-1,2i)} (-1)^{j-i} 2^{2i-j} \binom{i}{j-i} \lambda_{r-1}(j) h_i.
\end{aligned}$$

Moreover, by (13), (9), and (11),

$$
\begin{aligned}
L_r &= K_r - (UK_{r-1})\frac{1}{1-q} + UL_{r-1} \\
&= \sum_{i=1}^{r-1} \gamma_r(i)h_{i+1} - \sum_{i=1}^{r-1} \delta_{r-1}(i)h_{i+1} + UL_{r-1} \\
&= \sum_{i=2}^{r} \gamma_r(i-1)h_i - \sum_{i=2}^{r} \delta_{r-1}(i-1)h_i + UL_{r-1},
\end{aligned}
$$

so that (14) holds with

$$\lambda_r(1) = -\lambda_{r-1}(2) + 2\lambda_{r-1}(1), \tag{20}$$

and, for $2 \le i \le r$,

$$
\begin{aligned}
\lambda_r(i) &= \gamma_r(i-1) - \delta_{r-1}(i-1) \\
&\quad + \sum_{j=i}^{\min(r-1,2i)} (-1)^{j-i} 2^{2i-j} \binom{i}{j-i} \lambda_{r-1}(j).
\end{aligned} \tag{21}
$$

It follows that all the $\lambda_r(i)$ are integers. Furthermore, by (19) with $j=2$, (18) and (20), we get (15). Finally, (16) follows from (21), (10), (12), and (19). ■

3 Proof of Theorem 1

Throughout this section, the element $f(q)$ of $\mathbb{Z}[[q]]$ will simply be written as f. If the argument is not q, then we will, of course, include the argument in the notation.

By (7), we have

$$Uh_0 = h_0, \tag{22}$$
$$Uh_1 = 2h_1, \tag{23}$$
$$Uh_2 = 4h_2 - h_1. \tag{24}$$

Also notice that

$$U\frac{1}{1-q} = \frac{1}{1-q}. \tag{25}$$

Using (1) and (4), we find that

$$\begin{aligned}
UB &= \frac{1}{1-q}B - \frac{q}{1-q} \\
&= \frac{1}{1-q}\left(\frac{1}{1-q}B(q^2) - h_0(q^2)\right) - h_0 \\
&= \left(h_1 + \frac{1}{1-q}\right)B(q^2) - \frac{1}{1-q}h_0(q^2) - h_0.
\end{aligned}$$

Applying U once more, we get, using (4), (22), (23), and (25),

$$\begin{aligned}
U^2B &= \left(Uh_1 + U\frac{1}{1-q}\right)B - \left(U\frac{1}{1-q}\right)h_0 - Uh_0 \\
&= \left(2h_1 + \frac{1}{1-q}\right)B - h_1 - h_0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
U^2B - B &= (2h_1 + h_0)(B-1) + h_1 \\
&= (2h_1 + h_0)\left(\frac{1}{1-q}B(q^2) - \frac{1}{1-q^2}\right) + h_1 \\
&= (2h_2 + h_1)B(q^2) - (2h_1 + h_0)\frac{1}{1-q^2} + h_1,
\end{aligned}$$

so that, using (22), (23), (24), and (8),

$$\begin{aligned}
U^3B - UB &= (2Uh_2 + Uh_1)B - (2Uh_1 + Uh_0)\frac{1}{1-q} + Uh_1 \\
&= 8h_2B - 4h_2 + h_1 \\
&= K_2(B-1) + L_2.
\end{aligned}$$

Thus

$$U^{r+1}B - U^{r-1}B = K_r(B-1) + L_r \tag{26}$$

is true for $r = 2$. Suppose that (26) holds for some $r \geq 2$. Then we have

$$\begin{aligned}
U^{r+1}B - U^{r-1}B &= K_r\left(\frac{1}{1-q}B(q^2) - \frac{1}{1-q^2}\right) + L_r \\
&= \left(\frac{1}{1-q}K_r\right)B(q^2) - K_r\frac{1}{1-q^2} + L_r,
\end{aligned}$$

and applying U we get by (8) and (13),

$$\begin{aligned} U^{r+2}B - U^r B &= K_{r+1}B - (UK_r)\frac{1}{1-q} + UL_r \\ &= K_{r+1}(B-1) + L_{r+1}. \end{aligned}$$

Thus (26) holds for all $r \geq 2$.

For $r \geq 2$, we have, by Lemma 1,

$$K_r \equiv 0 \pmod{2^{r+1}},$$

and, by Lemma 3,

$$L_r \equiv 2^{r-2}h_1 \pmod{2^{r-1}},$$

so that, by (26) and (6),

$$\sum_{n=1}^{\infty}(b(2^{r+1}n) - b(2^{r-1}n))q^n \equiv 2^{r-2}\sum_{n=1}^{\infty} nq^n \pmod{2^{r-1}}.$$

Therefore,

$$b(2^{r+1}n) - b(2^{r-1}n) \equiv 2^{r-2}n \pmod{2^{r-1}},$$

and this completes the proof of Theorem 1.

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