Congruences for 3-core cubic partitions

Robson da Silva[1*] and James A. Sellers[2†]

[1]Departmento de Ciência e Tecnologia, Universidade Federal de São Paulo, São José dos Campos, 12247–014, São Paulo, Brazil.
[2]Department of Mathematics and Statistics, University of Minnesota Duluth, Duluth, 55812, Minnesota, USA.

*Corresponding author(s). E-mail(s): silva.robson@unifesp.br;
Contributing authors: jsellers@d.umn.edu;
†This author contributed equally to this work.

Abstract

In this paper, our goal is to significantly extend the list of proven arithmetic properties satisfied by the function that enumerates cubic partitions which are also 3-cores, namely $\boldsymbol{C_3(n)}$**,** which was studied extensively by Gireesh in 2017. Our proof techniques are elementary, including classical generating function manipulations and dissections.

Keywords: Congruence, cubic partition, 3-core partition, generating function

MSC Classification: 11P81 , 05A17

1 Introduction

A partition of an integer $n \geq 0$ is a non-increasing sequence of positive integers, $\lambda_1 \geq \cdots \geq \lambda_s$, such that $n = \lambda_1 + \cdots + \lambda_s$. The λ_is are called the parts of the partition. The number of partitions of n is denoted by $p(n)$, where $p(0)$ is defined as 1. The study of arithmetic properties of $p(n)$ emerged as a vibrant area of research since Ramanujan [23] proved a set of congruences for $p(n)$, including:

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

After Ramanujan, significant contributions were given by Watson [25], Atkin [4], Dyson [14], Andrews and Garvan [2], Ono [22] and Mahlburg [21], among others. One of the most impressive achievements in the study of arithmetic properties of $p(n)$ is the following result of Ono [22]: given a prime $p \geq 5$, there exist infinitely many congruences of the type $p(An + B) \equiv 0 \pmod{p}$. In a subsequent paper, Ahlgren [1] generalized this result for composite moduli M that are coprime to 6.

Recently, many authors have extended this study to other partition functions, see for example [3, 7–13]. Chen [9], for instance, proved many congruences for $a_t(n)$, the number of t-core partitions of n. We recall that a t-core partition of n is a partition having none of the hook numbers in its Ferrers graph divisible by t. As noted in [9], the generating function for the number of t-core partitions is given by

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;\ q^t)_\infty^t}{(q;\ q)_\infty},$$

where we use the standard q-series notation (for $\mathsf{q} < 1$)

$$(a;\ q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k).$$

There are many beautiful congruences for $a_t(n)$. For example, Garvan, Kim and Stanton [15] proved that if α is a positive integer and $\ell = 5, 7, 11$, then for all $n \geq 0$,

$$a_\ell\left(\ell^\alpha n - \frac{\ell^2 - 1}{24}\right) \equiv 0 \pmod{\ell^\alpha}.$$

In 2010, Hei-Chi Chan [7, 8] introduced the cubic partition function, $a(n)$, in connection with Ramanujan's cubic continued fraction. In doing so, he noted that the generating function for $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;\ q)_\infty (q^2;\ q^2)_\infty}. \tag{1}$$

Thanks to (1), we see that cubic partitions of an integer n can be interpreted as pairs consisting of two partitions whose parts sum to n, with the added property that all of the parts in the second partition must be even.

A number of authors have since considered arithmetic properties of the cubic partition function along with similar properties of a number of related functions. For example, in 2017, Gireesh [16] considered the function, denoted by $C_3(n)$, which gives the number of 3-core cubic partitions. Thus, $C_3(n)$ counts the number of cubic partitions, as described above, where none of the hook numbers in the Ferrers graph of either of the partitions in question is divisible by 3. As noted in [16, Eq. (6)], the generating function for the number of 3-core cubic partitions is given by

$$\sum_{n=0}^{\infty} C_3(n)q^n = \frac{(q^3;\ q^3)_\infty^3 (q^6;\ q^6)_\infty^3}{(q;\ q)_\infty (q^2;\ q^2)_\infty}, \tag{2}$$

In [16], the author proved a number of arithmetic properties satisfied by $C_3(n)$, including the fact that $C_3(3n+2) = 3C_3(n)$ for all n. In this paper, our goal is to significantly extend the list of proven arithmetic properties satisfied by $C_3(n)$ using elementary generating function manipulations and well–known q–series identities.

This paper is organized as follows. In Section 2, we establish some notation and recall some useful identities. The 2-, 3-, 4-, and 6-dissections for $C_3(n)$ are the content of Section 3. Section 4 is devoted to presenting and proving the main results of this paper.

2 Preliminaries

Throughout the remainder of this paper, we define

$$f_k := (q^k;\ q^k)_\infty$$

in order to shorten the notation. So, we can rewrite (2) as

$$\sum_{n=0}^{\infty} C_3(n)q^n = \frac{f_3^3 f_6^3}{f_1 f_2}. \tag{3}$$

We recall Ramanujan's theta functions

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}},\ \text{for } \mathsf{ab} < 1,$$

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}, \tag{4}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}. \tag{5}$$

We recall the classical Jacobi's identity (see [6, Theorem 1.3.9])

$$f_1^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2} \tag{6}$$

and Euler's identity (see [19, Eq. (1.6.1)])

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}. \tag{7}$$

We will also require the following 2– and 3–dissections of certain q–series:

Lemma 1 *The following identities hold:*

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{8}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{9}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{10}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{11}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}. \tag{12}$$

Proof By items (v) and (vi) of [5, Entry 25, p.40] we have $\phi(-q)^2 = \phi(q^2)^2 - 4q\psi(q^4)^2$. Using (4) and (5) it follows that

$$\frac{f_2^{10}}{(-q;\,-q)_\infty^4 f_4^4} = \frac{f_4^{10}}{f_2^4 f_8^4} - 4q\frac{f_8^4}{f_4^2}.$$

Using the fact that

$$(-q;\,-q)_\infty = \frac{f_2^3}{f_1 f_4},$$

we obtain (8) after simplifications.

Identities (9) and (10) are equivalent to Eq. (22.7.5) and (30.9.9) of [19], respectively. Replacing q by $-q$ in [19, (27.7.3)] yields (11). Finally, identity (12) is [19, (30.12.3)]. □

Lemma 2 *The following identities hold:*

$$\frac{1}{f_1 f_2} = \frac{f_9^9}{f_3^6 f_6^2 f_{18}^3} + q\frac{f_9^6}{f_3^5 f_6^3} + 3q^2\frac{f_9^3 f_{18}^3}{f_3^4 f_6^4} - 2q^3\frac{f_{18}^6}{f_3^3 f_6^5} + 4q^4\frac{f_{18}^9}{f_3^2 f_6^6 f_9^3}, \tag{13}$$

$$\frac{f_2^3}{f_1^3} = \frac{f_6}{f_3} + 3q\frac{f_6^4 f_9^5}{f_3^8 f_{18}} + 6q^2\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2 f_{18}^5}{f_3^6 f_9}. \tag{14}$$

Proof Identity (13) is equivalent to Eq. (39) of [20]. Identity (14) is Eq. (3.1) in [24]. □

3 Dissections for $C_3(n)$

We begin by providing an extremely elementary proof of the 2-dissection of (2).

Theorem 3 *We have*

$$\sum_{n=0}^{\infty} C_3(2n)q^n = \frac{f_2^3 f_3^5}{f_1^3 f_6}, \quad \textit{and} \tag{15}$$

$$\sum_{n=0}^{\infty} C_3(2n+1)q^n = \frac{f_3^3 f_6^3}{f_1 f_2}. \tag{16}$$

Proof Substituting (9) into (3), we obtain

$$\sum_{n=0}^{\infty} C_3(n)q^n = \frac{f_4^3 f_6^5}{f_2^3 f_{12}} + q\frac{f_6^3 f_{12}^3}{f_2 f_4}.$$

Extracting the even and the odd parts from this identity, we obtain

$$\sum_{n=0}^{\infty} C_3(2n)q^{2n} = \frac{f_4^3 f_6^5}{f_2^3 f_{12}},$$

$$\sum_{n=0}^{\infty} C_3(2n+1)q^{2n+1} = q\frac{f_6^3 f_{12}^3}{f_2 f_4}.$$

After dividing the last expression by q and replacing q^2 by q in both identities, we arrive at (15) and (16). □

Remark 1 As noted in [16, Eq. (41)], it follows from (3) and (16) that

$$C_3(2n+1) = C_3(n).$$

Now we 3-dissect (3).

Theorem 4 *We have*

$$\sum_{n=0}^{\infty} C_3(3n)q^n = \frac{f_2 f_3^9}{f_1^3 f_6^3} - 2q\frac{f_6^6}{f_2^2}, \tag{17}$$

$$\sum_{n=0}^{\infty} C_3(3n+1)q^n = \frac{f_3^6}{f_1^2} + 4q\frac{f_1 f_6^9}{f_2^3 f_3^3}, \quad \textit{and} \tag{18}$$

$$\sum_{n=0}^{\infty} C_3(3n+2)q^n = 3\frac{f_3^3 f_6^3}{f_1 f_2}. \tag{19}$$

Proof Substituting (13) into (3) yields

$$\sum_{n=0}^{\infty} C_3(n)q^n = \frac{f_6 f_9^9}{f_3^3 f_{18}^3} + q\frac{f_9^6}{f_3^2} + 3q^2\frac{f_9^3 f_{18}^3}{f_3 f_6} - 2q^3\frac{f_{18}^6}{f_6^2} + 4q^4\frac{f_3 f_{18}^9}{f_6^3 f_9^3}.$$

Extracting the terms of the form q^{3n}, q^{3n+1}, and q^{3n+2} we obtain

$$\sum_{n=0}^{\infty} C_3(3n)q^{3n} = \frac{f_6 f_9^9}{f_3^3 f_{18}^3} - 2q^3\frac{f_{18}^6}{f_6^2},$$

$$\sum_{n=0}^{\infty} C_3(3n+1)q^{3n+1} = q\frac{f_9^6}{f_3^2} + 4q^4\frac{f_3 f_{18}^9}{f_6^3 f_9^3},$$

$$\sum_{n=0}^{\infty} C_3(3n+2)q^{3n+2} = 3q^2\frac{f_9^3 f_{18}^3}{f_3 f_6}.$$

Dividing these three identities, respectively, by q^0, q, and q^2 and, then, replacing in the resulting identities q^3 by q, we obtain (17), (18), and (19). □

Remark 2 As noted in [16, Eq. (39)], it follows from (19) that

$$C_3(3n+2) = 3C_3(n).$$

The next theorem yields the 4-dissection of (3).

Theorem 5 *We have*

$$\sum_{n=0}^{\infty} C_3(4n)q^n = \frac{f_2^6 f_6^8}{f_1^6 f_{12}^4} - 12q^2\frac{f_2^2 f_3^2 f_{12}^4}{f_1^4}, \tag{20}$$

$$\sum_{n=0}^{\infty} C_3(4n+1)q^n = \frac{f_2^3 f_3^5}{f_1^3 f_6}, \tag{21}$$

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n = 3\frac{f_2^2 f_6^{12}}{f_1^4 f_3^2 f_{12}^4} - 4q\frac{f_2^6 f_3^4 f_{12}^4}{f_1^6 f_6^4}, \quad \textit{and} \tag{22}$$

$$\sum_{n=0}^{\infty} C_3(4n+3)q^n = \frac{f_3^3 f_6^3}{f_1 f_2}. \tag{23}$$

Proof Using (8) and (11), we can rewrite (15) as

$$\begin{aligned}\sum_{n=0}^{\infty} C_3(2n)q^n &= \frac{f_2^3}{f_6} f_3^4 \frac{f_3}{f_1^3} \\ &= \frac{f_2^3}{f_6}\left(\frac{f_{12}^{10}}{f_6^2 f_{24}^4} - 4q^3\frac{f_6^2 f_{24}^4}{f_{12}^2}\right)\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right).\end{aligned} \tag{24}$$

Expanding the product on the right hand side of (24) and extracting the even and the odd parts, we obtain

$$\sum_{n=0}^{\infty} C_3(4n)q^{2n} = \frac{f_4^6 f_{12}^8}{f_2^6 f_{24}^4} - 12q^4\frac{f_4^2 f_6^2 f_{24}^4}{f_2^4}, \tag{25}$$

$$\sum_{n=0}^{\infty} C_3(4n+2)q^{2n+1} = 3q\frac{f_4^2 f_{12}^{12}}{f_2^4 f_6^2 f_{24}^4} - 4q^3\frac{f_4^6 f_6^4 f_{24}^4}{f_2^6 f_{12}^4}. \tag{26}$$

Thus, (20) follows from (25) after replacing q^2 by q. In order to obtain (22), we divide (26) by q and, then, replace q^2 by q.

Identities (21) and (23) follow directly from (15) and (16), respectively, and the fact that $C_3(2n+1) = C_3(n)$. □

We close this section with the 6-dissection of (3).

Theorem 6 *We have*

$$\sum_{n=0}^{\infty} C_3(6n)q^n = f_1^4 + 12q\frac{f_2 f_6^5}{f_1 f_3}, \tag{27}$$

$$\sum_{n=0}^{\infty} C_3(6n+1)q^n = \frac{f_2 f_3^9}{f_1^3 f_6^3} - 2q\frac{f_6^6}{f_2^2}, \tag{28}$$

$$\sum_{n=0}^{\infty} C_3(6n+2)q^n = 3\frac{f_2^3 f_3^5}{f_1^3 f_6}, \tag{29}$$

$$\sum_{n=0}^{\infty} C_3(6n+3)q^n = \frac{f_3^6}{f_1^2} + 4q\frac{f_1 f_6^9}{f_2^3 f_3^3}, \tag{30}$$

$$\sum_{n=0}^{\infty} C_3(6n+4)q^n = 6\frac{f_2^2 f_3^2 f_6^2}{f_1^2}, \tag{31}$$

$$\sum_{n=0}^{\infty} C_3(6n+5)q^n = 3\frac{f_3^3 f_6^3}{f_1 f_2}. \tag{32}$$

Proof Substituting (14) into (15), we obtain

$$\sum_{n=0}^{\infty} C_3(2n)q^n = \frac{f_3^5}{f_6}\left(\frac{f_6}{f_3} + 3q\frac{f_6^4 f_9^5}{f_3^8 f_{18}} + 6q^2\frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 12q^3\frac{f_6^2 f_{18}^5}{f_3^6 f_9}\right).$$

Now we extract the terms involving q^{3n} and q^{3n+2}, being left respectively with

$$\sum_{n=0}^{\infty} C_3(6n)q^{3n} = f_3^4 + 12q^3\frac{f_6 f_{18}^5}{f_3 f_9},$$

$$\sum_{n=0}^{\infty} C_3(6n+4)q^{3n+2} = 6q^2\frac{f_6^2 f_9^2 f_{18}^2}{f_3^2}.$$

After simplifications, we arrive at (27) and (31).

Identities (28) and (30) follow from the fact that $C_3(2n+1) = C_3(n)$ and using (17) and (18), respectively. Similarly (29) and (32) are direct consequences of the fact that $C_3(3n+2) = 3C_3(n)$ and the identities (15) and (16), respectively. □

4 Arithmetic properties of $C_3(n)$

We let $d_{r,m}(n)$ denote the number of divisors d of n with $d \equiv r \pmod{m}$. In what follows, we let $r\{\triangle + 3\triangle\}(n)$ be the number of representations of n as the sum of a triangular number and three times a triangular number. We know (see [17, Eq. (1.8)]) that

$$r\{\triangle + 3\triangle\}(n) = d_{1,3}(2n+1) - d_{2,3}(2n+1).$$

We begin this section with a complete parity characterization of $C_3(2n)$.

Theorem 7 *For all $n \geq 0$, we have*

$$C_3(2n) \equiv r\{\triangle + 3\triangle\}(n) \pmod{2}.$$

Proof Using (6) it follows from (15) that

$$\sum_{n=0}^{\infty} C_3(2n)q^n \equiv f_1^3 f_3^3 \equiv \sum_{k,l=0}^{\infty} q^{k(k+1)/2+3l(l+1)/2} \pmod{2}, \tag{33}$$

from which the result follows. □

This theorem yields the following infinite family of Ramanujan–like congruences modulo 2.

Corollary 8 *Let p be a prime such that $p \equiv 5$ or $11 \pmod{12}$. Then for all $k, m \geq 0$ with $p \nmid m$, we have*

$$C_3\left(2p^{2k+1}m + p^{2k+2} - 1\right) \equiv 0 \pmod{2}.$$

Proof From (33) we have

$$\sum_{n=0}^{\infty} C_3(2n)q^{8n+4} \equiv \sum_{j,l=0}^{\infty} q^{(2j+1)^2+3(2l+1)^2} \pmod{2}.$$

Thus $C_3(2n) \equiv 0 \pmod{2}$ if $8n+4$ is not of the form x^2+3y^2. Since $p \equiv 5$ or $11 \pmod{12}$ it follows that $\left(\frac{-3}{p}\right) = -1$, which implies that $\nu_p(N)$ is even if N is of the form x^2+3y^2.

Since $n = p^{2k+1}m + (p^{2k+2}-1)/2$, we have

$$8n+4 = 8p^{2k+1}m + 4p^{2k+2} = p^{2k+1}(8m+4p).$$

Therefore $\nu_p(8n+4)$ is odd and the result follows. □

Thus, for example, the following congruences hold for all $n \geq 0$ thanks to the above:

$$\begin{aligned} C_3(50n+24+10r) &\equiv 0 \pmod 2, \text{ for } r \in \{1,2,3,4\}, \\ C_3(242n+120+22r) &\equiv 0 \pmod 2, \text{ for } r \in \{1,2,\ldots,10\}, \\ C_3(578n+288+34r) &\equiv 0 \pmod 2, \text{ for } r \in \{1,2,\ldots,16\}. \end{aligned}$$

The next theorem exhibits a complete parity characterization of $C_3(3n)$.

Theorem 9 *For all $n \geq 0$, we have*

$$C_3(3n) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = k(3k+2), \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

Proof We recall the identity (see [19, p. 273])

$$\Omega(q) = \sum_{k=-\infty}^{\infty} q^{k(3k+2)} = \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6},$$

which yields

$$\sum_{k=-\infty}^{\infty} q^{k(3k+2)} \equiv \frac{f_3 f_6}{f_1} \equiv \frac{f_3^3}{f_1} \pmod 2. \tag{34}$$

Thus, using (17), we have

$$\sum_{n=0}^{\infty} C_3(3n) q^n \equiv \frac{f_2 f_3^3}{f_1^3} \equiv \frac{f_3^3}{f_1} \equiv \sum_{k=-\infty}^{\infty} q^{k(3k+2)} \pmod 2,$$

which completes the proof. □

Corollary 10 *For all $n \geq 0$,*

$$\begin{aligned} C_3(12n+6) &\equiv 0 \pmod 2, \quad \text{and} \\ C_3(12n+9) &\equiv 0 \pmod 2. \end{aligned}$$

Proof Since $12n+6 = 3(4n+2)$, Theorem 9 implies that we need to know whether $4n+2$ can be written as $k(3k+2)$ for some integer k. This is equivalent to asking whether

$$3(4n+2)+1 = 9k^2+6k+1 = (3k+1)^2$$

for some k. If so, then $12n+7$ would have to be a square. However, $12n+7 \equiv 3 \pmod 4$, and there are no squares congruent to 3 modulo 4. Therefore, $4n+2$ can never be written as $k(3k+2)$ for any k, which implies the first congruence in this corollary.

Similarly, since $12n+9 = 3(4n+3)$, Theorem 9 implies that we need to know whether $4n+3$ can be written as $k(3k+2)$ for some integer k. This is equivalent to asking whether

$$3(4n+3)+1 = 9k^2+6k+1 = (3k+1)^2$$

for some k. If so, then $12n+10$ would have to be a square. However, $12n+10 \equiv 2 \pmod 4$, and there are no squares congruent to 2 modulo 4. Therefore, $4n+3$ can never be written as $k(3k+2)$ for any k, which implies the second congruence in this corollary. □

Corollary 11 *For all primes $p > 3$ and all $n \geq 0$, we have*

$$C_3(3(pn+r)) \equiv 0 \pmod 2,$$

if $3r+1$ is a quadratic nonresidue modulo p.

Proof Let $p > 3$ be a prime and $3r+1$ a quadratic nonresidue modulo p. If $pn+r = k(3k+2)$, then $r \equiv 3k^2+2k \pmod p$, which implies that $3r+1 \equiv (3k+1)^2 \pmod p$, a contradiction. Therefore the result follows from Theorem 9. □

Thus, for example, we know that, for all $n \geq 0$, the following congruences hold:

$$\begin{aligned} C_3(15n+6) &\equiv 0 \pmod 2, & C_3(33n+6) &\equiv 0 \pmod 2,\\ C_3(15n+12) &\equiv 0 \pmod 2, & C_3(33n+9) &\equiv 0 \pmod 2,\\ C_3(21n+9) &\equiv 0 \pmod 2, & C_3(33n+12) &\equiv 0 \pmod 2,\\ C_3(21n+12) &\equiv 0 \pmod 2, & C_3(33n+18) &\equiv 0 \pmod 2,\\ C_3(21n+18) &\equiv 0 \pmod 2, & C_3(33n+27) &\equiv 0 \pmod 2. \end{aligned}$$

Now we present a characterization result for the parity of $C_3(3n+1)$.

Theorem 12 *For all $n \geq 0$, we have*

$$C_3(3n+1) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = 2k(3k-2),\\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

Proof By (18) and (34), we have

$$\sum_{n=0}^{\infty} C_3(3n+1)q^n \equiv \frac{f_3^6}{f_1^2} \equiv \frac{f_6^3}{f_2} \equiv \sum_{k=-\infty}^{\infty} q^{2k(3k-2)} \pmod 2,$$

from which the result follows. □

Corollary 13 *For all primes $p > 3$ and all $n \geq 0$, we have*

$$C_3(3(pn + r) + 1) \equiv 0 \pmod 2,$$

if $(3r + 2)(p + 1)/2$ is a quadratic nonresidue modulo p.

Proof If we had $pn + r = 2k(3k - 2)$, then $r \equiv 6k^2 - 4k \pmod p$, which would imply that $3r + 2 \equiv 2(3k - 1)^2 \pmod p$. Thus, $(3r + 2)(p + 1)/2 \equiv (3k - 1)^2 \pmod p$. However $(3r + 2)(p + 1)/2$ is a quadratic nonresidue modulo p. Therefore the result follows from Theorem 12. □

Among the infinitely many Ramanujan–Like congruences that the corollary above yields we have, for example, for all $n \geq 0$, the following congruences:

$$\begin{aligned} C_3(15n + 10) &\equiv 0 \pmod 2, & C_3(33n + 4) &\equiv 0 \pmod 2,\\ C_3(15n + 13) &\equiv 0 \pmod 2, & C_3(33n + 13) &\equiv 0 \pmod 2,\\ C_3(21n + 4) &\equiv 0 \pmod 2, & C_3(33n + 19) &\equiv 0 \pmod 2,\\ C_3(21n + 16) &\equiv 0 \pmod 2, & C_3(33n + 22) &\equiv 0 \pmod 2,\\ C_3(21n + 19) &\equiv 0 \pmod 2, & C_3(33n + 25) &\equiv 0 \pmod 2. \end{aligned}$$

We next consider parity results for the arithmetic progressions $4n$ and $4n + 2$. We first prove a specific result for $C_3(4n)$ which provides additional Ramanujan–like congruences modulo 2.

Theorem 14 *For all $n \geq 0$, we have*

$$C_3(4n) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = k(k+1),\\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

Proof From Theorem 5 we know

$$\sum_{n=0}^{\infty} C_3(4n)q^n = \frac{f_2^6 f_6^8}{f_1^6 f_{12}^4} - 12q^2 \frac{f_2^2 f_3^2 f_{12}^4}{f_1^4}$$

Thus, we know

$$\sum_{n=0}^{\infty} C_3(4n)q^n \equiv \frac{f_2^6 f_6^8}{f_1^6 f_{12}^4} \equiv \frac{f_2^6 f_6^8}{f_2^3 f_6^8} \equiv f_2^3 \pmod 2. \tag{35}$$

Finally, thanks to (6), we obtain

$$\sum_{n=0}^{\infty} C_3(4n)q^n \equiv \sum_{k=0}^{\infty} q^{k(k+1)} \pmod 2,$$

which completes the proof. □

Corollary 15 *For all $n \geq 0$,*

$$C_3(8n + 4) \equiv 0 \pmod 2.$$

Proof By (35) and the fact that f_2^3 is an even function of q, we know that $C_3(4(2n+1)) = C_3(8n+4) \equiv 0 \pmod 2$ for all $n \geq 0$. □

Corollary 16 *Let $p > 2$ be a prime. Then, for all $n \geq 0$,*

$$C_3(4(pn+r)) \equiv 0 \pmod 2,$$

if $4r+1$ is a quadratic nonresidue modulo p.

Proof If $pn+r = k(k+1)$, then $r \equiv k^2+k \pmod p$. Thus, $4r+1 \equiv (2k+1)^2 \pmod p$, which is impossible since $4r+1$ is a quadratic nonresidue modulo p. Therefore the result follows from Theorem 14. □

Thus, for example, the following congruences hold for all $n \geq 0$ thanks to the above:

$$\begin{aligned} C_3(20n+4r) &\equiv 0 \pmod 2, \text{ for } r \in \{3,4\}, \\ C_3(28n+4r) &\equiv 0 \pmod 2, \text{ for } r \in \{1,3,4\}, \text{and} \\ C_3(44n+4r) &\equiv 0 \pmod 2, \text{ for } r \in \{3,4,5,7,10\}. \end{aligned}$$

We next turn our attention to the arithmetic progression $4n+2$ to yield an additional infinite family of congruences.

Theorem 17 *For all $n \geq 0$, we have*

$$C_3(4n+2) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = 3k(k+1), \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

Proof Taking (22) modulo 2 yields

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n \equiv \frac{f_2^2 f_6^{12}}{f_1^4 f_3^2 f_{12}^4} \equiv f_6^3 \pmod 2.$$

Thus, using (6), we obtain

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n \equiv \sum_{k=0}^{\infty} q^{3k(k+1)} \pmod 2,$$

which completes the proof. □

Corollary 18 *For all $n \geq 0$,*

$$\begin{aligned} C_3(8n+6) &\equiv 0 \pmod 2, \\ C_3(12n+6) &\equiv 0 \pmod 2, \quad \text{and} \\ C_3(12n+10) &\equiv 0 \pmod 2. \end{aligned}$$

Proof Note that $n = 3k(k+1)$ is an integer multiple of 6 (because $3k(k+1)$ is six times a triangular number). Therefore, for all $n \geq 0$,

$$C_3(4(2n+1)+2) = C_3(8n+6) \equiv 0 \pmod 2,$$
$$C_3(4(3n+1)+2) = C_3(12n+6) \equiv 0 \pmod 2, \quad \textit{and}$$
$$C_3(4(3n+2)+2) = C_3(12n+10) \equiv 0 \pmod 2$$

thanks to Theorem 17. □

Corollary 19 *Let $p > 3$ be a prime and let 3^{-1} be the inverse of 3 modulo p (that is, $3^{-1} = (2p+1)/3$ if $p \equiv 1 \pmod 3$ and $3^{-1} = (p+1)/3$ if $p \equiv 2 \pmod 3$). Then, for all $n \geq 0$,*

$$C_3(4(pn+r)+2) \equiv 0 \pmod 2,$$

if $3^{-1}(4r+3)$ is a quadratic nonresidue modulo p.

Proof If $pn + r = 3k(k+1)$, then $r \equiv 3k^2 + 3k \pmod p$, which implies $4r + 3 \equiv 3(2k+1)^2 \pmod p$. Thus, $3^{-1}(4r+3) \equiv (2k+1)^2 \pmod p$ which is not possible since $3^{-1}(4r+3)$ is a quadratic nonresidue modulo p. Therefore $pn + r$ is not of the form $3k(k+1)$ and the result follows thanks to Theorem 17. □

Thus, for example, we know the following specific congruences hold thanks to the above corollary:

$$C_3(20n+4r+2) \equiv 0 \pmod 2, \text{ for } r \in \{2,4\},$$
$$C_3(28n+4r+2) \equiv 0 \pmod 2, \text{ for } r \in \{2,3,5\}, \text{and}$$
$$C_3(44n+4r+2) \equiv 0 \pmod 2, \text{ for } r \in \{1,4,8,9,10\}.$$

The next result presents a complete characterization of $C_3(2n)$ modulo 3, from which we derive an infinite family of congruences modulo 3.

In what follows, given an integer j, we let $\pentagon_j$ denote the pentagonal number $j(3j-1)/2$. Given two positive integers α and β, we also define

$$s\{\alpha\pentagon + \beta\pentagon\}(n) = \sum_{\substack{k,l\in\mathbb{Z} \\ \alpha\pentagon_k + \beta\pentagon_l = n}} (-1)^{k+l}.$$

Theorem 20 *For all $n \geq 0$, we have*

$$C_3(2n) \equiv s\{6\pentagon + 18\pentagon\}(n) \pmod 3.$$

Proof The following congruence follows directly from (15):

$$\sum_{n=0}^{\infty} C_3(2n)q^n = \frac{f_2^3 f_3^5}{f_1^3 f_6} \equiv f_3^4 \equiv f_3 f_9 \pmod 3.$$

Thanks to (7) it follows that

$$\sum_{n=0}^{\infty} C_3(2n)q^n \equiv \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{3k(3k-1)/2+9l(3l-1)/2} \pmod 3, \tag{36}$$

which completes the proof. □

Corollary 21 *For all* $n \geq 0$,

$$C_3(6n+2) \equiv 0 \pmod 3, \quad \text{and}$$
$$C_3(6n+4) \equiv 0 \pmod 3.$$

Proof Note that $n = 3k(3k-1)/2 + 9l(3l-1)/2$ is clearly a multiple of 3. Therefore, for all $n \geq 0$,

$$C_3(2(3n+1)) = C_3(6n+2) \equiv 0 \pmod 3$$

and

$$C_3(2(3n+2)) = C_3(6n+4) \equiv 0 \pmod 3$$

thanks to Theorem 20. □

Of course, the above congruence results also follow immediately from Theorem 6.

As mentioned before, Theorem 20 yields an infinite family of congruences modulo 3.

Corollary 22 *Let* $p > 3$ *be a prime such that* $p \equiv 5$ *or* $11 \pmod{12}$*. Then for all* $k, m \geq 0$ *with* $p \nmid m$*, we have*

$$C_3\left(2p^{2k+1}m + p^{2k+2} - 1\right) \equiv 0 \pmod 3.$$

Proof From (36) we have

$$\sum_{n=0}^{\infty} C_3(2n)q^{8n+4} \equiv \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{(6k-1)^2+3(6l-1)^2} \pmod 3.$$

Thus $C_3(2n) \equiv 0 \pmod 3$ if $8n+4$ is not of the form x^2+3y^2. The rest of the proof follows the same steps as in the proof of Corollary 8. □

now turn our attention to characterizing $C_3(\alpha n + \beta)$, for some values of α and β, for higher moduli. We begin with the complete characterization of $C_3(4n+2)$ modulo 4.

Theorem 23 *For all $n \geq 0$, we have*

$$C_3(4n+2) \equiv 3\, r\{3\triangle + 3\triangle\}(n) \pmod 4.$$

Proof Taking (22) modulo 4 yields

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n \equiv 3\frac{f_2^2 f_6^{12}}{f_1^4 f_3^2 f_{12}^4} \equiv 3\frac{f_6^{12}}{f_3^2 f_6^8} \equiv 3\frac{f_6^4}{f_3^2} = 3\psi(q^3)^2 \pmod 4.$$

Now, using (5), we obtain

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n \equiv 3\sum_{k,l=0}^{\infty} q^{3k(k+1)/2+3l(l+1)/2} \pmod 4, \tag{37}$$

which completes the proof. □

We note that two congruences follow immediately from the proof of Theorem 23.

Corollary 24 *For all $n \geq 0$,*

$$C_3(12n+6) \equiv 0 \pmod 4 \quad \textit{and}$$
$$C_3(12n+10) \equiv 0 \pmod 4.$$

Proof Note from the proof of Theorem 23 that

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n \equiv 3\psi(q^3)^2 \pmod 4$$

so that, modulo 4, the generating function is a function of q^3. Therefore, for all $n \geq 0$,

$$C_3(4(3n+1)+2) = C_3(12n+6) \equiv 0 \pmod 4$$

and

$$C_3(4(3n+2)+2) = C_3(12n+10) \equiv 0 \pmod 4.$$

□

Theorem 23 yields an infinite family of congruences modulo 4.

Corollary 25 *Let $p > 3$ be a prime. Then, for all $k, m \geq 0$ with $p \nmid m$,*

$$C_3(12p^{2k+1}m + 3p^{2k+2} - 1) \equiv 0 \pmod 4.$$

Proof We start by noting that if $n = 3k(k+1)/2 + 3l(l+1)/2$, then $8n+6 = 3(2k+1)^2 + 3(2l+1)^2$. Thus, it follows from (37) that

$$\sum_{n=0}^{\infty} C_3(4n+2)q^{8n+6} \equiv 3 \sum_{k,l=0}^{\infty} q^{3(2k+1)^2+3(2l+1)^2} \pmod 4.$$

So, $C_3(4n+2) \equiv 0 \pmod 4$ if $8n+6$ is not of the form $3(x^2+y^2)$.

We have $n = 3p^{2k+1}m + 3((p^{2k+2}-1))/4$, which implies that $8n+6 = 3p^{2k+1}(8m+2p)$. Thus, $\nu_p(8n+6)$ is odd. Therefore $8n+6$ is not of the form $3(x^2+y^2)$ and the result follows. □

In what follows, ⬠ represents a pentagonal number. Thus, $r\{3\triangle + ⬠\}(n)$ is the number of representations of n as the sum of three times a triangular number and a pentagonal number. Thanks to [18, Eq. (1.3)] we know that

$$r\{3\triangle + ⬠\}(n) = d_{1,12}(12n+5) - d_{11,12}(12n+5).$$

Theorem 26 *For all $n \geq 0$, we have*

$$C_3(6n+4) \equiv 2\, r\{6\triangle + 2⬠\}(n) \pmod 4.$$

Proof It follows from (31) that

$$\sum_{n=0}^{\infty} C_3(6n+4)q^n = 6\frac{f_2^2 f_3^2 f_6^2}{f_1^2} \equiv 2f_2 f_6^3 \pmod 4.$$

Thanks to (6) and (7), we have

$$\sum_{n=0}^{\infty} C_3(6n+4)q^n \equiv 2 \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} q^{k(3k-1)+3l(l+1)} \pmod 4, \tag{38}$$

from which the result follows. □

Theorem 26 also yields an infinite family of congruences modulo 4.

Corollary 27 *Let $p > 3$ be a prime with $p \equiv 3 \pmod 4$. Then, for all $k, m \geq 0$ with $p \nmid m$,*

$$C_3(6p^{2k+1}m + 5p^{2k+2} - 1) \equiv 0 \pmod 4.$$

Proof It follows from (38) that

$$\sum_{n=0}^{\infty} C_3(6n+4)q^{12n+10} \equiv 2 \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} q^{(6k-1)^2+(6l+3)^2} \pmod 4.$$

Thus, $C_3(6n+4) \equiv 0 \pmod 4$ if $12n+10$ is not the sum of two squares. However, here we have $n = p^{2k+1}m + 5(p^{2k+2}-1)/6$ and, then,

$$12n+10 = 12p^{2k+1}m + 10p^{2k+2} = p^{2k+1}(12m+10p).$$

It follows that $\nu_p(12n+10)$ is odd, which implies that $12n+10$ is not the sum of two squares and the result follows. □

Our next theorem exhibits the complete characterization of $C_3(12n+1)$ modulo 4.

Theorem 28 *For all* $n \geq 0$*, we have*

$$C_3(12n+1) \equiv s\{2\pentagon + 2\pentagon\}(n) \pmod 4.$$

Proof We start by using (11) to extract the even part of (28), which yields

$$\sum_{n=0}^{\infty} C_3(12n+1)q^{2n} \equiv \frac{f_4^6 f_6^4}{f_2^8 f_{12}^2} \equiv f_2^2 \pmod 4.$$

By (7), we have

$$\sum_{n=0}^{\infty} C_3(12n+1)q^n \equiv \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{k(3k-1)+l(3l-1)} \pmod 4, \qquad (39)$$

from which the result follows. □

Corollary 29 *For all* $n \geq 0$, $C_3(24n+13) \equiv 0 \pmod 4$.

Proof Note that $k(3k-1)$ and $l(3l-1)$ are each, respectively, twice a pentagonal number. Thus, $n = k(3k-1) + l(3l-1)$ must be even. So, thanks to Theorem 28, we know that, for all $n \geq 0$,

$$C_3(12(2n+1)+1) = C_3(24n+13) \equiv 0 \pmod 4.$$

□

The above theorem also yields infinitely many Ramanujan–like congruences modulo 4.

Corollary 30 *Let* $p > 3$ *be a prime with* $p \equiv 3 \pmod 4$*. Then, for all* $k, m \geq 0$ *with* $p \nmid m$*,*

$$C_3(12p^{2k+1}m + 2p^{2k+2} - 1) \equiv 0 \pmod 4.$$

Proof From (39) we have

$$\sum_{n=0}^{\infty} C_3(12n+1)q^{12n+2} \equiv \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{(6k-1)^2+(6l-1)^2} \pmod 4.$$

Thus, $C_3(12n+1) \equiv 0 \pmod 4$ if $12n+2$ is not the sum of two squares. However we have $n = p^{2k+1}m + (p^{2k+2}-1)/6$, which implies that

$$12n+2 = 12p^{2k+1}m + 2p^{2k+2} = p^{2k+1}(12m+2p).$$

It follows that $\nu_p(12n+2)$ is odd and, then, $12n+2$ is not a sum of two squares and the result follows. □

Theorem 31 *For all $n \geq 0$, we have*

$$C_3(12n+9) \equiv 2\, r\{6\triangle + 2\pentagon\}(n) \pmod 4.$$

Proof It follows from (30) that

$$\sum_{n=0}^{\infty} C_3(6n+3)q^n \equiv \frac{f_3^6}{f_1^2} \equiv \frac{f_3^2 f_6^2}{f_1^2} \pmod 4.$$

Using (10) we can extract the odd part of both sides of this congruence:

$$\sum_{n=0}^{\infty} C_3(12n+9)q^n \equiv 2\frac{f_2 f_3^4 f_4 f_{12}}{f_1^4 f_6} \equiv 2 f_2 f_6^3 \pmod 4.$$

Finally, using (6) and (7), we have

$$\sum_{n=0}^{\infty} C_3(12n+9)q^n \equiv 2 \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} q^{k(3k-1)+3l(l+1)} \pmod 4, \tag{40}$$

from which the result follows. □

Corollary 32 *For all $n \geq 0$, $C_3(24n+21) \equiv 0 \pmod 4$.*

Proof Note that $n = k(3k-1) + 3l(l+1)$ must be even because $k(3k-1)$ is twice a pentagonal number while $3l(l+1)$ is six times a triangular number. Therefore, for all $n \geq 0$,

$$C_3(12(2n+1)+9) = C_3(24n+21) \equiv 0 \pmod 4$$

thanks to Theorem 31. □

Theorem 31 also yields an infinite family of congruences modulo 4. The proof of the next result will be omitted since it is analogous to the proof of Corollary 27.

Corollary 33 *Let $p > 3$ be a prime with $p \equiv 3 \pmod 4$. Then, for all $k, m \geq 0$ with $p \nmid m$,*

$$C_3(12p^{2k+1}m + 10p^{2k+2} - 1) \equiv 0 \pmod 4.$$

We now present the complete characterization of $C_3(18n+2)$ modulo 4.

Theorem 34 *For all $n \geq 0$, we have*

$$C_3(18n+2) \equiv 3\, s\{2\pentagon + 2\pentagon\}(n) \pmod 4.$$

Proof Initially we use (13) to extract from (19) the terms involving q^{3n}:

$$\sum_{n=0}^{\infty} C_3(9n+2)q^n = 3\frac{f_2 f_3^9}{f_1^3 f_6^3} - 6q\frac{f_6^6}{f_2^2} \equiv 3\frac{f_2 f_3 f_6}{f_1^3} - 2q\frac{f_6^6}{f_2^2} \pmod{4}.$$

Now using (11) we can extract the even part on both sides of the last congruence:

$$\sum_{n=0}^{\infty} C_3(18n+2)q^n = 3\frac{f_2^6 f_3^4}{f_1^8 f_6^2} \equiv 3f_2^2 \pmod{4}.$$

Thanks to (7) this yields

$$\sum_{n=0}^{\infty} C_3(18n+2)q^n = 3\sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{k(3k-1)+l(3l-1)} \pmod{4},$$

which completes the proof. □

Corollary 35 *For all* $n \geq 0$, $C_3(36n+20) \equiv 0 \pmod{4}$.

Proof Note that $n = k(3k-1) + l(3l-1)$ is even because each of $k(3k-1)$ and $l(3l-1)$ is twice a pentagonal number. Therefore, for all $n \geq 0$,

$$C_3(18(2n+1)+2) = C_3(36n+20) \equiv 0 \pmod{4}$$

thanks to Theorem 34. □

Theorem 34 also yields an infinite family of congruences modulo 4. The proof of the next corollary is analogous to the proof of Corollary 30 and will, therefore, be omitted.

Corollary 36 *Let* p *be a prime with* $p \equiv 7$ *or* $11 \pmod{12}$. *Then, for all* $k, m \geq 0$ *with* $p \nmid m$,

$$C_3(18p^{2k+1}m + 3p^{2k+2} - 1) \equiv 0 \pmod{4}.$$

The next theorem describes $C_3(12n+10)$ modulo 8 which also yields an infinite family of congruences modulo 8.

In the next theorem, $r\{3\triangle + 2\pentagon\}(n)$ is the number of representations of n as three times a triangular number and twice a pentagonal number. We know (see [18, Eq. (1.4)]) that

$$r\{3\triangle + 2\pentagon\}(n) = d_{1,8}(24n+11) - d_{7,8}(24n+11).$$

Theorem 37 *For all* $n \geq 0$, *we have*

$$C_3(12n+10) \equiv 4\,r\{6\triangle + 4\pentagon\}(n) \pmod{8}.$$

Proof We start using (10) to extract the odd parts on both sides of (31):

$$\sum_{n=0}^{\infty} C_3(12n+10)q^n = 12\frac{f_2 f_3^4 f_4 f_{12}}{f_1^2 f_6} \equiv 4 f_4 f_6^3 \pmod{8},$$

where we have used the elementary fact that $4f_j^2 \equiv 4f_{2j} \pmod{8}$. Using (6) and (7), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} C_3(12n+10)q^n &\equiv 4\sum_{k=-\infty}^{\infty}(-1)^k q^{2k(3k-1)}\sum_{l=0}^{\infty}(-1)^l(2l+1)q^{3l(l+1)}\\ &\equiv 4\sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} q^{2k(3k-1)+3l(l+1)} \pmod{8}.\end{aligned} \tag{41}$$

This completes the proof. □

Corollary 38 *For all* $n \geq 0$, $C_3(24n+22) \equiv 0 \pmod{8}$.

Proof Note that $n = 2k(3k-1) + 3l(l+1)$ is even because $2k(3k-1)$ is four times a pentagonal number and $3l(l+1)$ is six times a triangular number. Therefore, for all $n \geq 0$,

$$C_3(12(2n+1)+10) = C_3(24n+22) \equiv 0 \pmod{8}$$

thanks to Theorem 37. □

As noted above, this theorem yields an infinite family of congruences modulo 8.

Corollary 39 *Let* $p > 3$ *be a prime such that* $p \equiv 5$ *or* $7 \pmod{8}$. *Then for all* $k, m \geq 0$ *with* $p \nmid m$,

$$C_3\left(12p^{2k+1}m + 11p^{2k+2} - 1\right) \equiv 0 \pmod{8}.$$

Proof From (41) we have

$$\sum_{n=0}^{\infty} C_3(12n+10)q^{12n+11} \equiv 4\sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} q^{2(6k-1)^2+(6l+3)^2} \pmod{8}.$$

It follows that $C_3(12n+10) \equiv 0 \pmod{8}$ if $12n+11$ is not of the form $2x^2+y^2$. Since $p \equiv 5$ or $7 \pmod{8}$ we know that $\left(\frac{-2}{p}\right) = -1$, which implies that $\nu_p(N)$ is even if N is of the form $2x^2+y^2$. However, here $n = p^{2k+1}m + 11(p^{2k+2}-1)/12$. Then

$$12n + 11 = 12p^{2k+1}m + 11p^{2k+2} = p^{2k+1}(12m + 11p).$$

Therefore $\nu_p(12n+11)$ is odd and the result follows. □

We close this section with the following modulo 9 complete characterization of $C_3(6n+2)$ and, as a consequence, an infinite family of congruences.

Theorem 40 *For all $n \geq 0$, we have*

$$C_3(6n+2) \equiv 3\, s\{6\pentagon + 18\pentagon\}(n) \pmod{9}.$$

Proof By (29) and the elementary fact $3f_j^3 \equiv 3f_{3j} \pmod{9}$, we have

$$\sum_{n=0}^{\infty} C_3(6n+2)q^n \equiv 3f_3^4 \equiv 3f_3f_9 \pmod{9}.$$

Using (7), we obtain

$$\sum_{n=0}^{\infty} C_3(6n+2)q^n \equiv 3 \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{3k(3k-1)/2+9l(3l-1)/2} \pmod{9}. \tag{42}$$

This completes the proof. □

Corollary 41 *For all $n \geq 0$,*

$$C_3(18n+8) \equiv 0 \pmod{9}, \quad \text{and}$$
$$C_3(18n+14) \equiv 0 \pmod{9}.$$

Proof Note that $n = 3k(3k-1)/2 + 9l(3l-1)/2$ is clearly a multiple of 3. Therefore, for all $n \geq 0$,

$$C_3(6(3n+1)+2) = C_3(18n+8) \equiv 0 \pmod{9}$$

and

$$C_3(6(3n+2)+2) = C_3(18n+14) \equiv 0 \pmod{9}$$

thanks to Theorem 40. □

Of course, $C_3(18n+8) \equiv 0 \pmod{9}$ because $C_3(9n+8) = 9C_3(n)$. This theorem yields an infinite family of congruences modulo 9.

Corollary 42 *Let $p > 3$ be a prime such that $p \equiv 5$ or $11 \pmod{12}$. Then for all $k, m \geq 0$ with $p \nmid m$, we have*

$$C_3\left(6p^{2k+1}m + 3p^{2k+2} - 1\right) \equiv 0 \pmod{9}.$$

Proof From (42) we have

$$\sum_{n=0}^{\infty} C_3(6n+2)q^{8n+4} \equiv 3 \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{(6k-1)^2+3(6l-1)^2} \pmod{9}.$$

Thus $C_3(6n+2) \equiv 0 \pmod{9}$ if $8n+4$ is not of the form x^2+3y^2. The rest of the proof follows the same steps as in the proof of Corollary 8. □

5 Concluding remarks

While there are certainly more arithmetic properties satisfied by $C_3(n)$, we have successfully demonstrated a significantly extended list of straightforward results, beyond the arithmetic properties found by Gireesh [16], using elementary generating function manipulations and well–known q–series identities. The interested reader may wish to extend this study even further.

Acknowledgments. The first author was supported by São Paulo Research Foundation (FAPESP) (grant no. 2019/14796-8).

References

[1] S. Ahlgren, *The partition function modulo composite integers M*, Math. Ann. **318** (2000), 795–803.

[2] G.E. Andrews and F. Garvan, *Dyson's crank of a Partition*, Bull. Amer. Math. Soc. **10** (1988), 167–171.

[3] G.E. Andrews, M.D. Hirschhorn and J.A. Sellers, *Arithmetic properties of partitions with even parts distinct*, Ramanujan J. **23** (2010), 169–181.

[4] A.L.O. Atkin, *Proof of a conjecture of Ramanujan*, Glasg. Math. J. **8** (1967), 14–32.

[5] B.C. Berndt, *Ramanujan's notebooks, Part III*, Springer-Verlag (1991).

[6] B.C. Berndt, *Number Theory in the Spirit of Ramanujan*, Student Mathematical Library Vol. 34 American Mathematical Society (2006).

[7] H.-C. Chan, *Ramanujan's cubic continued fraction and an analogue of his 'Most beautiful identity'*, Int. J. Number Theory **6** (2010), 673–680.

[8] H.-C. Chan, *Ramanujan's cubic continued fraction and Ramanujan type congruences for a certain partition function*, Int. J. Number Theory **6** (2010), 819–834.

[9] S.-C. Chen, *Congruences for t-core partition functions*, J. Number Theory **133** (2013), 4036–4046.

[10] S. Chern, *New congruences for 2-color partitions*, J. Number Theory **163** (2016), 474–481.

[11] S. Chern and M.G. Dastidar, *Congruences and recursions for the cubic partition*, Ramanujan J. **44** (2017), 559–566.

[12] S.P. Cui and N.S.S. Gu, *Arithmetic properties of ℓ-regular partitions*, Adv. in Appl. Math. **51** (2013), 507–523.

[13] R. da Silva and J.A. Sellers, *Infinitely many congruences for k-regular partitions with designated summands*, Bull. Braz. Math. Soc. **51** (2020), 357–370.

[14] F.J. Dyson, *Guesses in the Theory of Partitions*, Eureka **8** (1944), 10–15.

[15] F. Garvan, D.B. Kim, and D. Stanton, *Cranks and t-cores*, Invent. Math., **101** (1990), 1–17.

[16] D.S. Gireesh, *Formulas for cubic partition with 3-cores*, J. Math. Anal. Appl. **453** (2017), 20–31.

[17] M.D. Hirschhorn, *The number of representations of a number by various forms*, Discrete Math. **298** (2005), 205–211.

[18] M.D. Hirschhorn, *The number of representations of a number by various forms involving triangles, squares, pentagons and octagons* in: Proc. of Ramanujan Rediscovered, Bangalore, India (2009), RMS Lecture Note Series, Vol. 14, pp. 113–124.

[19] M.D. Hirschhorn, *The power of q, a personal journey*, Developments in Mathematics, Vol. 49, New York: Springer (2017).

[20] M.S. Mahadeva Naika and D.S. Gireesh, *Congruences for 3-regular partitions with designated summands*, Integers **16** (2016), Article A25.

[21] K. Mahlburg, *Partition congruences and the Andrews-Garvan-Dyson crank*, Proc. Natl. Acad. Sci. **102** (2005), 15373–15376.

[22] K. Ono, *Distribution of the partition function modulo m*, Ann. Math. **151** (2000) 293–307

[23] Ramanujan S, *Collected Papers*, Cambridge University Press, London, 1927; reprinted: A. M. S. Chelsea, 2000 with new preface and extensive commentary by B. Berndt.

[24] P.C. Toh, *Ramanujan type identities and congruences for partition pairs*, Discrete Math. **312** (2012), 1244–1250.

[25] G.N. Watson, *Ramanujans Vermutung über Zerfällungsanzahlen*, J. Reine und Angew. Math. **179** (1938), 97–128.