CONGRUENCES FOR OVERPARTITIONS WITH RESTRICTED ODD DIFFERENCES

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Abstract. In a recent work, Bringmann, Dousse, Lovejoy, and Mahlburg defined the function $\overline{t}(n)$ to be the number of overpartitions of weight n where (i) the difference between two successive parts may be odd only if the larger part is overlined and (ii) if the smallest part is odd then it is overlined. In their work, they proved that $\overline{t}(n)$ satisfies an elegant congruence modulo 3, namely, for $n \geq 1$,

$$\overline{t}(n) \equiv \begin{cases} (-1)^{k+1} \pmod 3 & \text{if } n = k^2 \text{ for some integer } k, \\ 0 \pmod 3 & \text{otherwise.} \end{cases}$$

In this work, using elementary tools for manipulating generating functions, we prove that $\overline{t}$ satisfies a corresponding parity result. We prove that, for all $n \geq 1$,

$$\overline{t}(2n) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = (3k+1)^2 \text{ for some integer } k, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

We also provide a truly elementary proof of the mod 3 characterization of Bringmann, et.al., as well as a number of additional congruences satisfied by $\overline{t}(n)$ for various moduli.

2010 Mathematics Subject Classification: 05A17, 11P83

Keywords: congruences; overpartitions; restricted odd differences

1. Introduction

In a recent work, Bringmann, Dousse, Lovejoy, and Mahlburg [2] defined the function $\overline{t}(n)$ to be the number of overpartitions of weight n where (i) the difference between two successive parts may be odd only if the larger part is overlined and (ii) if the smallest part is odd then it is overlined. For example, $\overline{t}(4) = 8$ where the overpartitions in question are given by the following:

$$4, \overline{4}, 3+\overline{1}, \overline{3}+\overline{1}, 2+2, 2+\overline{2}, \overline{2}+1+\overline{1}, 1+1+1+\overline{1}$$

By considering certain q–difference equations, the authors prove that the generating function for $\overline{t}(n)$ is given by

$$\sum_{n\geq 0}\overline{t}(n)q^n = \frac{f_3}{f_1f_2}$$

where

$$f_k := (1-q^k)(1-q^{2k})(1-q^{3k})\dots$$

They also proved that $\overline{t}(n)$ satisfies an elegant congruence modulo 3.

Theorem 1.1. *For all $n \geq 1$,*

$$\overline{t}(n) \equiv \begin{cases} (-1)^{k+1} \pmod 3 & \text{if } n = k^2 \text{ for some integer } k, \\ 0 \pmod 3 & \text{otherwise.} \end{cases}$$

In this work, using elementary tools for manipulating generating functions, we prove that $\overline{t}$ satisfies a corresponding parity result.

Theorem 1.2. *For all $n \geq 1$,*

$$\overline{t}(2n) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = (3k+1)^2 \text{ for some integer } k, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

We also provide a truly elementary proof of the mod 3 characterization provided by Bringmann, et.al., as well as proofs of a number of additional congruences satisfied by $\overline{t}(n)$ for various moduli. We list these additional congruences here:

Theorem 1.3. *For all $n \geq 0$,*

$$\overline{t}(24n+4) \equiv 0 \pmod 4, \tag{1}$$

$$\overline{t}(32n+4) \equiv 0 \pmod 4, \tag{2}$$

$$\overline{t}(48n+36) \equiv 0 \pmod 4. \tag{3}$$

Theorem 1.4. *For all $n \geq 0$,*

$$\overline{t}(16n+14) \equiv 0 \pmod{12}, \tag{4}$$

$$\overline{t}(24n+22) \equiv 0 \pmod{12}, \tag{5}$$

$$\overline{t}(32n+28) \equiv 0 \pmod{12}, \tag{6}$$

$$\overline{t}(48n+24) \equiv 0 \pmod{12}, \tag{7}$$

$$\overline{t}(48n+40) \equiv 0 \pmod{12}, \tag{8}$$

$$\overline{t}(48n+42) \equiv 0 \pmod{12}. \tag{9}$$

Theorem 1.5. *For all $n \geq 0$,*

$$\overline{t}(8n+5) \equiv 0 \pmod 9.$$

2. Preliminary Tools

In the work below, we will utilize the following functions.

$$\begin{aligned}
\varphi(q) &:= \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2},\\
\psi(q) &:= \sum_{n=0}^{\infty} q^{n(n+1)/2} = \sum_{n=-\infty}^{\infty} q^{2n^2+n} = \frac{f_2^2}{f_1},\\
b(q) &:= \frac{f_1^3}{f_3},\\
\Pi(q) &:= \sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2} = \frac{f_2 f_3^2}{f_1 f_6},\\
\Omega(q) &:= \sum_{n=-\infty}^{\infty} q^{3n^2+2n} = \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6},\\
X(q) &:= \frac{f_1 f_6^3}{f_2 f_3^3}.
\end{aligned}$$

The functions $\varphi(q)$ and $\psi(q)$ are classical theta functions of Ramanujan, while $b(q)$ was introduced by Borwein, Borwein, and Garvan [1]; see [4, Chapter 22]. The functions $\Pi(q)$ and $\Omega(q)$ are featured in [4, Chapter 26]. The function $X(q)$ was introduced by Chan [3]; see [4, Section 14.3].

We will also make use of various lemmas. First, we note the following 2–dissections:

Lemma 2.1.

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}.$$

Proof. This follows directly from [4, (1.9.4)]. ∎

Lemma 2.2.

$$\frac{f_1}{f_3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}.$$

Proof. See [4, (30.10.1)]. ∎

Lemma 2.3.

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}.$$

Proof. See [4, (30.10.3)]. ∎

Lemma 2.4.

$$f_1f_3 = \frac{f_2f_8^2f_{12}^4}{f_4^2f_6f_{24}^2} - q\frac{f_4^4f_6f_{24}^2}{f_2f_8^2f_{12}^2}.$$

Proof. See [4, (30.12.1)]. ∎

Lemma 2.5.

$$b(q) = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2f_{12}^3}{f_4f_6^2}.$$

Proof. See [4, (22.1.13)]. ∎

Lemma 2.6.

$$\frac{1}{b(q)} = \frac{f_4^6f_6^3}{f_2^9f_{12}^2} + 3q\frac{f_4^2f_6f_{12}^2}{f_2^7}.$$

Proof.

$$\begin{aligned}\frac{1}{b(q)} &= \frac{b(-q)}{b(q)b(-q)}\\ &= \frac{f_4^3f_6^3}{f_2^9f_{12}}\left(\frac{f_4^3}{f_{12}} + 3q\frac{f_2^2f_{12}^3}{f_4f_6^2}\right) \quad \text{using Lemma 2.5}\\ &= \frac{f_4^6f_6^3}{f_2^9f_{12}^2} + 3q\frac{f_4^2f_6f_{12}^2}{f_2^7}.\end{aligned}$$

∎

Next, we call out three 3–dissections which we require.

Lemma 2.7.

$$\varphi(-q) = \frac{f_9^2}{f_{18}} - 2q\frac{f_3f_{18}^2}{f_6f_9}.$$

Proof. See [4, (14.3.3)]. ∎

Lemma 2.8.

$$\psi(q) = \Pi(q^3) + q\psi(q^9).$$

Proof. See [4, (14.3.3)]. ∎

Lemma 2.9.

$$f_1f_2 = f_9f_{18}\left(\frac{1}{X(q^3)} - q - 2q^2X(q^3)\right).$$

Proof. See [4, (14.3.1)]. ∎

We also make use of the following congruences.

Lemma 2.10.

$$f_1 \equiv \Pi(q) \pmod 2.$$

Proof. Euler's product [4, (1.6.1)] yields

$$\begin{aligned} f_1 &= \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2} \\ &\equiv \sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2} \pmod 2 \\ &= \Pi(q). \end{aligned}$$

∎

Lemma 2.11.

$$f_1^3 \equiv \psi(q) \pmod 4.$$

Proof. Jacobi's cube of Euler's product [4, (1.7.1)] yields

$$\begin{aligned} f_1^3 &= \sum_{n\geq 0} (-1)^n (2n+1) q^{(n^2+n)/2} \\ &= \sum_{n=-\infty}^{\infty} (4n+1) q^{2n^2+n} \\ &\equiv \sum_{n=-\infty}^{\infty} q^{2n^2+n} \pmod 4 \\ &= \psi(q). \end{aligned}$$

∎

Lemma 2.12.

$$\Omega(q) \equiv \frac{f_3^3}{f_1} \pmod 2.$$

Proof.

$$\begin{aligned} \Omega(q) &= \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6} \\ &\equiv \frac{f_3^3}{f_1} \pmod 2. \end{aligned}$$

∎

With the above tools in hand, we are prepared to prove all of the theorems mentioned above in elementary fashion.

3. Proofs of Theorems 1.1–1.5

We begin this section by providing a truly elementary proof of Theorem 1.1 which was originally proven in [2].

Proof. (of Theorem 1.1) We have

$$\begin{aligned}
\sum_{n\geq 0}\overline{t}(n)q^n &= \frac{f_3}{f_1f_2}\\
&\equiv \frac{f_1^3}{f_1f_2} \pmod 3\\
&= \frac{f_1^2}{f_2}\\
&= \varphi(-q)\\
&= \sum_{k=-\infty}^{\infty}(-1)^kq^{k^2}\\
&= 1+2\sum_{k\geq 1}(-1)^kq^{k^2}\\
&\equiv 1+\sum_{k\geq 1}(-1)^{k+1}q^{k^2} \pmod 3.
\end{aligned}$$

The result follows. ∎

Remark 3.1. *Theorem 1.1 provides an immediate proof of the modulo 3 "portion" of the congruences listed in Theorem 1.4. One simply needs to show that there are no squares in the arithmetic progressions in question; this requires a simple set of straightforward calculations. Hence, in what follows, we only focus on the modulo 4 portion of those congruences listed in Theorem 1.4.*

We next turn to an elementary proof of Theorem 1.2 in the spirit of the proof of Theorem 1.1 just provided.

Proof. (of Theorem 1.2) We have

$$\begin{aligned}
\sum_{n\geq 0}\overline{t}(n)q^n &= \frac{f_3}{f_1f_2}\\
&= \frac{1}{f_2}\frac{f_3}{f_1}\\
&= \frac{1}{f_2}\left(\frac{f_4f_6f_{16}f_{24}^2}{f_2^2f_8f_{12}f_{48}}+q\frac{f_6f_8^2f_{48}}{f_2^2f_{16}f_{24}}\right)
\end{aligned}$$

using Lemma 2.3. Therefore, modulo 2,

$$\begin{aligned}
\sum_{n\geq 0}\bar{t}(2n)q^n &= \frac{f_2f_3f_8f_{12}^2}{f_1^3f_4f_6f_{24}} \\
&\equiv \frac{f_1^3}{f_3} \\
&\equiv \frac{\psi(q)}{\Pi(q^3)} \text{ using Lemmas 2.10 and 2.11} \\
&= \frac{\Pi(q^3)+q\psi(q^9)}{\Pi(q^3)} \text{ using Lemma 2.8} \\
&= 1+q\frac{\psi(q^9)}{\Pi(q^3)} \\
&\equiv 1+q\frac{f_9^3}{f_3} \text{ using Lemmas 2.10 and 2.11} \\
&\equiv 1+q\Omega(q^3) \text{ using Lemma 2.12} \\
&= 1+q\sum_{n=-\infty}^{\infty} q^{3(3n^2+2n)} \\
&= 1+\sum_{n=-\infty}^{\infty} q^{(3n+1)^2}.
\end{aligned}$$

∎

We note, in passing, that the work above implies that

$$\sum_{n\geq 0}\bar{t}(2n+1)q^n = \frac{f_3f_4^2f_{24}}{f_1^3f_8f_{12}}. \tag{10}$$

We will use this fact later in our proof of Theorem 1.5.

We now turn to the proofs of Theorems 1.3–1.4 which require a number of generating function dissections.

Proof. (of Theorems 1.3–1.4) We continue to compute a variety of generating function dissections.

$$\begin{aligned}
\sum_{n\geq 0}\bar{t}(2n)q^n &= \frac{f_2f_3f_8f_{12}^2}{f_1^3f_4f_6f_{24}} \\
&= \frac{f_2f_8f_{12}^2}{f_4f_6f_{24}}\frac{f_3}{f_1^3} \\
&= \frac{f_2f_8f_{12}^2}{f_4f_6f_{24}}\frac{1}{b(q)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{f_2 f_8 f_{12}^2}{f_4 f_6 f_{24}} \left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7} \right) \text{ using Lemma 2.6} \\
&= \frac{f_4^5 f_6^2 f_8}{f_2^8 f_{24}} + 3q \frac{f_4 f_8 f_{12}^4}{f_2^6 f_{24}}.
\end{aligned}
$$

So

$$
\begin{aligned}
&\sum_{n\geq 0} \bar{t}(4n+2)q^n \\
&= 3\frac{f_2 f_4 f_6^4}{f_1^6 f_{12}} \\
&= 3\frac{f_2 f_4 f_6^4}{f_{12}} \left(\frac{1}{f_1^2} \right)^3 \\
&= 3\frac{f_4 f_6^4}{f_2^{14} f_{12}} \left(\frac{f_8^5}{f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_8} \right)^3 \text{ using Lemma 2.1} \\
&= 3\frac{f_4 f_6^4}{f_2^{14} f_{12}} \left(\frac{f_8^{15}}{f_{16}^6} + 6q \frac{f_4^2 f_8^9}{f_{16}^2} + 12q^2 f_4^4 f_8^3 f_{16}^2 + 8q^3 \frac{f_4^6 f_{16}^6}{f_8^3} \right) \\
&= 3\left(\frac{f_4 f_6^4 f_8^{15}}{f_2^{14} f_{12} f_{16}^6} + 6q \frac{f_4^3 f_6^4 f_8^9}{f_2^{14} f_{12} f_{16}^2} + 12q^2 \frac{f_4^5 f_6^4 f_8^3 f_{16}^2}{f_2^{14} f_{12}} + 8q^3 \frac{f_4^7 f_6^4 f_{16}^6}{f_2^{14} f_8^3 f_{12}} \right).
\end{aligned}
$$

Therefore,

$$
\sum_{n\geq 0} \bar{t}(8n+2)q^n = 3\frac{f_2 f_3^4 f_4^{15}}{f_1^{14} f_6 f_8^6} + 36q \frac{f_2^5 f_3^4 f_4^3 f_8^2}{f_1^{14} f_6} \tag{11}
$$

and

$$
\sum_{n\geq 0} \bar{t}(8n+6)q^n = 18\frac{f_2^3 f_3^4 f_4^9}{f_1^{14} f_6 f_8^2} + 24q \frac{f_2^7 f_3^4 f_8^6}{f_1^{14} f_4^3 f_6}.
$$

It follows that, modulo 12,

$$
\sum_{n\geq 0} \bar{t}(8n+6)q^n \equiv 6\frac{f_2^3 f_3^4 f_4^9}{f_1^{14} f_6 f_8^2}.
$$

Now,

$$
\frac{f_2^3 f_3^4 f_4^9}{f_1^{14} f_6 f_8^2} \equiv f_2^6 f_6 \pmod{2}.
$$

This implies

$$
\bar{t}(16n+14) \equiv 0 \pmod{12}.
$$

See Theorem 1.4, (4).

Note also that, modulo 4,

$$\sum_{n\geq 0}\overline{t}(8n+6)q^n \equiv 2\frac{f_2^3f_3^4f_4^9}{f_1^{14}f_6f_8^2} = 2\frac{f_3^4}{f_6}\left(\frac{f_2^3f_4^9}{f_1^{14}f_8^2}\right).$$

Now, modulo 2,

$$\begin{aligned}\frac{f_2^3f_4^9}{f_1^{14}f_8^2} &\equiv f_1^{12} \equiv f_2^6 \equiv \psi(q^2)^2\\ &= \left(\frac{f_{12}f_{18}^2}{f_6f_{36}} + q^2\frac{f_{36}^2}{f_{18}}\right)^2 \text{ using Lemma 2.8}\\ &= \frac{f_{12}^2f_{18}^4}{f_6^2f_{36}^2} + 2q^2\frac{f_{12}f_{18}f_{36}}{f_6} + q^4\frac{f_{36}^4}{f_{18}^2}.\end{aligned}$$

It follows that, modulo 4,

$$\sum_{n\geq 0}\overline{t}(8n+6)q^n \equiv 2\frac{f_3^4}{f_6}\left(\frac{f_{12}^2f_{18}^4}{f_6^2f_{36}^2} + 2q^2\frac{f_{12}f_{18}f_{36}}{f_6} + q^4\frac{f_{36}^4}{f_{18}^2}\right).$$

If we extract the terms of the form q^{3n+2}, we find that

$$\sum_{n\geq 0}\overline{t}(24n+22)q^n \equiv 0 \pmod 4.$$

See Theorem 1.4, (5).

For the remainder of this proof, unless explicitly stated otherwise, all congruences are computed modulo 4.

Next, we see that

$$\begin{aligned}\sum_{n\geq 0}\overline{t}(4n)q^n &\equiv \frac{f_2^5f_3^2f_4}{f_2^4f_{12}}\\ &= \frac{f_2f_3^2f_4}{f_{12}}\\ &= \frac{f_2f_4}{f_{12}}f_3^2\\ &= \frac{f_2f_4f_6}{f_{12}}\frac{f_3^2}{f_6}\\ &= \frac{f_2f_4f_6}{f_{12}}\varphi(-q^3)\\ &= \frac{f_2f_4f_6}{f_{12}}\left(\varphi(q^{12}) - 2q^3\psi(q^{24})\right) \text{ using [4, (1.9.4)]}\\ &\equiv \frac{f_2f_4f_6}{f_{12}}\left(\varphi(-q^{12}) - 2q^3\psi(q^{24})\right)\end{aligned}$$

$$\begin{aligned} &= \frac{f_2f_4f_6}{f_{12}}\left(\frac{f_{12}^2}{f_{24}} - 2q^3\frac{f_{48}^2}{f_{24}}\right) \\ &= \frac{f_2f_4f_6f_{12}}{f_{24}} - 2q^3\frac{f_2f_4f_6f_{48}^2}{f_{12}f_{24}}. \end{aligned}$$

From the above 2–dissection, we know

$$\sum_{n\geq 0}\overline{t}(8n)q^n \equiv \frac{f_1f_2f_3f_6}{f_{12}}$$

and

$$\sum_{n\geq 0}\overline{t}(8n+4)q^n \equiv 2q\frac{f_1f_2f_3f_{24}^2}{f_6f_{12}}.$$

Note that

$$\begin{aligned} \sum_{n\geq 0}\overline{t}(8n)q^n &\equiv \frac{f_2f_6}{f_{12}}f_1f_3 \\ &= \frac{f_2f_6}{f_{12}}\left(\frac{f_2f_8^2f_{12}^4}{f_4^2f_6f_{24}^2} - q\frac{f_4^4f_6f_{24}^2}{f_2f_8^2f_{12}^2}\right) \text{ using Lemma 2.4.} \end{aligned}$$

So

$$\sum_{n\geq 0}\overline{t}(8n)q^n \equiv \frac{f_2^2f_8^2f_{12}^3}{f_4^2f_{24}^2} - q\frac{f_4^4f_6^2f_{24}^2}{f_8^2f_{12}^3}.$$

This means

$$\sum_{n\geq 0}\overline{t}(16n)q^n \equiv \frac{f_1^2f_4^2f_6^3}{f_2^2f_{12}^2}.$$

and

$$\sum_{n\geq 0}\overline{t}(16n+8)q^n \equiv 3\frac{f_2^4f_3^2f_{12}^2}{f_4^2f_6^3}.$$

This implies

$$\begin{aligned} \sum_{n\geq 0}\overline{t}(16n)q^n &\equiv \frac{f_1^2f_2^2f_4^2f_6^3}{f_2^4f_{12}^2} \\ &\equiv \frac{f_1^2f_2^2f_6^3}{f_{12}^2} \\ &\equiv \frac{f_1^2f_2^2f_6^4}{f_6f_{12}^2} \end{aligned}$$

$$\equiv \frac{f_1^2 f_2^2}{f_6}.$$

Also,

$$\begin{aligned}\sum_{n\geq 0} \bar{t}(16n+8)q^n &\equiv 3\frac{f_2^4 f_3^2 f_{12}^2}{f_4^2 f_6^3} \\ &\equiv 3\frac{f_3^2 f_6 f_{12}^2}{f_6^4} \\ &\equiv 3f_3^2 f_6.\end{aligned}$$

This yields

$$\sum_{n\geq 0} \bar{t}(48n+24)q^n \equiv 0$$

and

$$\sum_{n\geq 0} \bar{t}(48n+40)q^n \equiv 0.$$

See Theorem 1.4, (7) and (8).

In a different vein,

$$\begin{aligned}&\sum_{n\geq 0} \bar{t}(8n+4)q^n \\ \equiv\ & 2q\frac{f_1 f_2 f_3 f_{24}^2}{f_6 f_{12}} \\ \equiv\ & 2q\frac{f_1 f_2 f_3 f_{24}^2}{f_3^2 f_{12}} \\ \equiv\ & 2q\frac{f_1 f_2 f_{48}}{f_3 f_{12}} \\ \equiv\ & 2q\frac{f_2 f_{48}}{f_{12}}\frac{f_1}{f_3} \\ \equiv\ & 2q\frac{f_2 f_{48}}{f_{12}}\left(\frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}\right) \text{ using Lemma 2.2} \\ =\ & 2q\frac{f_2^2 f_{16} f_{24}^2}{f_6^2 f_8 f_{12}} + 2q^2\frac{f_2^2 f_8^2 f_{48}^2}{f_4 f_6^2 f_{16} f_{24}}.\end{aligned}$$

Hence,

$$\sum_{n\geq 0} \bar{t}(16n+4)q^n \equiv 2q\frac{f_1^2 f_4^2 f_{24}^2}{f_2 f_3^2 f_8 f_{12}}$$

$$\begin{aligned} &\equiv 2q\frac{f_{24}^2}{f_3^2f_{12}} \\ &\equiv 2q\frac{f_{12}^3}{f_6} \\ &\equiv 2qf_6^5. \end{aligned}$$

Therefore,

$$\sum_{n\geq 0}\bar{t}(32n+4)q^n \equiv 0$$

and

$$\sum_{n\geq 0}\bar{t}(48n+36)q^n \equiv 0.$$

See Theorem 1.3, (2) and (3).

Also from above, we know

$$\sum_{n\geq 0}\bar{t}(16n+12)q^n \equiv 2\frac{f_1^2f_8f_{12}^2}{f_3^2f_4f_6}.$$

Moreover,

$$\begin{aligned} \frac{f_1^2f_8f_{12}^2}{f_3^2f_4f_6} &\equiv \frac{f_2f_4^2f_6^4}{f_4f_6^3} \pmod 2 \\ &\equiv f_2f_4f_{12} \pmod 2 \end{aligned}$$

which is an even function of q. Therefore, for all $n\geq 0$,

$$\bar{t}(32n+28)\equiv 0.$$

See Theorem 1.4, (6).

From a different perspective,

$$\begin{aligned} \sum_{n\geq 0}\bar{t}(8n+4)q^n &\equiv 2q\frac{f_3f_{24}^2}{f_6f_{12}}f_1f_2 \\ &\equiv 2q\frac{f_3f_{24}^2}{f_6f_{12}}\left(f_9f_{18}\left(\frac{1}{X(q^3)}-q-2q^2X(q^3)\right)\right) \end{aligned}$$

using Lemma 2.9. Thus,

$$\sum_{n\geq 0}\bar{t}(24n+4)q^n \equiv 0.$$

See Theorem 1.3, (1).

Thanks to (11),

$$\begin{aligned}\sum_{n\geq 0}\overline{t}(8n+2)q^n &\equiv 3\frac{f_2f_3^4f_4^{15}}{f_1^{14}f_6f_8^6}\\ &\equiv 3\frac{f_1^2f_2f_6}{f_4}\\ &\equiv 3\frac{f_2f_6}{f_4}f_1^2\\ &\equiv 3\frac{f_2^2f_6}{f_4}\varphi(-q)\\ &\equiv 3\frac{f_2^2f_6}{f_4}\left(\varphi(q^4)-2q\psi(q^8)\right)\ \text{using [4, (1.9.4)].}\end{aligned}$$

So

$$\begin{aligned}\sum_{n\geq 0}\overline{t}(16n+10)q^n &\equiv 2\frac{f_1^2f_3f_8^2}{f_2f_4}\\ &\equiv 2f_3\frac{f_1^2f_8^2}{f_2f_4}\\ &\equiv 2f_3\varphi(-q)\psi(q^4)\\ &\equiv 2f_3\left(\frac{f_9^2}{f_{18}}-2q\frac{f_3f_{18}^2}{f_6f_9}\right)\left(\frac{f_{24}f_{36}^2}{f_{12}f_{72}}+q^4\frac{f_{72}^2}{f_{36}}\right)\end{aligned}$$

using Lemmas 2.7 and 2.8. This yields

$$\sum_{n\geq 0}\overline{t}(48n+42)q^n \equiv 0.$$

See Theorem 1.4, (9).

Thanks to the above work on congruences modulo 4, as well as Remark 3.1 which provides us with the necessary congruences modulo 3, we see that the proofs of Theorems 1.3 and 1.4 are now complete. ∎

We now provide a proof of Theorem 1.5 by appropriately dissecting the generating function for $\overline{t}(2n+1)$ which was obtained earlier.

Proof. (of Theorem 1.5) Thanks to (10) above,

$$\begin{aligned}\sum_{n\geq 0}\overline{t}(2n+1)q^n &= \frac{f_3f_4^2f_{24}}{f_1^3f_8f_{12}}\\ &= \frac{f_4^2f_{24}}{f_8f_{12}}\frac{1}{b(q)}\\ &= \frac{f_4^2f_{24}}{f_8f_{12}}\left(\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}\right) \quad \text{using Lemma 2.6}\\ &= \frac{f_4^8f_6^3f_{24}}{f_2^9f_8f_{12}^3}+3q\frac{f_4^4f_6f_{12}f_{24}}{f_2^7f_8}.\end{aligned}$$

So

$$\begin{aligned}\sum_{n\geq 0}\overline{t}(4n+1)q^n &= \frac{f_2^8f_3^3f_{12}}{f_1^9f_4f_6^3}\\ &= \frac{f_2^8f_{12}}{f_4f_6^3}\left(\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}\right)^3 \quad \text{using Lemma 2.6.}\end{aligned}$$

Thus,

$$\sum_{n\geq 0}\overline{t}(8n+5)q^n = 9\frac{f_2^{13}f_3^4}{f_1^{17}f_6}+27q\frac{f_2^5f_6^7}{f_1^{13}}.$$

This implies that, for all $n\geq 0$,

$$\overline{t}(8n+5)\equiv 0 \pmod 9.$$

∎

4. Closing Thoughts

Computational evidence indicates that additional congruences are satisfied by $\overline{t}$ for various moduli. Proofs of such congruences are left to the interested reader.

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