INFINITELY MANY CONGRUENCES MODULO 5 FOR 4–COLORED FROBENIUS PARTITIONS

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Abstract. In his 1984 AMS Memoir, Andrews introduced the family of functions $c\phi_k(n)$, which denotes the number of generalized Frobenius partitions of n into k colors. Recently, Baruah and Sarmah, Lin, Sellers, and Xia established several Ramanujan–like congruences for $c\phi_4(n)$ relative to different moduli. In this paper, employing classical results in q–series, the well–known theta functions of Ramanujan, and elementary generating function manipulations, we prove a characterization of $c\phi_4(10n+1)$ modulo 5 which leads to an infinite set of Ramanujan–like congruences modulo 5 satisfied by $c\phi_4$. This work greatly extends the recent work of Xia on $c\phi_4$ modulo 5.

2010 Mathematics Subject Classification: 05A17, 11P83

Keywords: congruences; generalized Frobenius partitions; theta functions; generating functions

1. Introduction

In his 1984 AMS Memoir, George Andrews [1] defined the family of k–colored generalized Frobenius partition functions which are denoted by $c\phi_k(n)$ where $k \geq 1$ is the number of colors in question. Among many things, Andrews [1, Corollary 10.1] proved that, for all $n \geq 0$, $c\phi_2(5n+3) \equiv 0 \pmod 5$. Over the years, many authors proved similar congruence properties for various k–colored generalized Frobenius partition functions, typically for a small number of colors k. See, for example, [4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16].

In 2011, Baruah and Sarmah [2] proved a number of congruence properties for $c\phi_4$, all with moduli which are powers of 4. In [14], Sellers proved a new congruence result modulo 5 for $c\phi_4$:

Theorem 1.1. *For all* $n \geq 0$, $c\phi_4(10n+6) \equiv 0 \pmod 5$.

Quite recently, Xia [15] proved an additional congruence result for $c\phi_4$ modulo 5:

Theorem 1.2. *For all* $n \geq 0$, $c\phi_4(20n+11) \equiv 0 \pmod 5$.

In this note, we significantly extend the study of congruences satisfied by $c\phi_4$ mod 5. By employing classical results in q–series, the well–known theta functions of Ramanujan, and elementary generating function manipulations, we prove a characterization of $c\phi_4(10n+1)$ modulo 5 which leads to an infinite set of Ramanujan–like congruences modulo 5 satisfied by $c\phi_4$. In particular, we shall prove the following:

Theorem 1.3. *For all* $n \geq 0$,

$$c\phi_4(10n+1) \equiv \begin{cases} k+1 \pmod 5 & \text{if } n = k(3k+1) \text{ for some integer } k, \\ 0 \pmod 5 & \text{otherwise.} \end{cases}$$

With Theorem 1.3, we can easily prove Theorem 1.2 as well as the following corollary (which provides infinitely many Ramanujan–like congruences satisfied by $c\phi_4$ modulo 5).

Corollary 1.4. *Let $p \geq 5$ be prime and let r be an integer, $1 \leq r \leq p-1$, such that $12r+1$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$,*

$$c\phi_4(10pn + 10r + 1) \equiv 0 \pmod 5.$$

2. Some Necessary Tools

In this section, we collect a number of definitions and lemmas which are needed to prove the main results of this paper.

First, recall Ramanujan's theta functions

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

Using Jacobi's Triple Product Identity [3, Entry 19], we have the following well–known product representations for $\varphi(q)$ and $\psi(q)$:

$$\varphi(q) = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2} \tag{1}$$

and

$$\psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \tag{2}$$

where $(a;b)_\infty := (1-a)(1-ab)(1-ab^2)(1-ab^3)\dots$

Next, we note an important q–series result of Ramanujan which is easily proven using the Quintuple Product Identity. Such a proof can be found in Berndt [3, Corollary 1.3.21].

Lemma 2.1.

$$\sum_{n=-\infty}^{\infty} (6n+1) q^{3n^2+n} = \frac{(q^2;q^2)_\infty^5}{(q^4;q^4)_\infty^2}$$

Lastly, we require one pivotal lemma.

Lemma 2.2. *Let $F(q) = \psi(q)(q^4;q^4)_\infty^3$ where $\psi(q)$ is one of Ramanujan's theta functions defined above. Let*

$$F(q) = F_0(q) + F_1(q) + F_2(q) + F_3(q) + F_4(q)$$

be the 5–dissection of $F(q)$ where $F_i(q)$ contains all the terms in $F(q)$ in which the power of q is $i \pmod 5$ (whenever $F(q)$ is written as a power series in q). Then

$$F_0(q) = (q^5;q^5)_\infty^3 \varphi(-q^{10}) - 5q^{15}(q^{100};q^{100})_\infty^3 \psi(q^{25}).$$

Proof. Note that

$$\begin{aligned} F(q) &= \psi(q)(q^4;q^4)_\infty^3 \\ &= \sum_{m=-\infty}^{\infty} q^{2m^2+m} \sum_{n=-\infty}^{\infty} (4n+1) q^{8n^2+4n}. \end{aligned}$$

Then, in order to complete the square, we have

$$q^5F(q^8) = \sum_{m,n=-\infty}^{\infty} (4n+1)q^{(4m+1)^2+4(4n+1)^2} = \sum_{x,y=-\infty}^{\infty} yq^{x^2+4y^2},$$

where $x,\ y \equiv 1 \pmod 4$.

Using the notation provided in the statement of the lemma, we then see that

$$q^5F_0(q^8) = \sum_{x,y=-\infty}^{\infty} yq^{x^2+4y^2}$$

where $x,\ y \equiv 1 \pmod 4$ **and** $x^2+4y^2 \equiv 0 \pmod 5$.

Now the solution of $x^2+4y^2 \equiv 0 \pmod 5$ is $x \equiv \pm y \pmod 5$. So

$$\begin{aligned} q^5F_0(q^8) \quad = \quad & \sum_{x\equiv y \pmod 5} yq^{x^2+4y^2} + \sum_{x\equiv -y \pmod 5} yq^{x^2+4y^2} \\ & - \sum_{x\equiv y\equiv 0 \pmod 5} yq^{x^2+4y^2}, \end{aligned}$$

where it must also be the case that $x,\ y \equiv 1 \pmod 4$.

In the first sum, we have $x = 4m+1$, $y = 4n+1$ and $x \equiv y \pmod 5$. So $4m+1 \equiv 4n+1 \pmod 5$, $4(m-n) \equiv 0 \pmod 5$, $m-n \equiv 0 \pmod 5$, $m+4n = 5w$, $m = w+4v$, $n = w-v$ and

$$x = 4w+16v+1, \quad y = 4w-4v+1.$$

In the second sum, $x = 4m+1$, $y = 4n+1$ and $x+y \equiv 0 \pmod 5$. So $4(m+n)+2 \equiv 0 \pmod 5$, $4(m+n) \equiv -2 \pmod 5$, $m+n \equiv 2 \pmod 5$, $m+n = 5u+2$, $m-4n = 5v+2$, $m = 4u+v+2$, $n = u-v$ and

$$x = 16u+4v+9, \quad y = 4u-4v+1.$$

In the third sum, $x = 20u+5, \quad y = 20v+5$.

It follows that

$$\begin{aligned} q^5F_0(q^8) \quad = \quad & \sum_{u,v=-\infty}^{\infty} (4u-4v+1)q^{(4u+16v+1)^2+4(4u-4v+1)^2} \\ & + \sum_{u,v=-\infty}^{\infty} (4u-4v+1)q^{(16u+4v+9)^2+4(4u-4v+1)^2} \\ & - \sum_{u,v=-\infty}^{\infty} (20v+5)q^{(20u+5)^2+4(20v+5)^2} \\ = \quad & \sum_{u,v=-\infty}^{\infty} (4u-4v+1)q^{80u^2+320v^2+40u+5} \\ & + \sum_{u,v=-\infty}^{\infty} (4u-4v+1)q^{320u^2+80v^2+320u+40v+85} \\ & - \sum_{u,v=-\infty}^{\infty} (20v+5)q^{400u^2+1600v^2+200u+800v+125} \end{aligned}$$

Therefore, we know that

$$\begin{aligned}
F_0(q) &= \sum_{-\infty}^{\infty}(4u+1)q^{10u^2+5u}\sum_{-\infty}^{\infty}q^{40v^2} \\
&\quad -\sum_{-\infty}^{\infty}4vq^{4v^2}\sum_{-\infty}^{\infty}q^{10u^2+5u} \\
&\quad -q^{10}\sum_{-\infty}^{\infty}(4v+1)q^{10v^2+5v}\sum_{-\infty}^{\infty}q^{40u^2+40u} \\
&\quad +q^{10}\sum_{-\infty}^{\infty}(4u+2)q^{40u^2+40u}\sum_{-\infty}^{\infty}q^{10v^2+5v} \\
&\quad -5q^{15}\sum_{-\infty}^{\infty}(4v+1)q^{200v^2+100v}\sum_{-\infty}^{\infty}q^{50u^2+25u} \\
&= (q^5;q^5)_\infty^3\varphi(q^{40}) - 2q^{10}(q^5;q^5)_\infty^3\psi(q^{80}) - 5q^{15}(q^{100};q^{100})_\infty^3\psi(q^{25}) \\
&= (q^5;q^5)_\infty^3\Big(\varphi(q^{40}) - 2q^{10}\psi(q^{80})\Big) - 5q^{15}(q^{100};q^{100})_\infty^3\psi(q^{25}) \\
&= (q^5;q^5)_\infty^3\varphi(-q^{10}) - 5q^{15}(q^{100};q^{100})_\infty^3\psi(q^{25}).
\end{aligned}$$

∎

3. Proofs of the Main Results

With the above tools in hand, we now provide an elementary proof of Theorem 1.3.

Proof. (Of Theorem 1.3) Lin [9, Equation (1.3)] notes that

$$\sum_{n=0}^{\infty} c\phi_4(2n+1)q^n = 16\frac{(q^2;q^2)_\infty^{17}}{(q;q)_\infty^{16}(q^4;q^4)_\infty^2}.$$

Via elementary generating function manipulations, we have the following:

$$\begin{aligned}
\sum_{n=0}^{\infty} c\phi_4(2n+1)q^n &= 16\frac{(q^2;q^2)_\infty^{17}}{(q;q)_\infty^{16}(q^4;q^4)_\infty^2} \\
&= 16\frac{(q^2;q^2)_\infty^{15}(q^2;q^2)_\infty^2}{(q;q)_\infty^{15}(q;q)_\infty(q^4;q^4)_\infty^2} \\
&= 16\frac{(q^2;q^2)_\infty^{15}\psi(q)}{(q;q)_\infty^{15}(q^4;q^4)_\infty^2} \\
&\equiv \frac{(q^{10};q^{10})_\infty^3\psi(q)}{(q^5;q^5)_\infty^3(q^4;q^4)_\infty^2} \pmod 5 \\
&\equiv \frac{(q^{10};q^{10})_\infty^3\psi(q)(q^4;q^4)_\infty^3}{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty} \pmod 5.
\end{aligned}$$

From here, we now wish to find a representation of the generating function for $c\phi_4(10n+1)$. Thanks to Lemma 2.2, we see that, modulo 5, such a generating function is given by

$$\sum_{n=0}^{\infty} c\phi_4(10n+1)q^{5n} \equiv \frac{(q^{10};q^{10})_\infty^3 (q^5;q^5)_\infty^3 \varphi(-q^{10})}{(q^5;q^5)_\infty^3 (q^{20};q^{20})_\infty} \pmod{5}$$

or

$$\sum_{n=0}^{\infty} c\phi_4(10n+1)q^{n} \equiv \frac{(q^{2};q^{2})_\infty^3 \varphi(-q^{2})}{(q^4;q^4)_\infty} \pmod{5}. \tag{3}$$

Thanks to (1), we see that (3) implies

$$\begin{aligned}\frac{(q^{2};q^{2})_\infty^3 \varphi(-q^{2})}{(q^4;q^4)_\infty} &= \frac{(q^{2};q^{2})_\infty^3}{(q^4;q^4)_\infty} \times \frac{(q^{2};q^{2})_\infty^2}{(q^4;q^4)_\infty} \\ &= \frac{(q^{2};q^{2})_\infty^5}{(q^4;q^4)_\infty^2}\end{aligned}$$

The result follows thanks to Lemma 2.1. ∎

With Theorem 1.3 in hand, we can now quickly prove Theorem 1.2 and Corollary 1.4.

Proof. (of Theorem 1.2) This result holds because $k(3k+1)$ is twice a pentagonal number and, therefore, even for every integer k. This means that $2n+1$ can never be represented as $k(3k+1)$ for any integer k. This means that, for all $n \geq 0$,

$$c\phi_4(10(2n+1)+1) \equiv 0 \pmod{5}.$$

∎

We remark that this proof of Theorem 1.2 is significantly shorter and more elementary than the proof given by Xia [15].

Proof. (of Corollary 1.4) Let p and r be chosen as in the statement of the corollary. Then, by Theorem 1.3, we must ask whether there exists an integer k such that

$$pn + r = k(3k+1).$$

By completing the square, this is equivalent to asking whether there is an integer k such that

$$12(pn+r)+1 = (6k+1)^2.$$

This would imply that $12r+1 \equiv (6k+1)^2 \pmod{p}$. However, r has been explicitly chosen so that $12r+1$ is a quadratic nonresidue modulo p. Hence, $12r+1$ cannot be congruent to a square modulo p. This implies that $pn+r$ can never be represented as $k(3k+1)$ for some integer, and the corollary then follows thanks to Theorem 1.3. ∎

Clearly, for each prime $p \geq 5$, Corollary 1.4 provides $(p-1)/2$ different congruences modulo 5 satisfied by $c\phi_4$. Hence, we now have infinitely many nontrivial Ramanujan–like congruences modulo 5 for 4–colored generalized Frobenius partitions.

4. Closing Thoughts

As we close, it is worth noting that Theorem 1.2 and Corollary 1.4 imply infinitely many congruences modulo 5 for the function $\phi_4(n)$ (which is the number of generalized Frobenius partitions of n which allow up to 4 repetitions of an integer in either row). See Andrews [1] for more details.

Corollary 4.1. *For all $n \geq 0$,*

$$\phi_4(20n + 11) \equiv 0 \pmod 5.$$

Corollary 4.2. *Let $p \geq 5$ be prime and let r be an integer, $1 \leq r \leq p - 1$, such that $12r + 1$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$,*

$$\phi_4(10pn + 10r + 1) \equiv 0 \pmod 5.$$

Proof. Both corollaries follow from a result of Garvan [5] which states that, for any prime p,

$$\phi_{p-1}(n) \equiv c\phi_{p-1}(n) \pmod p$$

for any integer $n \geq 0$. ∎

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