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# PARITY RESULTS FOR PARTITIONS WHEREIN EACH PART APPEARS AN ODD NUMBER OF TIMES

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**Abstract**

In this brief note, we consider the function $f(n)$ which enumerates partitions of weight $n$ wherein each part appears an odd number of times. In a recent work, Chern noted that such partitions can be placed in one–to–one correspondence with the partitions of $n$ which he calls unlimited parity alternating partitions with smallest part odd. Our goal is to study the parity of $f(n)$ in detail. In particular, we prove a characterization of $f(2n)$ modulo 2 which implies infinitely many Ramanujan–like congruences modulo 2 satisfied by the function $f$. All of the proof techniques utilized are elementary and involve classical generating function dissection tools.

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## 1. Introduction

In a recent note, Chern [2] defined the function $pa_o(n)$ to be the number of unlimited parity alternating partitions of $n$ with smallest part odd. Chern's work is motivated by work of Andrews [1] who defined a partition of $n$ as "parity alternating" if the parts of the partition in question alternate in parity.

Chern notes in passing that $pa_o(n)$ also counts the number of partitions of $n$ in which each part appears an odd number of times. (Indeed, one can place the unlimited parity alternating partitions of $n$ with smallest part odd and the partitions of $n$ in which each part appears an odd number of times in one–to–one corresponce via conjugation.)

In order to simplify notation, we let $f(n)$ be the number of partitions of $n$ in which each part appears an odd number of times. Our primary goal in this note is to prove the following characterization of $f(2n)$ modulo 2:

THEOREM 1.1. *For all $n \geq 0$,*

$$f(2n) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = k^2 \text{ for some integer } k \text{ with } 3 \nmid k, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

At the conclusion of the note, we will highlight infinite families of Ramanujan–like congruences modulo 2 which are satisfied by $f$. We will also note how Theorem 1.1 implies a characterization modulo 2 of $a_3(n)$, the number of 3–cores of $n$. [4].

## 2. An Elementary Generating Function Proof

In order to prove Theorem 1.1, we will utilize some well–known generating function results and elementary manipulations thereof. We provide this foundation here.

We begin by setting some standard notation. In particular, we define $(a;q)_\infty$, which is the usual Pochhammer symbol, to be

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)(1-aq^3)\dots$$

Next, we provide three important lemmas.

Lemma 2.1.

$$\frac{(q;q)_\infty}{(q^3;q^3)_\infty} = \frac{(q^2;q^2)_\infty}{(q^6;q^6)_\infty^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n}.$$

Proof.

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n} &= (q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty \\ &= \frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}.\end{aligned}$$

The result follows. ∎

Lemma 2.2.

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} \equiv \sum_{n=-\infty}^{\infty} q^{3n^2-2n} \ (mod\ 2).$$

Proof.

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{3n^2-2n} &\equiv \sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n} \pmod 2\\ &= \frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}\\ &\equiv \frac{(q;q)_\infty(q^3;q^3)_\infty^4}{(q;q)_\infty^2(q^3;q^3)_\infty} \pmod 2\\ &= \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.\end{aligned}$$

∎

As an aside, we note that Lemma 2.2 yields a mod 2 characterization for the number of 3-core partitions of $n$ [4]. We will return to this observation at the end of this paper.

LEMMA 2.3. *If, as usual,*

$$\psi(q) = \sum_{n\geq 0} q^{(n^2+n)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \quad \textit{and} \quad \Pi(q) = \sum_{n=-\infty}^{\infty} q^{(3n^2-n)/2}$$

*then*

$$\psi(q) = \Pi(q) + q\psi(q^9).$$

PROOF. See [3, Chapter 1]. ∎

We are now in a position to prove Theorem 1.1.

PROOF. (of Theorem 1.1)

$$\begin{aligned}
\sum_{n\geq 0} f(n)q^n &= \prod_{n\geq 1}\left(1+\frac{q^n}{1-q^{2n}}\right) \\
&= \prod_{n\geq 1}\frac{1+q^n-q^{2n}}{1-q^{2n}} \\
&\equiv \prod_{n\geq 1}\frac{1+q^n+q^{2n}}{1-q^{2n}} \pmod 2 \\
&= \prod_{n\geq 1}\frac{(1-q^{3n})}{(1-q^n)(1-q^{2n})} \\
&= \frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^2;q^2)_\infty} \\
&= \frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^2(q^2;q^2)_\infty}\cdot\frac{(q;q)_\infty}{(q^3;q^3)_\infty} \\
&\equiv \frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty^2}\cdot\frac{(q;q)_\infty}{(q^3;q^3)_\infty} \pmod 2 \\
&= \frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty^2}\cdot\frac{(q^2;q^2)_\infty}{(q^6;q^6)_\infty^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n} \quad \text{by Lemma 2.1} \\
&= \frac{1}{(q^2;q^2)_\infty(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n} \\
&= \frac{1}{(q^2;q^2)_\infty(q^6;q^6)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{12n^2-4n} - q\sum_{n=-\infty}^{\infty} q^{12n^2-8n}\right).
\end{aligned}$$

It follows that

$$\sum_{n\geq 0} f(2n)q^n \equiv \frac{1}{(q;q)_\infty(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty} q^{6n^2-2n} \pmod 2$$

$$
\begin{aligned}
&\equiv \frac{1}{(q;q)_\infty (q^3;q^3)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{6n^2-2n} \pmod{2} \\
&= \frac{1}{(q;q)_\infty (q^3;q^3)_\infty} (q^4;q^4)_\infty \\
&\equiv \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty (q^3;q^3)_\infty} \pmod{2} \\
&= \frac{\psi(q)}{(q^3;q^3)_\infty} \\
&= \frac{\Pi(q^3) + q\psi(q^9)}{(q^3;q^3)_\infty} \quad \text{by Lemma 2.3} \\
&\equiv \frac{(q^3;q^3)_\infty + q\psi(q^9)}{(q^3;q^3)_\infty} \pmod{2} \\
&= 1 + q\frac{(q^{18};q^{18})_\infty^2}{(q^3;q^3)_\infty (q^9;q^9)_\infty} \\
&\equiv 1 + q\frac{(q^9;q^9)_\infty^4}{(q^3;q^3)_\infty (q^9;q^9)_\infty} \pmod{2} \\
&= 1 + q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} \\
&\equiv 1 + q\sum_{n=-\infty}^{\infty} q^{9n^2-6n} \pmod{2} \quad \text{Lemma 2.2} \\
&= 1 + \sum_{n=-\infty}^{\infty} q^{(3n-1)^2} \\
&= 1 + \sum_{n>0,\ 3\nmid n} q^{n^2}.
\end{aligned}
$$

The result follows. ∎

Several comments are in order as we close.

First, note that we can now prove a variety of corollaries which provide infinitely many Ramanujan–like congruences modulo 2 involving $f(2n)$. We simply need to make sure we avoid arguments of the form $2n$ where $n$ is square. So, although not exhaustive, we provide two such corollaries here.

Corollary 2.4. *Let $p \geq 3$ be prime and let $r$ be a quadratic nonresidue modulo $p$. Then, for all $M \geq 1$ and $n \geq 0$,*

$$f(2M^2(pn + r)) \equiv 0 \pmod{2}.$$

Proof. Thanks to Theorem 1.1, we need to see whether $pn + r$ can be written as $pn + r = k^2$ with $3 \nmid k$. However, note that $pn + r = k^2$ implies that $r \equiv k^2 \pmod{p}$. This contradicts the definition of $r$ given in the corollary. And we know that $M^2(pn+r)$

cannot be square because it is the product of a square and a non–square. The result follows. ∎

Corollary 2.5. *For all $M \geq 1$ and $n \geq 0$,*

$$f(2M^2(4n+2)) \equiv 0 \pmod 2.$$

Proof. Note that, for $M = 1$, the result follows because $4n + 2$ is never square. (All squares are congruent to either 0 or 1 modulo 4.) Next, we need to ask whether $M^2(4n + 2)$ can ever be square. Clearly, this also cannot be the case given that $M^2(4n + 2)$ is the product of a square with a non–square. ∎

Secondly, we highlight an unrelated observation about the parity of $a_3(n)$, the number of 3–core partitions of $n$ [4]. Since the generating function for $a_3(n)$ is given by

$$\sum_{n\geq 0} a_3(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty},$$

it is clear that Lemma 2.2 yields the following result:

Theorem 2.6. *For all $n \geq 0$,*

$$a_3(n) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = 3m^2 + 2m \text{ for some integer } m, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

Finally, we note that a combinatorial proof of Theorem 1.1 would be very illuminating.

## 3. Acknowledgements

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