Arithmetic Relations for Overpartitions

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Abstract

In recent work, Corteel and Lovejoy extensively studied overpartitions as a means of better understanding and interpreting various q–series identities. Our goal in this article is quite different. We wish to prove a number of arithmetic relations satisfied by the overpartition function. Employing elementary generating function dissection techniques, we will prove identities such as

$$\sum_{n\geq 0} \overline{p}(8n+7)q^n = 64\frac{(q^2)_\infty^{22}}{(q)_\infty^{23}}$$

and congruences such as

$$\overline{p}(9n+6) \equiv 0 \pmod 8$$

where $\overline{p}(n)$ denotes the number of overpartitions of n.

1 Introduction and statement of results

In recent work, Corteel and Lovejoy [3] revisited the combinatorial objects known as overpartitions. First studied by MacMahon [4], an **overpartition** of the nonnegative integer n is a nonincreasing sequence of natural numbers whose sum is n, and where the first occurrence of parts of each size may be overlined. For example, the eight overpartitions of the integer 3 are

$$3,\ \overline{3},\ 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1,\ \overline{1}+1+1.$$

We denote the number of overpartitions of n by $\overline{p}(n)$, and define $\overline{p}(0)$ to be 1.

As noted in [3], the generating function for $\overline{p}(n)$ is given by

$$\sum_{n\geq 0}\overline{p}(n)q^n = \prod_{n\geq 1}\frac{1+q^n}{1-q^n} = \frac{(q^2)_\infty}{(q)_\infty^2}$$

where $(a)_\infty = (1-a)(1-a^2)(1-a^3)\cdots$. This generating function, when written as a power series in q, begins

$$\sum_{n\geq 0}\overline{p}(n)q^n = 1+2q+4q^2+8q^3+14q^4+24q^5+40q^6+64q^7+\ \cdots\ .$$

Thanks to `MAPLE`, we were able to expand this generating function to obtain the first several hundred values of $\overline{p}(n)$ in order to search for arithmetic relations. Our main goal in this article is to prove numerous arithmetic relations satisfied by $\overline{p}(n)$ which we identified. These will take two forms: the first will be generating function formulas for $\overline{p}(n)$ on certain arithmetic progressions, while the second will be congruences satisfied by $\overline{p}(n)$ on certain arithmetic progressions. Both types of results are in the spirit of Ramanujan's results for the partition function $p(n)$. The techniques we employ are elementary, involving dissections of q–series.

We shall prove a number of results, beginning with the following 2–, 3– and 4–dissections of the generating function for $\overline{p}(n)$.

Theorem 1

$$\sum_{n\geq 0}\overline{p}(n)q^n = \frac{(q^8)_\infty^5}{(q^2)_\infty^4(q^{16})_\infty^2} + 2q\frac{(q^4)_\infty^2(q^{16})_\infty^2}{(q^2)_\infty^4(q^8)_\infty}, \tag{1}$$

$$\sum_{n\geq 0}\overline{p}(n)q^n = \frac{(q^6)_\infty^4(q^9)_\infty^6}{(q^3)_\infty^8(q^{18})_\infty^3} + 2q\frac{(q^6)_\infty^3(q^9)_\infty^3}{(q^3)_\infty^7} + 4q^2\frac{(q^6)_\infty^2(q^{18})_\infty^3}{(q^3)_\infty^6}, \tag{2}$$

and

$$\begin{aligned}\sum_{n\geq 0}\overline{p}(n)q^n &= \frac{(q^8)_\infty^{19}}{(q^4)_\infty^{14}(q^{16})_\infty^6} + 2q\frac{(q^8)_\infty^{13}}{(q^4)_\infty^{12}(q^{16})_\infty^2}\\ &\quad +4q^2\frac{(q^8)_\infty^7(q^{16})_\infty^2}{(q^4)_\infty^{10}} + 8q^3\frac{(q^8)_\infty(q^{16})_\infty^6}{(q^4)_\infty^8}. \end{aligned} \tag{3}$$

From there we shall prove the following generating function identities (which clearly imply various Ramanujan–like congruences):

Theorem 2

$$\sum_{n\geq 0} \overline{p}(2n+1)q^n = 2\frac{(q^2)_\infty^2 (q^8)_\infty^2}{(q)_\infty^4 (q^4)_\infty}, \tag{4}$$

$$\sum_{n\geq 0} \overline{p}(3n+2)q^n = 4\frac{(q^2)_\infty^2 (q^6)_\infty^3}{(q)_\infty^6}, \tag{5}$$

$$\sum_{n\geq 0} \overline{p}(4n+3)q^n = 8\frac{(q^2)_\infty (q^4)_\infty^6}{(q)_\infty^8}, \tag{6}$$

and

$$\sum_{n\geq 0} \overline{p}(8n+7)q^n = 64\frac{(q^2)_\infty^{22}}{(q)_\infty^{23}}. \tag{7}$$

Lastly, we prove the following miscellaneous congruences:

Theorem 3 For all $n \geq 0$,

$$\overline{p}(9n+3) \equiv 0 \pmod 8, \tag{8}$$

$$\overline{p}(9n+6) \equiv 0 \pmod 8, \tag{9}$$

$$\overline{p}(27n+18) \equiv 0 \pmod 4, \tag{10}$$

$$\overline{p}(5n+2) \equiv 0 \pmod 4, \tag{11}$$

and

$$\overline{p}(5n+3) \equiv 0 \pmod 4. \tag{12}$$

2 Proofs

We require a few definitions and lemmas.

Let

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \psi(q) = \sum_{n\geq 0} q^{(n^2+n)/2} \text{ and } X(q) = \sum_{n=-\infty}^{\infty} q^{3n^2+2n}.$$

Then we can prove the following identities which are invaluable in proving Theorem 1.

Lemma 1

$$\phi(q) = \frac{(q^2)_\infty^5}{(q)_\infty^2 (q^4)_\infty^2}, \quad \psi(q) = \frac{(q^2)_\infty^2}{(q)_\infty}, \quad X(q) = \frac{(q^2)_\infty^2 (q^3)_\infty (q^{12})_\infty}{(q)_\infty (q^4)_\infty (q^6)_\infty},$$

$$\phi(q) = \phi(q^4)+2q\psi(q^8),\ \ \phi(q)^2 = \phi(q^2)^2+4q\psi(q^4)^2,\ \ \phi(q)\phi(-q) = \phi(-q^2)^2,$$
$$\psi(q)^2 = \phi(q)\psi(q^2), \quad \text{and} \quad \phi(q) = \phi(q^9) + 2qX(q^3).$$

Proof All of the above follow from Jacobi's triple product identity [1, Theorem 2.8] and straightforward q–series manipulations.

Before proving Theorems 1–3, we note one additional observation made by George E. Andrews [2]. Namely,

$$\sum_{n\geq 0} \overline{p}(n)q^n = 1/\phi(-q)$$

or

$$\sum_{n\geq 0} \overline{p}(n)(-q)^n = 1/\phi(q).$$

We utilize this equality heavily in our proof of Theorem 1.

Proof (of Theorem 1) We see that

$$\begin{aligned}
\sum_{n\geq 0} \overline{p}(n)(-q)^n &= \frac{1}{\phi(q)} \\
&= \frac{\phi(-q)}{\phi(q)\phi(-q)} \\
&= \frac{\phi(-q)}{\phi(-q^2)^2} \quad \text{using Lemma 1} \\
&= \frac{1}{\phi(-q^2)^2}\left(\phi(q^4) - 2q\psi(q^8)\right) \quad \text{using Lemma 1 again.}
\end{aligned}$$

This is equivalent to (1).

Next, we let $\omega = e^{2\pi i/3}$. Then we have

$$\begin{aligned}
\sum_{n\geq 0} \overline{p}(n)(-q)^n &= \frac{1}{\phi(q)} \\
&= \frac{\phi(\omega q)\phi(\omega^2 q)}{\phi(q)\phi(\omega q)\phi(\omega^2 q)} \\
&= \frac{\phi(q^9)}{\phi(q^3)^4}\left(\phi(q^9) + 2\omega qX(q^3)\right)\left(\phi(q^9) + 2\omega^2 qX(q^3)\right) \\
&\qquad\qquad \text{using Lemma 1} \\
&= \frac{\phi(q^9)}{\phi(q^3)^4}\left(\phi(q^9)^2 - 2q\phi(q^9)X(q^3) + 4q^2X(q^3)^2\right).
\end{aligned}$$

This is equivalent to (2).

Finally, we have

$$\begin{aligned}
\sum_{n\geq 0}\overline{p}(n)(-q)^n &= \frac{1}{\phi(q)} \\
&= \frac{\phi(iq)\phi(-q)\phi(-iq)}{\phi(q)\phi(iq)\phi(-q)\phi(-iq)} \\
&= \frac{\phi(-q)\phi(q^2)^2}{\phi(-q^2)^2\phi(q^2)^2} \\
&= \frac{\phi(-q)\phi(q^2)^2}{\phi(-q^4)^4} \\
&= \frac{1}{\phi(-q^4)^4}\left(\phi(q^4)-2q\psi(q^8)\right)\left(\phi(q^4)^2+4q^2\psi(q^8)^2\right) \\
&= \frac{1}{\phi(-q^4)^4}\left(\phi(q^4)^3-2q\phi(q^4)^2\psi(q^8)\right. \\
&\qquad \left.+4q^2\phi(q^4)\psi(q^8)^2-8q^3\psi(q^8)^3\right).
\end{aligned}$$

This is equivalent to (3), and Theorem 1 is now proven.

With Theorem 1 proved, we can now prove all portions of Theorem 2 in a straightforward manner.

Proof (of Theorem 2) First off, note that (4), (5), and (6) are obtained by reading off the appropriate portions of the dissections (1), (2), and (3) respectively. Next, we see that

$$\begin{aligned}
\sum_{n\geq 0}\overline{p}(4n+3)q^n &= 8\frac{\psi(q^2)^3}{\phi(-q)^4} \\
&= 8\frac{\psi(q^2)^3\phi(q)^4}{\phi(q)^4\phi(-q)^4} \\
&= 8\frac{\psi(q^2)^3\left(\phi(q^2)^2+4q\psi(q^4)^2\right)^2}{\phi(-q^2)^8}.
\end{aligned}$$

Hence,

$$\sum_{n\geq 0}\overline{p}(8n+7)q^n = 64\frac{\psi(q)^3\phi(q)^2\psi(q^2)^2}{\phi(-q)^8}$$

$$= 64\frac{\psi(q)^7}{\phi(-q)^8},$$

which is equivalent to (7). This completes the proof of Theorem 2.

We now turn our attention to the proof of Theorem 3, which also employs a number of elementary generating function manipulations.

Proof (of Theorem 3) We see from the above that

$$\begin{aligned}
\sum_{n\geq 0}\overline{p}(3n)(-q)^n &= \frac{\phi(q^3)^3}{\phi(q)^4} \\
&= \phi(q^3)^3\left(\frac{\phi(q^9)}{\phi(q^3)^4}\left(\phi(q^9)^2 - 2q\phi(q^9)X(q^3) + 4q^2X(q^3)^2\right)\right)^4 \\
&= \frac{\phi(q^9)^4}{\phi(q^3)^{13}}\left(\phi(q^9)^2 - 2q\phi(q^9)X(q^3) + 4q^2X(q^3)^2\right)^4 \\
&= \frac{\phi(q^9)^4}{\phi(q^3)^{13}}\big(\phi(q^9)^8 - 8q\phi(q^9)^7X(q^3) + 40q^2\phi(q^9)^6X(q^3)^2 \\
&\qquad -128q^3\phi(q^9)^5X(q^3)^3 + 304q^4\phi(q^9)^4X(q^3)^4 \\
&\qquad -512q^5\phi(q^9)^3X(q^3)^5 + 640q^6\phi(q^9)^2X(q^3)^6 \\
&\qquad -512q^7\phi(q^9)X(q^3)^7 + 256q^8X(q^3)^8\big).
\end{aligned}$$

It follows that

$$\sum_{n\geq 0}\overline{p}(9n)(-q)^n = \frac{\phi(q^3)^4}{\phi(q)^{13}}\left(\phi(q^3)^8 - 128q\phi(q^3)^5X(q)^3 + 640q^2\phi(q^3)^2X(q)^6\right),$$

$$\begin{aligned}
\sum_{n\geq 0}\overline{p}(9n+3)(-q)^n &= \frac{\phi(q^3)^4}{\phi(q)^{13}}\big(8\phi(q^3)^7X(q) - 304q\phi(q^3)^4X(q)^4 \\
&\qquad +512q^2\phi(q^3)X(q)^7\big),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n\geq 0}\overline{p}(9n+6)(-q)^n &= \frac{\phi(q^3)^4}{\phi(q)^{13}}\big(40\phi(q^3)^6X(q)^2 - 512q\phi(q^3)^3X(q)^5 \\
&\qquad +256q^2X(q)^8\big).
\end{aligned}$$

The last two equalities above imply (8) and (9).

Moreover, modulo 8, we have

$$
\begin{aligned}
\sum_{n\geq 0}\overline{p}(9n)(-q)^n &\equiv \frac{\phi(q^3)^{12}}{\phi(q)^{13}}\\
&\equiv \phi(q^3)^{12}\left(\frac{\phi(q^9)}{\phi(q^3)^4}\left(\phi(q^9)^2-2q\phi(q^9)X(q^3)+4q^2X(q^3)^2\right)\right)^{13}.
\end{aligned}
$$

If we expand this and extract those terms in which the power of q is 2 modulo 3, we find that, modulo 8,

$$\sum_{n\geq 0}\overline{p}(27n+18)(-q)^n \equiv 4\frac{\phi(q^3)^{37}X(q)^2}{\phi(q)^{40}}.$$

We thus obtain (10).

Finally, let $\rho=e^{2\pi i/5}$. Then

$$
\begin{aligned}
\sum_{n\geq 0}\overline{p}(n)(-q)^n &= \frac{1}{\phi(q)}\\
&= \frac{\phi(\rho q)\phi(\rho^2 q)\phi(\rho^3 q)\phi(\rho^4 q)}{\phi(q)\phi(\rho q)\phi(\rho^2 q)\phi(\rho^3 q)\phi(\rho^4 q)}\\
&= \frac{\phi(q^{25})}{\phi(q^5)^6}\phi(\rho q)\phi(\rho^2 q)\phi(\rho^3 q)\phi(\rho^4 q)\\
&= \frac{\phi(q^{25})}{\phi(q^5)^6}\left(\phi(q^{25})+2\rho qD(q^5)+2\rho^4q^4E(q^5)\right)\\
&\qquad \times\left(\phi(q^{25})+2\rho^2 qD(q^5)+2\rho^3q^4E(q^5)\right)\\
&\qquad \times\left(\phi(q^{25})+2\rho^3 qD(q^5)+2\rho^2q^4E(q^5)\right)\\
&\qquad \times\left(\phi(q^{25})+2\rho^4 qD(q^5)+2\rho q^4E(q^5)\right)\\
&= \frac{\phi(q^{25})}{\phi(q^5)^6}\left(\phi(q^{25})^4-2q\phi(q^{25})^3D(q^5)+4q^2\phi(q^{25})^2\right.\\
&\qquad D(q^5)^2-8q^3\phi(q^{25})D(q^5)^3+q^4\left(16D(q^5)^4\right.\\
&\qquad \left.-2\phi(q^{25})^3E(q^5)\right)-12q^5\phi(q^{25})^2D(q^5)E(q^5)\\
&\qquad +16q^6\phi(q^{25})D(q^5)^2E(q^5)-16q^7D(q^5)^3E(q^5)\\
&\qquad +4q^8\phi(q^{25})^2E(q^5)^2+16q^9\phi(q^{25})D(q^5)E(q^5)^2\\
&\qquad +16q^{10}D(q^5)^2E(q^5)^2-8q^{12}\phi(q^{25})E(q^5)^3\\
&\qquad \left.-16q^{13}D(q^5)E(q^5)^3+16q^{16}E(q^5)^4\right)
\end{aligned}
$$

where

$$D(q) = \sum_{n=-\infty}^{\infty} q^{5n^2+2n} \text{ and } E(q) = \sum_{n=-\infty}^{\infty} q^{5n^2+4n}.$$

It follows that

$$\begin{aligned}\sum_{n\geq 0} \overline{p}(5n+2)(-q)^n &= \frac{\phi(q^5)}{\phi(q)^6}\left(4\phi(q^5)^2 D(q)^2\right.\\ &\qquad \left.-16qD(q)^3E(q) - 8q^2\phi(q^5)E(q)^3\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{n\geq 0} \overline{p}(5n+3)(-q)^n &= \frac{\phi(q^5)}{\phi(q)^6}\left(8\phi(q^5)D(q)^3 - 4q\phi(q^5)^2E(q)^2\right.\\ &\qquad \left.+16q^2D(q)E(q)^3\right).\end{aligned}$$

We thus obtain (11) and (12).

3 Closing Remarks

We close by noting that $\overline{p}(n)$ appears to satisfy numerous other arithmetic properties, including the following: For all $n \geq 0$,

$$\overline{p}(27n+18) \equiv 0 \pmod{12} \tag{13}$$

and

$$\overline{p}(40n+35) \equiv 0 \pmod{40} \tag{14}$$

We further conjecture that if p is prime and r is a quadratic nonresidue modulo p then

$$\overline{p}(pn+r) = \begin{cases} 0 \pmod{8} & \text{if } p \equiv \pm 1 \pmod{8}, \\ 0 \pmod{4} & \text{if } p \equiv \pm 3 \pmod{8}. \end{cases}$$

These conjectures are based on the calculation and subsequent analysis of a large number of values of $\overline{p}(n)$ via `MAPLE`.

References

[1] G. E. Andrews, *The theory of partitions*, Cambridge University Press (1984).

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[4] P. A. MacMahon, *Combinatory Analysis*, Chelsea, New York (1960).