ON A PROBLEM OF LEHMER ON PARTITIONS INTO SQUARES

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Abstract. In 1948, D. H. Lehmer published a brief work discussing the difference between representations of the integer n as a sum of squares and partitions of n into square summands. In this article, we return to this topic and consider four partition functions involving square parts and prove various arithmetic properties of these functions. These results provide a natural extension to the work of Lehmer.

1. Introduction

Just over half a century ago, D. H. Lehmer [5] initiated the study of partitions of a number into a fixed number of squares. In a very readable introduction, he points out the distinction between the number of **representations** of n as a sum of four squares, which we denote by $r_4(n)$, and the number of **partitions** of n into four squares, which we denote by $p_{4\square}(n)$. Lehmer gives as an example the case $n = 98$, where the number of representations is 1368 and the number of partitions is 7.

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With the goal of better understanding the significant difference between $r_4(98)$ and $p_{4\square}(98)$, consider the following. The single partition $98 = 81 + 16 + 1 + 0$ gives rise to 192 distinct representations which are counted by $r_4(98)$. This is because one can permute the squares $81, 16, 1$ and 0 in 24 different ways and then one can think of 81 as $(\pm 9)^2$, 16 as $(\pm 4)^2$, and 1 as $(\pm 1)^2$. So there are 24×2^3 or 192 different representations of 98 generated by the single partition $81 + 16 + 1 + 0$.

Given this situation, Lehmer goes on to say "... it is a fair question to ask how one may obtain the reasonable number 7 from the obviously inflated value 1368. This problem is unfortunately one of extreme difficulty. At least no solution has ever been given." These comments appear to be true even today. He goes on "The difficulty lies in the unequal numbers of representations derivable from a single partition. It is not difficult to see that there are eleven different types of partitions as far as we are concerned ...".

In the next section, we describe Lehmer's eleven types of partitions and find the generating functions for each one of these. We show how these may be combined to obtain the generating function for $p_{4\square}(n)$ as well as for the related partition functions $p^{+}_{4\square}(n)$, the number of partitions of n into four positive squares, $p^{d}_{4\square}(n)$, the number of partitions of n into four distinct squares, and $p^{d+}_{4\square}(n)$, the number of partitions of n into four distinct positive squares. We then state and prove various arithmetic properties of these four partition functions.

2. The generating functions

We begin by reproducing Lehmer's list of eleven types of partitions into four squares. (This list appears in Table 1.) The values in the third column of this table show the number of representations derivable from each partition type. Thus, in the first row, each partition of n in the form $a^2 + b^2 + c^2 + d^2$ with a, b, c, d positive and distinct $(1 \le a < b < c < d)$ yields $4! \times 2^4$ or 384 representations of n as a sum of four squares because there are 4! ways to permute a^2, b^2, c^2, and d^2, and there are 2^4 variations of positive and negative signs that can be used. As another example, in the fifth row, each partition of the form $0^2+a^2+a^2+b^2$ with a, b positive and distinct corresponds to $\binom{4}{2}\binom{2}{1}\binom{1}{1}(2^3)$ or 96 representations.

I	$a^2+b^2+c^2+d^2$	384
II	$0^2+a^2+b^2+c^2$	192
III	$a^2+a^2+b^2+c^2$	192
IV	$a^2+a^2+b^2+b^2$	96
V	$0^2+a^2+a^2+b^2$	96
VI	$a^2+a^2+a^2+b^2$	64
VII	$0^2+0^2+a^2+b^2$	48
VIII	$0^2+a^2+a^2+a^2$	32
IX	$0^2+0^2+a^2+a^2$	24
X	$a^2+a^2+a^2+a^2$	16
XI	$0^2+0^2+0^2+a^2$	8

Table 1: Lehmer's Eleven Types of Partitions of n into Four Squares

We shall define the generating functions of each of these eleven types of partitions in a fairly self–explanatory fashion. Thus, for example, the generating function for partitions of type V will be denoted by $F(a^2+a^2+b^2,q)$.

It then follows that

(1)

$$\begin{aligned}\sum_{n\geq 0} p_{4\square}(n)q^n &= 1+F(a^2+b^2+c^2+d^2,q)+F(a^2+b^2+c^2,q)\\ &+F(a^2+a^2+b^2+c^2,q)+F(a^2+a^2+b^2+b^2,q)\\ &+F(a^2+a^2+b^2,q)+F(a^2+a^2+a^2+b^2,q)\\ &+F(a^2+b^2,q)+F(a^2+a^2+a^2,q)+F(a^2+a^2,q)\\ &+F(a^2+a^2+a^2+a^2,q)+F(a^2,q),\end{aligned}$$

while

$$\begin{aligned}\sum_{n\geq 0} r_4(n)q^n &= 1+384F(a^2+b^2+c^2+d^2,q)+192F(a^2+b^2+c^2,q)\\ &+192F(a^2+a^2+b^2+c^2,q)+96F(a^2+a^2+b^2+b^2,q)\\ &+96F(a^2+a^2+b^2,q)+64F(a^2+a^2+a^2+b^2,q)\\ &+48F(a^2+b^2,q)+32F(a^2+a^2+a^2,q)+24F(a^2+a^2,q)\\ &+16F(a^2+a^2+a^2+a^2,q)+8F(a^2,q).\end{aligned}$$

Next, we introduce the well–known function $\phi(q)$ given by

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}.$$

This function arises naturally in the area of representations as sums of squares because, for example, $\sum_{n\geq 0} r_4(n)q^n = \phi(q)^4$.

We now write each of the generating functions corresponding to the eleven partition types in Table 1 in terms of $\phi(q)$ and obtain the following theorem. We include the proofs of the various results in the statement of the theorem, as the work involved is straightforward.

Theorem 1. *The following generating function identities hold.*

$$\begin{aligned}
F(a^2, q) &= \frac{1}{2}(\phi(q) - 1), \\
F(a^2 + a^2, q) &= \frac{1}{2}(\phi(q^2) - 1), \\
F(a^2 + a^2 + a^2, q) &= \frac{1}{2}(\phi(q^3) - 1), \\
F(a^2 + a^2 + a^2 + a^2, q) &= \frac{1}{2}(\phi(q^4) - 1), \\
F(a^2 + b^2, q) &= \frac{1}{2}\left(F(a^2, q)^2 - F(a^2 + a^2, q)\right) \\
&= \frac{1}{8}\left(\phi(q)^2 - 2\phi(q) - 2\phi(q^2) + 3\right), \\
F(a^2 + a^2 + b^2 + b^2, q) &= \frac{1}{8}\left(\phi(q^2)^2 - 2\phi(q^2) - 2\phi(q^4) + 3\right), \\
F(a^2 + a^2 + b^2, q) &= F(a^2 + a^2, q)F(a^2, q) - F(a^2 + a^2 + a^2, q) \\
&= \frac{1}{4}\left(\phi(q)\phi(q^2) - \phi(q) - \phi(q^2) - 2\phi(q^3) + 3\right), \\
F(a^2 + a^2 + a^2 + b^2, q) &= F(a^2 + a^2 + a^2, q)F(a^2, q) - F(a^2 + a^2 + a^2 + a^2, q) \\
&= \frac{1}{4}\left(\phi(q)\phi(q^3) - \phi(q) - \phi(q^3) - 2\phi(q^4) + 3\right), \\
F(a^2 + b^2 + c^2, q) &= \frac{1}{3}\left(F(a^2 + b^2, q)F(a^2, q) - F(a^2 + a^2 + b^2, q)\right)
\end{aligned}$$

$$= \frac{1}{48}\big(\phi(q)^3 - 3\phi(q)^2 - 6\phi(q)\phi(q^2) + 9\phi(q) + 6\phi(q^2)$$
$$+8\phi(q^3) - 15\big)\,,$$
$$F(a^2+a^2+b^2+c^2,q) = F(a^2+a^2,q)F(a^2+b^2,q) - F(a^2+a^2+a^2+b^2,q)$$
$$= \frac{1}{16}\big(\phi(q)^2\phi(q^2) - \phi(q)^2 - 2\phi(q)\phi(q^2) - 4\phi(q)\phi(q^3)$$
$$-2\phi(q^2)^2 + 6\phi(q) + 5\phi(q^2) + 4\phi(q^3) + 8\phi(q^4) - 15\big)$$

and

$$F(a^2+b^2+c^2+d^2,q) = \frac{1}{4}\big(F(a^2+b^2+c^2,q)F(a^2,q) - F(a^2+a^2+b^2+c^2,q)\big)$$
$$= \frac{1}{384}\big(\phi(q)^4 - 4\phi(q)^3 - 12\phi(q)^2\phi(q^2) + 18\phi(q)^2$$
$$+24\phi(q)\phi(q^2) + 32\phi(q)\phi(q^3) + 12\phi(q^2)^2 - 60\phi(q)$$
$$-36\phi(q^2) - 32\phi(q^3) - 48\phi(q^4) + 105\big)\,. \blacksquare$$

Notice that we can substitute these eleven expressions into (1) to yield

$$\sum_{n\geq 0} p_{4\Box}(n)q^n = \frac{1}{384}\big(\phi(q)^4 + 4\phi(q)^3 + 12\phi(q)^2\phi(q^2) + 18\phi(q)^2 + 24\phi(q)\phi(q^2)$$
$$+32\phi(q)\phi(q^3) + 12\phi(q^2)^2 + 60\phi(q) + 36\phi(q^2) + 32\phi(q^3)$$
$$+48\phi(q^4) + 105\big)\,. \tag{2}$$

This result (2) was obtained via another method by Hirschhorn [3]. We can also obtain similar generating function identities for $p^{+}_{4\Box}$, $p^{d}_{4\Box}$, and $p^{d+}_{4\Box}$.

Theorem 2. *(including proof)*

$$\sum_{n\geq 0} p^{+}_{4\Box}(n)q^n = F(a^2+a^2+a^2+a^2,q) + F(a^2+a^2+b^2+b^2,q)$$
$$+ F(a^2+a^2+a^2+b^2,q) + F(a^2+a^2+b^2+c^2,q)$$
$$+ F(a^2+b^2+c^2+d^2,q)$$
$$= \frac{1}{384}\big(\phi(q)^4 - 4\phi(q)^3 + 12\phi(q)^2\phi(q^2) - 6\phi(q)^2 - 24\phi(q)\phi(q^2)$$
$$+32\phi(q)\phi(q^3) + 12\phi(q^2)^2 - 12\phi(q)$$

$$-12\phi(q^2) - 32\phi(q^3) + 48\phi(q^4) - 15\big) ,$$

$$\sum_{n\geq 0} p^{d}_{4\square}(n)q^n = F(a^2+b^2+c^2, q) + F(a^2+b^2+c^2+d^2, q)$$

$$= \frac{1}{384}\big(\phi(q)^4 + 4\phi(q)^3 - 12\phi(q)^2\phi(q^2) - 6\phi(q)^2$$

$$-24\phi(q)\phi(q^2) + 32\phi(q)\phi(q^3) + 12\phi(q^2)^2 + 12\phi(q)$$

$$+12\phi(q^2) + 32\phi(q^3) - 48\phi(q^4) - 15\big) ,$$

and

$$\sum_{n\geq 0} p^{d+}_{4\square}(n)q^n = F(a^2+b^2+c^2+d^2)$$

$$= \frac{1}{384}\big(\phi(q)^4 - 4\phi(q)^3 - 12\phi(q)^2\phi(q^2) + 18\phi(q)^2$$

$$+24\phi(q)\phi(q^2) + 32\phi(q)\phi(q^3) + 12\phi(q^2)^2 - 60\phi(q)$$

$$-36\phi(q^2) - 32\phi(q^3) - 48\phi(q^4) + 105\big) . \blacksquare$$

3. Some arithmetic relations

In [5], Lehmer proved that $p_{4\square}(8n) = p_{4\square}(2n)$ for all $n \geq 0$. Our goal was to search for other such arithmetic relations involving $p_{4\square}$, $p^{+}_{4\square}$, $p^{d}_{4\square}$, and $p^{d+}_{4\square}$.

Now that we have the generating functions for the four partition functions mentioned above, it is an easy matter to expand them using MAPLE or a similar package. We did so, and made the following discoveries.

Theorem 3. *For all $n \geq 0$,*

(i) $$p_{4\square}(8n) = p_{4\square}(2n),$$

(ii) $$p^{+}_{4\square}(8n) = p^{+}_{4\square}(2n),$$

(iii) $$p^{d}_{4\square}(8n) = p^{d}_{4\square}(2n),$$

and

(iv) $$p^{d+}_{4\square}(8n) = p^{d+}_{4\square}(2n),$$

(v) $$p_{4\square}(8n+4) = 2p_{4\square}(2n+1) + p^{+}_{4\square}(2n+1),$$

(vi) $$p^{+}_{4\square}(8n+4) = 2p^{+}_{4\square}(2n+1) + p_{4\square}(2n+1),$$

(vii) $$p^{d}_{4\square}(8n+4) = 2p^{d}_{4\square}(2n+1) + p^{d+}_{4\square}(2n+1),$$

and

(viii) $$p^{d+}_{4\square}(8n+4) = 2p^{d+}_{4\square}(2n+1) + p^{d}_{4\square}(2n+1),$$

(ix) $$p_{4\square}(32n+28) = 3p_{4\square}(8n+7)$$

and

(x) $$p^{d}_{4\square}(32n+28) = 3p^{d}_{4\square}(8n+7),$$

(xi) $$p_{4\square}(72n+69) \equiv 0 \pmod 2,$$

(xii) $$p^{+}_{4\square}(72n+69) \equiv 0 \pmod 2,$$

(xiii) $$p^{d}_{4\square}(72n+69) \equiv 0 \pmod 2,$$

and

(xiv) $$p^{d+}_{4\square}(72n+69) \equiv 0 \pmod 2.$$

Lehmer gave a straightforward proof of (i) by establishing a one–to–one correspondence between partitions of $2n$ into four squares and partitions of $8n$ into four squares. The same proof establishes (ii), (iii) and (iv). We shall give different proofs of (i) – (iv), utilizing generating functions.

In an earlier paper [4] we gave a proof of (ix) which can be extended to prove (x). However, note that parts (ix) and (x) are corollaries of parts (v) – (viii) together with the fact that if $8n+7$ is the sum of four squares, the squares are all positive. We shall prove parts (v) – (viii) below.

Lastly, we note that our proof in [4] of (xi) can be extended to prove (xii) – (xiv).

4. Proofs

In order to attack the proof of Theorem 3, we require the following facts.

Lemma 4. *Let*

$$\psi(q) = \sum_{n\geq 0} q^{(n^2+n)/2}.$$

Then we have the following:

$$\phi(q) = \phi(q^4) + 2q\psi(q^8),$$

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2,$$

$$\phi(q)\psi(q^2) = \psi(q)^2$$

and

$$\psi(q)\psi(q^3) = \phi(q^6)\psi(q^4) + q\phi(q^2)\psi(q^{12}).$$

For proofs of these identities, see [1, page 40, Entry 25] and [2, Lemma 14]. ■

We now turn to the proofs of (i) and (v). Using (2) and Lemma 4, we have

$$\begin{aligned}\sum_{n\geq 0} p_{4\square}(n)q^n = \frac{1}{384}\Big(&\left(\phi(q^4) + 2q\psi(q^8)\right)^4 + 4\left(\phi(q^4) + 2q\psi(q^8)\right)^3 \\ &+ 12\left(\phi(q^4) + 2q\psi(q^8)\right)^2\left(\phi(q^8) + 2q^2\psi(q^{16})\right) + 18\left(\phi(q^4) + 2q\psi(q^8)\right)^2 \\ &+ 24\left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^8) + 2q^2\psi(q^{16})\right) \\ &+ 32\left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^{12}) + 2q^3\psi(q^{24})\right) \\ &+ 12\left(\phi(q^8) + 2q^2\psi(q^{16})\right)^2 + 60\left(\phi(q^4) + 2q\psi(q^8)\right) + 36\left(\phi(q^8) + 2q^2\psi(q^{16})\right) \\ &+ 32\left(\phi(q^{12}) + 2q^3\psi(q^{24})\right) + 48\left(\phi(q^{16}) + 2q^4\psi(q^{32})\right) + 105\Big).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n\geq 0} p_{4\square}(4n)q^n = \frac{1}{384}\Big(&\phi(q)^4 + 16q\psi(q^2)^4 + 4\phi(q)^3 + 12\phi(q)^2\phi(q^2) + 96q\psi(q^2)^2\psi(q^4) \\ &+ 18\phi(q)^2 + 24\phi(q)\phi(q^2) + 32\phi(q)\phi(q^3) + 128q\psi(q^2)\psi(q^6) + 12\phi(q^2)^2 \\ &+ 48q\psi(q^4)^2 + 60\phi(q) + 36\phi(q^2) + 32\phi(q^3) + 48\phi(q^4) + 96q\psi(q^8) + 105\Big)\end{aligned}$$

and so

$$\begin{aligned}\sum_{n\geq 0} \left(p_{4\square}(4n) - p_{4\square}(n)\right) q^n &= \frac{1}{384}\Big(16q\psi(q^2)^4 + 96q\psi(q^2)^2\psi(q^4) + 128q\psi(q^2)\psi(q^6) \\ &\qquad + 48q\psi(q^4)^2 + 96q\psi(q^8)\Big) \\ &= \frac{1}{24}q\Big(\psi(q^2)^4 + 6\psi(q^2)^2\psi(q^4) + 8\psi(q^2)\psi(q^6) \\ &\qquad + 3\psi(q^4)^2 + 6\psi(q^8)\Big).\end{aligned}$$

This is an odd function of q, so

$$\sum_{n\geq 0} \left(p_{4\square}(8n) - p_{4\square}(2n)\right) q^n = 0,$$

which proves (i).

Moreover,

$$\sum_{n\geq 0}\left(p_{4\square}(8n+4)-p_{4\square}(2n+1)\right)q^n=\frac{1}{24}\Big(\psi(q)^4+6\psi(q)^2\psi(q^2)+8\psi(q)\psi(q^3)+3\psi(q^2)^2 + 6\psi(q^4)\Big).$$

This is the first step in the proof of (v). Next, we have

$$\begin{aligned}\sum_{n\geq 0}\left(p_{4\square}(n)+p^{+}_{4\square}(n)\right)q^n &= \frac{1}{384}\Big(2\phi(q)^4+24\phi(q)^2\phi(q^2)+12\phi(q)^2+64\phi(q)\phi(q^3)\\ &\quad +24\phi(q^2)^2+48\phi(q)+24\phi(q^2)+96\phi(q^4)+90\Big)\\ &= \frac{1}{192}\Big(\left(\phi(q^2)^2+4q\psi(q^4)^2\right)^2+12\left(\phi(q^2)^2+4q\psi(q^4)^2\right)\phi(q^2)\\ &\quad +6\left(\phi(q^2)^2+4q\psi(q^4)^2\right)\\ &\quad +32\left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{12})+2q^3\psi(q^{24})\right)\\ &\quad +12\phi(q^2)^2+24\left(\phi(q^4)+2q\psi(q^8)\right)+12\phi(q^2)+48\phi(q^4)+45\Big).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n\geq 0}\left(p_{4\square}(2n+1)+p^{+}_{4\square}(2n+1)\right)q^n &= \frac{1}{192}\Big(8\phi(q)^2\psi(q^2)^2+48\phi(q)\psi(q^2)^2+24\psi(q^2)^2\\ &\quad +64\phi(q^6)\psi(q^4)+64q\phi(q^2)\psi(q^{12})+48\psi(q^4)\Big)\\ &= \frac{1}{24}\Big(\psi(q)^4+6\psi(q)^2\psi(q^2)+3\psi(q^2)^2+8\psi(q)\psi(q^3)\\ &\quad +6\psi(q^4)\Big)\\ &= \sum_{n\geq 0}\left(p_{4\square}(8n+4)-p_{4\square}(2n+1)\right)q^n.\end{aligned}$$

This proves (v).

We now closely mimic the proofs of (i) and (v) above to prove (ii) – (iv) and (vi) – (viii). First, we find that

$$\sum_{n\geq 0}\left(p^{+}_{4\square}(4n)-p^{+}_{4\square}(n)\right)q^n=\frac{1}{24}\Big(q\psi(q^2)^4+6q\psi(q^2)^2\psi(q^4)+8q\psi(q^2)\psi(q^6)$$

$$+ 3q\psi(q^4)^2 + 6q\psi(q^8)\Big),$$

from which both (ii) and (vi) follow.

Also,

$$\begin{aligned}\sum_{n\geq 0}\left(p^{d}_{4\square}(4n) - p^{d}_{4\square}(n)\right) q^n &= \sum_{n\geq 0}\left(p^{d+}_{4\square}(4n) - p^{d+}_{4\square}(n)\right) q^n \\ &= \frac{1}{24}\Big(q\psi(q^2)^4 - 6q\psi(q^2)^2\psi(q^4) + 8q\psi(q^2)\psi(q^6) + 3q\psi(q^4)^2 \\ &\quad - 6q\psi(q^8)\Big)\end{aligned}$$

from which (iii) and (iv) follow, and

$$\begin{aligned}\sum_{n\geq 0}\left(p^{d}_{4\square}(8n+4) - p^{d}_{4\square}(2n+1)\right) q^n &= \sum_{n\geq 0}\left(p^{d+}_{4\square}(8n+4) - p^{d+}_{4\square}(2n+1)\right) q^n \\ &= \frac{1}{24}\Big(\psi(q)^4 - 6\psi(q)^2\psi(q^2) + 8\psi(q)\psi(q^3) + 3\psi(q^2)^2 \\ &\quad - 6\psi(q^4)\Big) \\ &= \sum_{n\geq 0}\left(p^{d}_{4\square}(2n+1) + p^{d+}_{4\square}(2n+1)\right) q^n\end{aligned}$$

from which (vii) and (viii) follow.

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