New Proofs of Identities of Lebesgue and Göllnitz via Tilings

David P. Little and James A. Sellers

Department of Mathematics, Penn State University, University Park, PA, 16802 USA

Abstract

In 1840, V. A. Lebesgue proved the following two series–product identities:

$$\sum_{n\geq 0}\frac{(-1;q)_n}{(q)_n}q^{\binom{n+1}{2}} = \prod_{n\geq 1}\frac{1+q^{2n-1}}{1-q^{2n-1}}$$

$$\sum_{n\geq 0}\frac{(-q;q)_n}{(q)_n}q^{\binom{n+1}{2}} = \prod_{n\geq 1}\frac{1-q^{4n}}{1-q^n}$$

These can be viewed as specializations of the following more general result:

$$\sum_{n\geq 0}\frac{(-z;q)_n}{(q)_n}q^{\binom{n+1}{2}} = \prod_{n\geq 1}(1+q^n)(1+zq^{2n-1})$$

There are numerous combinatorial proofs of this identity, all of which describe a bijection between different types of integer partitions. Our goal is to provide a new, novel combinatorial proof that demonstrates how both sides of the above identity enumerate the same collection of "weighted Pell tilings". In the process, we also provide a new proof of the Göllnitz identities.

Key words: Pell numbers, Lebesgue identities, Göllnitz identities,
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   Email addresses: dlittle@math.psu.edu (David P. Little),
sellersj@math.psu.edu (James A. Sellers).

1 Introduction

The objects of study in this work are the following two series–product identities:

$$\sum_{n\geq 0} \frac{(-1;q)_n}{(q)_n} q^{\binom{n+1}{2}} = \prod_{n\geq 1} \frac{1+q^{2n-1}}{1-q^{2n-1}} \tag{1}$$

$$\sum_{n\geq 0} \frac{(-q;q)_n}{(q)_n} q^{\binom{n+1}{2}} = \prod_{n\geq 1} \frac{1-q^{4n}}{1-q^n} \tag{2}$$

where $(z;q)_n = (1-z)(1-zq)(1-zq^2)\cdots(1-zq^{n-1})$ and $(q)_n = (q;q)_n$ for $n \geq 1$. Identities (1) and (2) were originally proved by V. A. Lebesgue [5] in 1840. (For additional discussion, see Andrews' work [2].) At the outset, it is worth noting that the right–hand sides of (1) and (2) can be interpreted "naturally" as the generating functions of certain integer partition functions; namely, the right–hand side of (1) is the generating function for overpartitions into odd parts [4] and the right–hand side of (2) is the generating function for 4–regular partitions or partitions wherein no part is a multiple of four. Because of these partition function interpretations of the product sides, (1) and (2) have been of interest for quite some time to those studying properties of integer partitions, especially those interested in identities of the Rogers–Ramanujan type. For example, in 1952, Slater [9] proved the above identities and included them in her well–known list of 130 identities of Rogers–Ramanujan type. Note that Lebesgue's and Slater's proofs are analytic in nature.

More recently, a number of combinatorial proofs of (1) and (2) have been published. These have appeared in the works of Bessenrodt [3] and Alladi and Gordon [1], and the interested reader may wish to reference the work of Pak [6] where both of these combinatorial proofs are discussed. Within the last year, Rowell [7] has also proven Lebesgue's identities combinatorially. In all of these works, bijections between different sets of integer partitions have been utilized. That is to say, previous authors have found different sets of objects (typically restricted partitions) whose generating functions are the left–hand side and the right–hand side, respectively, and then have found a bijection between these two different sets of objects.

In 2002, Santos and Sills [8] considered finite analogues for which (1) and (2) are limiting cases. In the process they were able, to a degree, to demonstrate connections between Lebesgue's identities and q–Pell sequences.

Our goal in this paper, simply stated, is to provide a new and fundamentally different combinatorial proof of Lebesgue's identities which more naturally connects them with q–Pell sequences. The most striking aspect is that the work is seated in the context of domino tilings of a $1 \times \infty$ board rather than integer

partitions. (Such a context makes explicit the connection to Pell numbers P_n since the number of ways to tile a $1 \times n$ board with two colors of squares and one color of domino is precisely P_n where $P_0 = 1$, $P_1 = 2$, and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$.)

With this goal in mind, we note that (1) and (2) as stated can be viewed as specializations of the following identity:

$$\sum_{n \geq 0} \frac{(-z; q)_n}{(q)_n} q^{\binom{n+1}{2}} = \prod_{n \geq 1} (1 + q^n)(1 + zq^{2n-1}) \tag{3}$$

Specifically, (1) follows from (3) by setting $z = 1$ and applying Euler's classic result

$$\prod_{n \geq 1} \frac{1}{1 - q^{2n-1}} = \prod_{n \geq 1} (1 + q^n),$$

and (2) follows from (3) by setting $z = q$ and noting that

$$\frac{1 - q^{4n}}{1 - q^n} = (1 + q^n)(1 + q^{2n}).$$

Therefore, our primary goal for the remainder of this work is to show how both sides of (3) count the same set of "weighted Pell tilings." Once this goal is complete, we can also provide new proofs of the following two identities of Göllnitz:

$$\sum_{n \geq 0} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n^2+n} = \prod_{n \geq 1} (1 + q^{4n-2})(1 + q^{4n-1})(1 + q^{4n}) \tag{4}$$

$$\sum_{n \geq 0} \frac{(-1/q; q^2)_n}{(q^2; q^2)_n} q^{n^2+n} = \prod_{n \geq 1} (1 + q^{4n-3})(1 + q^{4n-2})(1 + q^{4n}) \tag{5}$$

These two follow immediately from Corollary 8 below by certain replacements of the parameter z.

2   Pell Tilings

We begin this section by describing the combinatorial objects which are to be used to prove (3). Consider tilings of a $1 \times \infty$ board using white squares, black squares, and dominoes (i.e., 1×2 tiles). Let $\mathcal{T}$ equal the set of all such tilings with a finite number of black squares and dominoes and let $t \in T$ refer to a tile (a white square, a black square, or a domino) in tiling T. Define the

weight of tile t as

$$w(t) = \begin{cases} aq^i & \text{if } t \text{ is a black square covering position } i \\ bq^i & \text{if } t \text{ is a domino covering positions } i \text{ and } i+1 \\ 1 & \text{if } t \text{ is a white square covering position } i \end{cases}$$

where a and b are free parameters. The weight of a tiling T is defined as

$$w(T) = \prod_{t \in T} w(t)$$

and the generating function that counts all tilings by weight is denoted by

$$F_q(a, b) = \sum_{T \in \mathcal{T}} w(T).$$

The following theorem provides the motivation for combinatorially proving (3) via tilings rather than other objects such as integer partitions.

Theorem 1

$$F_q(a, b) = \sum_{n \geq 0} \frac{(a+b)(a+bq) \cdots (a+bq^{n-1})}{(q)_n} q^{\binom{n+1}{2}}$$

Proof. We consider the following construction of a tiling T. First, select $n \geq 0$, which represents the total number of weighted tiles (i.e., black squares and/or dominoes) to be used in T. Next, select the initial positions of the weighted tiles. This corresponds to choosing a strictly increasing sequence $1 \leq i_1 < i_2 < \cdots < i_n$ which accounts for a q-weight of $q^{i_1 + i_2 + \cdots + i_n}$. And finally, place a black square or a domino in each of these positions. If a domino is placed in position i_j, then each of the $n - j$ weighted tiles to its right must be shifted one position to the right to guarantee room for the domino. In other words, the factor $a + bq^{n-j}$ represents the choice of making the jth weighted tile a black square or a domino, respectively. The factor of q^{n-j} accounts for the shift in position of the last $n - j$ weighted tiles. Therefore, we have

$$F_q(a, b) = \sum_{n \geq 0} \sum_{1 \leq i_1 < i_2 < \cdots < i_n} q^{i_1 + i_2 + \cdots + i_n}(a+b)(a+bq) \cdots (a+bq^{n-1})$$

$$= \sum_{n \geq 0} \frac{(a+b)(a+bq) \cdots (a+bq^{n-1})}{(q)_n} q^{\binom{n+1}{2}}$$

as desired. $\square$

As an example, consider the following construction of a tiling with $n = 9$ weighted tiles. First, select the initial positions of the weighted tiles to be $1, 2, 3, 6, 8, 9, 10, 11$, and 12. This can be realized by starting with a tiling that

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has black squares in these nine positions and white squares everywhere else, as illustrated below.

Second, select which of these weighted tiles will be converted into dominoes. Specifically, suppose we select the third, fifth and eighth weighted tiles, from left to right, to become dominoes. Convert the eighth weighted tile into a domino after shifting all tiles to its right by one position to the right.

Now convert the fifth weighted tile into a domino after shifting all tiles to its right by one position.

And finally, convert the third weighted tile into a domino after shifting all tiles to its right by one position.

This completes the construction.

We end this section by noting an obvious connection to a result of Euler. In particular, setting $b = 0$ eliminates all tilings that have at least one domino and yields
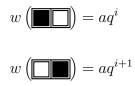
$$F_q(a, 0) = \prod_{n \geq 1} (1 + aq^n). \tag{6}$$

The right–hand side follows from the fact that to construct a tiling with no dominoes, one only needs to go through each position $n \geq 1$ and decide whether or not to place a black square in that position. (This is an alternative to the typical interpretation of (6) in terms of partitions into distinct parts.)

3   Recursive Formulas Involving $F_q(a, b)$

In this section, we consider the effect of replacing a with aq and b with bq in $F_q(a, b)$. If both of these replacements are made simultaneously, then each weighted tile is simply moved one position to the right, resulting in a tiling that must have a white square in the first position. We desire to understand each of these replacements individually with the ultimate goal of proving useful recurrences involving $F_q(a, b)$.

Lemma 2 *The generating function for tilings where at least $n \geq 0$ white squares and/or dominoes appear before the first black square, if any, is given by $F_q(aq^n, b)$.*

Proof. We proceed by induction. Clearly, $F_q(a, b)$ is the generating function for tilings where at least zero white squares and/or dominoes appear before the first black square. This is the basis case for our proof by induction. Next, we assume that $F_q(aq^n, b)$ is the generating function for tilings where at least n white squares and/or dominoes appear before the first black square. Now we replace a with aq in $F_q(aq^n, b)$, resulting in $F_q(aq^{n+1}, b)$. Combinatorially, this translates into increasing the q-weight of a tiling by the number of black squares. We will accomplish this by starting with the last (right-most) black square and working right to left. Suppose that the black square in position i is immediately followed by a white square in position $i + 1$. Switching the order of these two tiles increases the q-weight of the black square by a factor of q, as illustrated below.

$$w\left(\blacksquare\square\right) = aq^i$$

$$w\left(\square\blacksquare\right) = aq^{i+1}$$

On the other hand, if the black square in position i is immediately followed by a domino covering positions $i + 1$ and $i + 2$, then switching the order of these two tiles increases the q-weight of the black square by a factor of q^2 and decreases the q-weight of the domino by a factor of q. The cumulative effect is to increase the q-weight by a factor of q, as illustrated below.

$$w\left(\blacksquare\ \square\!\square\right) = aq^i \cdot bq^{i+1}$$
$$= abq^{2i+1}$$

$$w\left(\square\!\square\ \blacksquare\right) = bq^i \cdot aq^{i+2}$$
$$= abq^{2i+2}$$

By starting with the last black square and working right to left, we ensure that the above two cases will be the only cases encountered and in total, the q-weight will increase by a factor of q for each black square. Furthermore, every black square now has exactly one more white square and/or domino appearing to its left, including the first black square, if there is one.

It remains to show that this process is reversible. However, this is clearly achieved by working left to right and switching each black square with the tile immediately to its left. Therefore, $F_q(aq^{n+1}, b)$ must be the generating function

for tilings where at least $n + 1$ white squares and/or dominoes appear before the first black square, as required. $\square$

We can now use Lemma 2 as part of the following result which yields a useful recurrence for $F_q(a, b)$.

Lemma 3 $F_q(a, b) = F_q(aq, b) + aqF_q(aq, bq)$

Proof. Using the previous lemma, we know that the first term counts all tilings where the first position in the tiling is covered by a white square or a domino. It remains to count tilings where the first position is covered by a black square. To do so, start with any tiling and shift all weighted tiles one position to the right (i.e., replace a with aq and b with bq in $F_q(a, b)$). Now cover the first position of the tiling with a black square (i.e., multiply by aq). The result is $aqF_q(aq, bq)$, the generating function for tilings where the first position is covered by a black square. $\square$

For our next recursive formula, we will consider replacing b with bq in $F_q(a, b)$. However, instead of attempting to directly increase the weight of each domino in the tiling, we will shift every weighted tile to the right (i.e., replace a with aq and b with bq) and then shift each black square back to the left (i.e., replace a with a/q) as described at the end of the proof of Lemma 2.

Lemma 4 *The generating function for tilings where at least $n \geq 0$ white squares appear before the first domino, if any, is given by $F_q(a, bq^n)$.*

Proof. First note that $F_q(aq^n, bq^n)$ is the generating function for tilings where the first n positions (or more) are covered by white squares. Now switch the order of each black square with the tile immediately to its left, starting with the first (left-most) black square and working left to right. This is precisely the inverse of the procedure described in the proof of Lemma 2 and results in $F_q(aq^{n-1}, bq^n)$. Applying this procedure n times results in $F_q(a, bq^n)$. Since the relative order of the white squares and dominoes remains unchanged throughout the process, at least n white squares must appear before the first domino, as claimed. $\square$

Lemma 5 $F_q(a, b) = F_q(a, bq) + bqF_q(aq, bq^2)$

Proof. Using Lemma 4, the first term on the right–hand side of this identity counts all tilings where at least one white square appears before the first domino. It remains to count tilings where at least one domino appears before the first white square. To do this, start with a tiling where at least one white square appears before the first domino and then shift all weighted tiles one position to the right (i.e., replace a with aq and b with bq in $F_q(a, bq)$). The result is that $F_q(aq, bq^2)$ is the generating function for tilings where the first position is covered by a white square and at least two white squares (including

the white square in the first position) appear before the first domino.

Now suppose that a tiling starts with a white square, followed by $i \geq 0$ consecutive black squares, followed by another white square. Replace this sequence of tiles with i consecutive black squares followed by a domino. The cumulative effect of this process is to multiply the weight of the tiling by bq, as illustrated below.

$$w\left(\boxed{\square\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\square}\right) = aq^2 \cdot aq^3 \cdots aq^{i+1}$$
$$= a^i q^{\binom{i+2}{2}-1}$$

$$w\left(\boxed{\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\;\;}\right) = aq^1 \cdot aq^2 \cdots aq^i \cdot bq^{i+1}$$
$$= a^i bq^{\binom{i+2}{2}}$$

Since this process is completely reversible, $bqF_q(aq, bq^2)$ counts tilings where at least one domino appears before the first white square. $\square$

4   Proof of the Lebesgue and Göllnitz Identities

We are now in a position to prove (3). One can easily manipulate Lemmas 3 and 5 to discover the following theorem; however, we offer a purely combinatorial argument which will be used to shed some light on the product side of (3).

Theorem 6 *For $a \neq 0$,*

$$F_q(a,b) = (1 + bq/a)F_q(a, bq^2) + bq(1 - 1/a)F_q(aq, bq^2) \tag{7}$$

Proof.   We will prove the above identity by counting tilings based on how many white squares appear before the first domino. Using Lemma 4, we know that $F_q(a, bq^2)$ counts tilings where at least two white squares appear before the first domino, if any. From the proof of Lemma 5, we know that $bqF_q(aq, bq^2)$ counts tilings where at least one domino appears before the first white square.

It remains to count tilings where exactly one white square appears before the first domino. To this end, we point out that $F_q(a, bq^2) - F_q(aq, bq^2)$ is the generating function for tilings where the first position is occupied by a black square and at least two white squares appear before the first domino. Consider such a tiling where the first $i \geq 1$ positions are covered by black squares, followed by the first white square, followed by $j \geq 0$ black squares, followed by the second white square. Replace this sequence of $i + j + 2$ tiles with $i - 1$ consecutive black squares, followed by a white square, followed by

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j black squares, followed by a domino. The cumulative effect of this process is to multiply the weight of the tiling by bq/a, as illustrated below.

$$w\left(\blacksquare\blacksquare\blacksquare\square\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\square\right) = aq^1 \cdot aq^2 \cdots aq^i \cdot aq^{i+2} \cdots aq^{i+j+1}$$
$$= a^{i+j}q^{\binom{i+1}{2}+ij+\binom{j+2}{2}-1}$$

$$w\left(\blacksquare\blacksquare\square\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\square\right) = aq^1 \cdot aq^2 \cdots aq^{i-1} \cdot aq^{i+1} \cdots aq^{i+j}bq^{i+j+1}$$
$$= a^{i+j-1}bq^{\binom{i+1}{2}+ij+\binom{j+2}{2}}$$

And since this process is reversible, $\frac{bq}{a}(F_q(a, bq^2) - F_q(aq, bq^2))$ must count tilings where exactly one white square appears before the first domino. Thus

$$F_q(a, b) = F_q(a, bq^2) + bqF_q(aq, bq^2) + \frac{bq}{a}(F_q(a, bq^2) - F_q(aq, bq^2))$$
$$= (1 + bq/a)F_q(a, bq^2) + bq(1 - 1/a)F_q(aq, bq^2).$$

$\square$

The identities of Lebesgue and Göllnitz mentioned above are now immediate consequences of Theorem 6.

Corollary 7 *(Lebesgue)*

$$\sum_{n\geq 0} \frac{(-z; q)_n}{(q)_n}q^{\binom{n+1}{2}} = \prod_{n\geq 1}(1 + q^n)(1 + zq^{2n-1})$$

Proof. The left–hand side of this equality is simply $F_q(1, z)$. Setting $a = 1$ in (7), we have

$$F_q(1, z) = (1 + zq)F_q(1, zq^2)$$

Iterating this recursion yields

$$F_q(1, z) = F_q(1, 0) \prod_{n\geq 1}(1 + zq^{2n-1})$$

provided $|q| < 1$. Applying equation (6) completes the proof. $\square$

Corollary 8 *(Göllnitz)*

$$\sum_{n\geq 0} \frac{(-z; q^2)_n}{(q^2; q^2)_n}q^{n^2+n} = \prod_{n\geq 1}(1 + q^{4n-2})(1 + zq^{4n-2})(1 + q^{4n})$$

Proof. This result immediately follows from Corollary 7 by replacing q with q^2. $\square$

Combinatorially speaking, replacing q with q^2 means that the left–hand side of Corollary 8 is the generating function for tilings where weighted tiles only appear in even–numbered positions. In the case of each domino, the left half of the domino falls on an even–numbered position and must be followed by two white squares. Furthermore, the specific replacements required for equations (4) and (5), namely $z = q$ and $z = 1/q$, amount to shifting every domino one position to the right or left, respectively. Both of these operations are feasible since every domino must have a white square immediately before and after it.

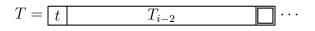## 5   Interpreting the Product Side of Lebesgue's Identity

Early in this work, we noted how the series side of (3) is easily interpreted as the generating function for the number of ways to tile a $1 \times \infty$ board using two colors of squares and one color of domino. However, in order to complete our purely combinatorial proof of (3), we need to provide a similar interpretation of the product side of (3) in terms of tilings. In this section, we will use the recursion from Theorem 6 to do exactly this. In particular, the proofs of Lemma 5 and Theorem 6 suggest that the black squares should be placed in the tiling before using the white squares as a guide for inserting dominoes. Furthermore, inserting a domino to the left of a white square in position i means that the tiles in the first $i - 1$ positions must be shifted to the left by one position, making room for a domino in positions $i - 1$ and i. Thus we are led to the following lemma, stated under the assumption that $a = 1$.

Lemma 9 *Suppose that the kth white square is in position $i > 1$ of tiling T and appears before the first domino. Let T' be formed by shifting the first $i - 1$ tiles to the left by one position (effectively removing the first tile of T), and covering positions $i - 1$ and i with a single domino. Then $w(T') = bq^{k-1}w(T)$.*
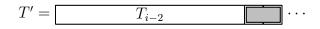
Proof. Shifting the first $i - 1$ tiles decreases the q-weight by one for each of the $i - k$ black squares that appear before the kth white square. Placing a domino in position $i - 1$ increases the q–weight of the tiling by $i - 1$. Thus the cumulative effect is to increase the weight of the tiling by a factor of bq^{k-1}, as claimed.   □

Note that in the process of shifting the first $i - 1$ tiles, we allow the first tile, t, to simply "fall off" of the board. However, if we apply the above lemma for certain values of k, then the number of white squares that appear before the first domino in T' can be used to encode the value of t. More specifically, if Lemma 9 is applied only when k is even, then t can be recovered from T' in the following manner. Suppose that the kth white square of tiling T is in position i where k is even and appears before the first domino. Then T can

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be decomposed as

$$T = \boxed{t \quad \boxed{ T_{i-2} } \quad \square} \cdots$$

where t is a black or white square and T_{i-2} is a tiling of a $1 \times (i-2)$ board that does not contain any dominoes. Accordingly, T' can then be decomposed in the following manner.

$$T' = \boxed{\boxed{ T_{i-2} } \quad \blacksquare} \cdots$$

Therefore, if T' contains an even number of white squares before the first domino (i.e., if T_{i-2} contains an even number of white squares), then tile t must have been a white square. On the other hand, if T' contains an odd number of white squares before the first domino, then tile t must have been a black square. This simple observation leads to our second proof of (3).

Proof of (3) – "Purely Combinatorial". We can construct any tiling in the following manner. Starting with an empty board, go through each position $n \geq 1$ and independently decide whether it should be covered by a white square or a black square, resulting in tiling T. This justifies each factor of $(1 + q^n)$.

The next step is to select a collection of distinct even integers, say $I = \{2n_1, 2n_2, \ldots, 2n_l\}$ where $0 < n_1 < n_2 < \cdots < n_l$. This collection will guide us in the construction of the following sequence of tilings

$$T = T^{(0)}, T^{(1)}, T^{(2)}, \ldots, T^{(l)}$$

where $T^{(i+1)}$ is obtained by applying Lemma 9 to $T^{(i)}$ with $k = 2n_{l-i}$. Note that $T^{(i+1)}$ will have at least $2n_{l-i} - 2 \geq 2n_{l-(i+1)}$ white squares before the first domino and thus Lemma 9 can be applied at each step. In other words, each factor of $(1 + bq^{2n-1})$ simply expresses the decision of whether or not to include $2n$ in I. The tiling $T^{(l)}$ is the final result of our construction.

Since this process of inserting dominoes is reversible, as described above, every tiling can be constructed in this manner. Thus

$$F_q(1, b) = \prod_{n \geq 1} (1 + q^n)(1 + bq^{2n-1})$$

as required. $\square$

At this point, an example should make our construction clear. Suppose that we take the term

$$q^2 \cdot q^4 \cdot q^5 \cdot q^9 \cdot q^{12} \cdot q^{13} \cdot q^{15} \cdot bq^{2-1} \cdot bq^{6-1} \cdot bq^{8-1} = b^3 q^{73} \tag{8}$$

from the expansion of the product

$$\prod_{n \geq 1}(1 + q^n)(1 + bq^{2n-1}).$$

We begin the construction of the corresponding tiling by placing black squares in positions $2, 4, 5, 9, 12, 13,$ and 15 and white squares in every other position.

$$T = \ \square\blacksquare\square\blacksquare\blacksquare\square\square\square\blacksquare\square\square\blacksquare\blacksquare\square\blacksquare\square \ \cdots$$

Now we insert three dominoes according to the set $\{2, 6, 8\}$. Applying Lemma 9 with $k = 8$ to T produces tiling $T^{(1)}$.

$$T^{(1)} = \ \blacksquare\square\blacksquare\blacksquare\square\square\square\blacksquare\square\square\blacksquare\blacksquare\boxed{}\blacksquare\square \ \cdots$$

Applying Lemma 9 with $k = 6$ to $T^{(1)}$ produces tiling $T^{(2)}$.

$$T^{(2)} = \ \square\blacksquare\blacksquare\square\square\square\blacksquare\square\boxed{}\blacksquare\blacksquare\boxed{}\blacksquare\square \ \cdots$$

And finally, applying Lemma 9 with $k = 2$ to $T^{(2)}$ produces tiling $T^{(3)}$

$$T^{(3)} = \ \blacksquare\blacksquare\boxed{}\square\square\blacksquare\square\boxed{}\blacksquare\blacksquare\boxed{}\blacksquare\square \ \cdots$$

which has weight

$$q^1 \cdot q^2 \cdot bq^3 \cdot q^7 \cdot bq^9 \cdot q^{11} \cdot q^{12} \cdot bq^{13} \cdot q^{15} = b^3 q^{73}.$$

The deconstruction process would proceed as follows. To reconstruct $T^{(2)}$ from $T^{(3)}$, note that there are zero white squares prior to the first domino in $T^{(3)}$, which covers positions 3 and 4. Thus the first position of $T^{(2)}$ is covered by a white square, followed by the first two tiles of $T^{(3)}$, followed by a white square. In other words, this first domino accounts for the factor of bq^1 in (8).

To reconstruct $T^{(1)}$ from $T^{(2)}$, note that $T^{(2)}$ has five white squares before the first domino, which covers positions 9 and 10. Thus the first position of $T^{(1)}$ is covered by a black square, followed by the first eight tiles of $T^{(2)}$, followed by a white square. In other words, the second domino accounts for the factor of bq^5 in (8).

And finally, to reconstruct T from $T^{(1)}$, note that $T^{(1)}$ has six white squares before the first domino, which covers positions 13 and 14. Thus the first position of T is covered by a white square, followed by the first twelve tiles of $T^{(1)}$, followed by a white square. In other words, the third domino accounts for the factor of bq^7 in (8).

6   Closing Remarks

It should be pointed out that the constructions presented in these proofs are closely related to the constructions in Alladi and Gordon [1]. In their work, each side of (3) is interpreted as the generating function for one of two different collections of bipartitions. Subsequently, each of these generating functions is shown to be equivalent to the generating function for partitions of n into distinct parts with "gaps". It is these partitions with distinct parts that are analogous to tilings, where dominoes mark the position of the "gaps".

By eliminating the need to convert partitions with distinct parts into bipartitions, we have greatly simplified the combinatorial description of Lebesgue's identity. In an upcoming paper, we will examine many other q–series identities, including some which appear in Slater's list, in the context of weighted tilings.

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