On the Dimension of the Space of Magic Squares Over a Field

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Abstract

We study various spaces of magic squares over a field, and determine their dimensions. These results generalize the main result of Small from 1988.

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1. Introduction

A *magic square of order* n over a field F is an $n \times n$ matrix with entries in F with the property that every row, every column, and the two main diagonals all have the same sum, called the *magic sum*. The set of all such magic squares over a given field F forms a vector space. In [4], Small proved that for $n \geq 5$, the dimension of this space of magic squares of order n is $n^2 - 2n$, independent of the field F. For $n < 5$, the results depend upon the characteristic of the field and are summarized in [4, page 622].

This definition of a magic square differs from the more traditional definition in which a magic square of order n is an $n \times n$ square which uses each of the numbers $0, 1, \ldots, n^2 - 1$ exactly once and for which each row, each column, and each of the two main diagonals has the constant sum $n(n^2 - 1)/2$, called the *magic sum*. Sometimes the square is based on the symbols $1, 2, \ldots, n^2$, in which case the magic sum is $n(n^2 + 1)/2$; see [1, pages 524-528] for properties and methods of construction for various kinds magic squares. Thompson [5] considers multiplicative properties of sets of magic squares (under normal matrix multiplication).

Let F be a field, $t \in F$, n a positive integer, and $0 \leq k \leq n$. Let $\mathcal{M}_{n,k}(t)$ be the set of all $n \times n$ matrices $[a_{ij}]_{0 \leq i,j \leq n-1}$ over F satisfying the following

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conditions:

$$\begin{aligned}
\sum_{j=0}^{n-1} a_{ij} &= t \quad \text{for all} \quad 0 \le i \le n-1 \quad \text{(row sums)} \\
\sum_{i=0}^{n-1} a_{ij} &= t \quad \text{for all} \quad 0 \le j \le n-1 \quad \text{(column sums)} \\
\sum_{j+i=-l-1} a_{ij} &= t \quad \text{for all} \quad 0 \le l \le k-1 \quad \text{(antidiagonal sums)} \\
\sum_{j-i=l} a_{ij} &= t \quad \text{for all} \quad 0 \le l \le k-1 \quad \text{(diagonal sums)}
\end{aligned}$$

where the subscripts are taken modulo n. Define $\mathcal{M}_{n,k} = \bigcup_{t \in F} \mathcal{M}_{n,k}(t)$.

Thus when $k = 0$, we have a square which is magic for all rows and columns but not necessarily for the two main diagonals. When $k = 1$, we have a magic square in the sense of Small [4]; and when $k = n$, we have a square in which each row, each column, and each of the $2n$ wrap around diagonals has the same sum. Following terminology from Latin squares, such a square might be called a *pandiagonal magic square*.

The problem which we wish to address in this paper is easily stated – for any field F and any $0 \le k \le n$, determine $\dim_F \mathcal{M}_{n,k}$. It is clear that

$$\dim_F \mathcal{M}_{n,k} = \dim_F \mathcal{M}_{n,k}(0) + \begin{cases} 1 & \text{if } \mathcal{M}_{n,k}(1) \neq \emptyset, \\ 0 & \text{if } \mathcal{M}_{n,k}(1) = \emptyset. \end{cases}$$

Our main results provide the determination of $\dim_F \mathcal{M}_{n,k}(0)$. The main theorems are stated in Section 2 and proved in Section 3. The remaining question is when $\mathcal{M}_{n,k}(1) \neq \emptyset$. We do not have the complete answer to the question. In Section 4, we include some sufficient conditions for $\mathcal{M}_{n,n}(1)$ to be nonempty.

2. Statement of main results

In this section, we give the main results of our paper. We break the results into two theorems based on whether the characteristic of F, which we denote by $\operatorname{char} F$, divides n or does not divide n.

Theorem 2.1. *Let $n \ge 0$ and let F be a field such that $\operatorname{char} F \nmid n$. Also, define $\alpha(n)$ by*

$$\alpha(n) := \begin{cases} 3 & \text{if } 2 \nmid n, \\ 4 & \text{if } 2 \mid n. \end{cases}$$

Then

$$\dim \mathcal{M}_{n,k}(0) = \begin{cases} n^2 - 4n + \alpha(n) & \text{if } n-1 \le k \le n, \\ n^2 - 2n - 2k + 1 & \text{if } 0 \le k \le n-2. \end{cases}$$

Theorem 2.2. *Let $n \geq 0$ and let F be a field such that* $\operatorname{char} F \mid n$*. Also, define $\beta_1(n)$ and $\beta_2(n)$ by*

$$\beta_1(n) := \begin{cases} 6 & \textit{if } 2 \nmid n, \\ 5 & \textit{if } 2 \mid n \textit{ and } \operatorname{char} F \nmid \frac{n}{2}, \\ 7 & \textit{if } 2 \mid n \textit{ and } \operatorname{char} F \mid \frac{n}{2} \end{cases}$$

and

$$\beta_2(n) := \begin{cases} 7 & \textit{if } 2 \nmid n \textit{ or } \operatorname{char} F \nmid \frac{n}{2}, \\ 8 & \textit{if } 2 \mid n \textit{ and } \operatorname{char} F \mid \frac{n}{2}. \end{cases}$$

Then

$$\dim \mathcal{M}_{n,k}(0) = \begin{cases} n^2 - 4n + \beta_1(n) & \textit{if } n-2 \leq k \leq n, \\ n^2 - 4n + \beta_2(n) & \textit{if } k = n-3, \\ n^2 - 2n - 2k + 1 & \textit{if } 0 \leq k \leq n-4. \end{cases}$$

3. Proofs of Theorems 2.1 and 2.2

We begin with the proof of Theorem 2.1.

Proof of Theorem 2.1

Throughout this proof, let F be a field such that char $F \nmid n$. Let $E_{ij} \in \mathrm{M}_{n\times n}(F)$ be the matrix whose (i,j) entry is 1 and whose other entries are 0. Put

$$A_i = \sum_{j=0}^{n-1} E_{ij}, \quad B_j = \sum_{i=0}^{n-1} E_{ij}, \quad C_l = \sum_{j+i=-l-1} E_{ij}, \quad D_l = \sum_{j-i=l} E_{ij}.$$

Endow $\mathrm{M}_{n\times n}(F)$ with the standard inner product

$$\langle [a_{ij}], [b_{ij}] \rangle = \sum_{i,j} a_{ij} b_{ij}.$$

Then clearly,

$$\mathcal{M}_{n,k}(0) = \{A_0, \ldots, A_{n-1}, B_0, \ldots, B_{n-1}, C_0, \ldots, C_{k-1}, D_0, \ldots, D_{k-1}\}^{\perp}.$$

Thus

$$\dim \mathcal{M}_{n,k}(0) = n^2 - \dim\langle A_0, \ldots, A_{n-1}, B_0, \ldots, B_{n-1}, C_0, \ldots, C_{k-1}, D_0, \ldots, D_{k-1}\rangle, \tag{1}$$

where $\langle \cdots \rangle$ denotes linear span.

Consider the equation

$$\sum_{i=0}^{n-1} \big[a(i)A_i + b(i)B_i + c(i)C_i + d(i)D_i\big] = 0, \tag{2}$$

where a, b, c, d are functions from $\mathbb{Z}_n$ to F to be determined. Entry wise, equation (2) is equivalent to

$$a(i) + b(j) + c(-j - i - 1) + d(j - i) = 0, \quad i, j \in \mathbb{Z}_n. \tag{3}$$

Let

$$\mathcal{S}_n = \{(a, b, c, d) : a, b, c, d : \mathbb{Z}_n \to F \text{ satisfy (3)}\},$$

and

$$\mathcal{S}_{n,k} = \{(a, b, c, d) \in \mathcal{S}_n : c(i) = d(i) = 0 \text{ for } k \le i < n\}.$$

Then

$$\dim\langle A_0, \ldots, A_{n-1}, B_0, \ldots, B_{n-1}, C_0, \ldots, C_{k-1}, D_0, \ldots, D_{k-1}\rangle = 2n+2k-\dim \mathcal{S}_{n,k}. \tag{4}$$

By (1) and (4),

$$\dim \mathcal{M}_{n,k}(0) = n^2 - 2n - 2k + \dim \mathcal{S}_{n,k}. \tag{5}$$

Now we see that the essential question is to solve the functional equation (3).

Lemma 3.1. *For $(a, b, c, d) \in \mathcal{S}_n$, the functions a, b, c, and d satisfy the following: $a(i) = \alpha$ for all $i \in \mathbb{Z}_n$, where $\alpha \in F$, $b(j) = \beta$ for all $j \in \mathbb{Z}_n$, where $\beta \in F$, and*

$$c(-j - 2i - 1) + d(j) = -\alpha - \beta. \tag{6}$$

Proof of Lemma 3.1. Let $(a, b, c, d) \in \mathcal{S}_n$. By (3),

$$\begin{aligned} na(i) &= -\sum_{j\in\mathbb{Z}_n} \big[b(j) + c(-j - i - 1) + d(j - i)\big] \\ &= -\sum_{j\in\mathbb{Z}_n} \big[b(j) + c(j) + d(j)\big]. \end{aligned}$$

So a is a constant function: $a(i) = \alpha$ for all $i \in \mathbb{Z}_n$, where $\alpha \in F$. In the same way, $b(j) = \beta$ for all $j \in \mathbb{Z}_n$, where $\beta \in F$. By (3),

$$c(-j - 2i - 1) + d(j) = -\alpha - \beta.$$

□

At this stage, we break our proof into two lemmas which depend on the parity of n.

Lemma 3.2. *If $2 \nmid n$, then*

$$\dim \mathcal{M}_{n,k}(0) = \begin{cases} n^2 - 4n + 3 & \text{if } k = n, \\ n^2 - 2n - 2k + 1 & \text{if } 0 \le k \le n - 1. \end{cases}$$

Proof of Lemma 3.2. By (6),

$$nd(j) = -n(\alpha+\beta) - \sum_{i\in\mathbb{Z}_n} c(-j-2i-1) = -(\alpha+\beta) - \sum_{i\in\mathbb{Z}_n} c(i).$$

Thus $d(j) = \delta$ for all $j \in \mathbb{Z}_n$, where $\delta \in F$. Hence the solutions of (3) are

$$\begin{cases} a(i) = \alpha, \\ b(i) = \beta, \\ c(i) = \gamma, \\ d(i) = \delta, \end{cases} \qquad i \in \mathbb{Z}_n,$$

where $\alpha, \beta, \gamma, \delta \in F$ and $\alpha+\beta+\gamma+\delta = 0$. When $0 \le k \le n-1$, the solutions of (3) in $\mathcal{S}_{n,k}$ are

$$\begin{cases} a(i) = \alpha, \\ b(i) = \beta, \\ c(i) = 0, \\ d(i) = 0, \end{cases} \qquad i \in \mathbb{Z}_n,$$

where $\alpha+\beta = 0$. It is clear that

$$\dim \mathcal{S}_{n,k} = \begin{cases} 3 & \text{if } k = n, \\ 1 & \text{if } 0 \le k \le n-1. \end{cases}$$

Thus equation (5) completes the proof. $\square$

Lemma 3.3. *If* $2 \mid n$, *then*

$$\dim \mathcal{M}_{n,k}(0) = \begin{cases} n^2 - 4n + 4 & \text{if } k = n \text{ or } n-1, \\ n^2 - 2n - 2k + 1 & \text{if } 0 \le k \le n-2. \end{cases}$$

Proof of Lemma 3.3. By (6),

$$nd(j) = -n(\alpha+\beta) - \sum_{i\in\mathbb{Z}_n} c(-j-2i-1) = -n(\alpha+\beta) - 2\left(\sum_{i\equiv -j-1 \pmod 2} c(i)\right).$$

So we have

$$d(j) = \begin{cases} \delta_0 & \text{if } j \equiv 0 \pmod 2, \\ \delta_1 & \text{if } j \equiv 1 \pmod 2, \end{cases}$$

where $\delta_0, \delta_1 \in F$. Therefore the solutions of (3) are

$$\begin{cases} a(i) = \alpha, \\ b(i) = \beta, \\ c(i) = \begin{cases} \gamma_0 & \text{if } i \equiv 0 \pmod 2, \\ \gamma_1 & \text{if } i \equiv 1 \pmod 2, \end{cases} \\ d(i) = \begin{cases} \delta_0 & \text{if } i \equiv 0 \pmod 2, \\ \delta_1 & \text{if } i \equiv 1 \pmod 2, \end{cases} \end{cases} \qquad i \in \mathbb{Z}_n,$$

where $\alpha, \beta, \gamma_0, \gamma_1, \delta_0, \delta_1 \in F$ and $\alpha + \beta + \gamma_0 + \delta_1 = 0$, $\alpha + \beta + \gamma_1 + \delta_0 = 0$.

When $k = n - 1$, the solutions of (3) in $\mathcal{S}_{n,k} = \mathcal{S}_{n,n-1}$ are

$$\begin{cases} a(i) = \alpha, \\ b(i) = \beta, \\ c(i) = \begin{cases} \gamma_0 & \text{if } i \equiv 0 \pmod 2, \\ 0 & \text{if } i \equiv 1 \pmod 2, \end{cases} \\ d(i) = \begin{cases} \delta_0 & \text{if } i \equiv 0 \pmod 2, \\ 0 & \text{if } i \equiv 1 \pmod 2, \end{cases} \end{cases} \qquad i \in \mathbb{Z}_n,$$

where $\alpha + \beta + \gamma_0 = 0$, $\alpha + \beta + \delta_0 = 0$. When $0 \le k \le n-2$, the solutions of (3) in $\mathcal{S}_{n,k}$ are

$$\begin{cases} a(i) = \alpha, \\ b(i) = \beta, \\ c(i) = 0, \\ d(i) = 0, \end{cases} \qquad i \in \mathbb{Z}_n,$$

where $\alpha + \beta = 0$. It is easy to see that

$$\dim \mathcal{S}_{n,k} = \begin{cases} 4 & \text{if } k = n, \\ 2 & \text{if } k = n-1, \\ 1 & \text{if } 0 \le k = n-2. \end{cases}$$

Thus (5) gives

$$\dim \mathcal{M}_{n,k}(0) = \begin{cases} n^2 - 4n + 4 & \text{if } k = n \text{ or } n-1, \\ n^2 - 2n - 2k + 1 & \text{if } 0 \le k \le n-2. \end{cases}$$

$\square$

Combining Lemmas 3.2 and 3.3 completes the proof of Theorem 2.1.

We now proceed to a proof of Theorem 2.2. Thus, we let F be a field such that $\operatorname{char} F \mid n$.

Lemma 3.4. *For $(a, b, c, d) \in \mathcal{S}_n$, we have*

$$(\Delta^2 d)(j) = (\Delta^2 c)(-j + 2i), \quad i, j \in \mathbb{Z}_n, \tag{7}$$

where $(\Delta c)(i) = c(i+1) - c(i)$.

Proof of Lemma 3.4. Let $f : \mathbb{Z}_n \times \mathbb{Z}_n \to F$ be a function. The equation

$$a(i) + b(j) = f(i, j), \quad i, j \in \mathbb{Z}_n \tag{8}$$

has a solution $a, b : \mathbb{Z}_n \to F$ if and only if

$$f(i,j) - f(i+1,j) - f(i,j+1) + f(i+1,j+1) = 0 \quad \text{for all } i, j \in \mathbb{Z}_n. \tag{9}$$

When (9) is satisfied, the solutions of (8) are

$$\begin{cases} a(i) = f(i,0) + \sigma, \\ b(i) = f(0,i) + \tau, \end{cases} \qquad i \in \mathbb{Z}_n, \tag{10}$$

where $\sigma + \tau = -f(0,0)$.

Assume that $(a,b,c,d) \in \mathcal{S}_n$. Put

$$f(i,j) = -c(-j-i-1) - d(j-i). \tag{11}$$

Then (9) becomes

$$\begin{aligned} 0 &= f(i,j) - f(i+1,j) - f(i,j+1) + f(i+1,j+1) \\ &= -c(-j-i-1) + 2c(-j-i-2) - c(-j-i-3) \\ &\quad + d(j-i+1) - 2d(j-i) + d(j-i-1) \\ &= (\Delta^2 d)(j-i-1) - (\Delta^2 c)(-j-i-3), \end{aligned}$$

where $(\Delta c)(i) = c(i+1) - c(i)$. The above equation is equivalent to

$$(\Delta^2 d)(j) = (\Delta^2 c)(-j+2i), \quad i,j \in \mathbb{Z}_n. \tag{12}$$

□

As with the proof of Theorem 2.1, we now break our proof into two lemmas which depend on the parity of n.

Lemma 3.5. *If $2 \nmid n$, then*

$$\dim \mathcal{M}_{n,k}(0) = \begin{cases} n^2 - 4n + 6 & \text{if } n-2 \le k \le n, \\ n^2 - 2n - 2k + 1 & \text{if } 0 \le k \le n-3. \end{cases}$$

Proof of Lemma 3.5. By (7) we have

$$(\Delta^2 d)(j) = (\Delta^2 c)(j) = \alpha \qquad \text{for all } j \in \mathbb{Z}_n,$$

where $\alpha \in F$. Thus

$$\begin{cases} c(j) = \frac{\alpha}{2} j^2 + \beta j + \gamma, \\ d(j) = \frac{\alpha}{2} j^2 + \beta' j + \gamma', \end{cases} \qquad j \in \mathbb{Z}_n, \tag{13}$$

where $\beta, \beta', \gamma, \gamma' \in F$. By (10), (11) and (13),

$$a(i) = f(i,0) + \sigma = -\alpha i^2 + (-\alpha + \beta + \beta')i - \frac{\alpha}{2} + \beta - \gamma - \gamma' + \sigma,$$

$$b(i) = f(0,i) + \tau = -\alpha i^2 + (-\alpha + \beta - \beta')i - \frac{\alpha}{2} + \beta - \gamma - \gamma' + \tau,$$

where

$$\sigma + \tau = -f(0,0) = \frac{\alpha}{2} - \beta + \gamma + \gamma'.$$

Thus the solutions of (3) are

$$\begin{cases} a(i) = -\alpha i^2 + (-\alpha + \beta + \beta')i + \sigma', \\ b(i) = -\alpha i^2 + (-\alpha + \beta - \beta')i + \tau', \\ c(i) = \frac{\alpha}{2} i^2 + \beta i + \gamma, \\ d(i) = \frac{\alpha}{2} i^2 + \beta' i + \gamma', \end{cases} \qquad i \in \mathbb{Z}_n, \tag{14}$$

where $\alpha, \beta, \beta', \gamma, \gamma', \sigma', \tau' \in F$ and $\sigma' + \tau' = -\frac{\alpha}{2} + \beta - \gamma - \gamma'$.

The F-map

$$\begin{array}{rccl} \phi : & \left\{ \begin{array}{c} (\alpha, \beta, \beta', \gamma, \gamma', \sigma', \tau') \in F^7 \\ : \sigma' + \tau' = -\frac{\alpha}{2} + \beta - \gamma - \gamma' \end{array} \right\} & \longrightarrow & \mathcal{S}_n \\ & (\alpha, \ldots, \tau') & \longmapsto & \text{the solution in (14)} \end{array} \tag{15}$$

is onto with $\ker \phi = 0$. Thus

$$\dim \mathcal{S}_n = 6.$$

When $k = n-1$, the solutions of (3) in $\mathcal{S}_{n,k} = \mathcal{S}_{n,n-1}$ are given by (14) subject to the conditions

$$\begin{cases} \sigma' + \tau' = -\frac{\alpha}{2} + \beta - \gamma - \gamma', \\ \frac{\alpha}{2} - \beta + \gamma = 0, \\ \frac{\alpha}{2} - \beta' + \gamma' = 0. \end{cases}$$

An argument similar to (15) gives

$$\dim \mathcal{S}_{n,n-1} = 4.$$

When $k = n-2$, the solutions of (3) in $\mathcal{S}_{n,k} = \mathcal{S}_{n,n-2}$ are given by (14) subject to the conditions

$$\begin{cases} \sigma' + \tau' = -\frac{\alpha}{2} + \beta - \gamma - \gamma', \\ \frac{\alpha}{2} - \beta + \gamma = 0, \\ \frac{\alpha}{2} - \beta' + \gamma' = 0, \\ 2\alpha - 2\beta + \gamma = 0, \\ 2\alpha - 2\beta' + \gamma' = 0. \end{cases} \tag{16}$$

The system (16) is equivalent to

$$\begin{cases} \sigma' + \tau' = -\frac{\alpha}{2} + \beta - \gamma - \gamma', \\ \beta = \beta' = \frac{3}{2}\alpha, \\ \gamma = \gamma' = \alpha. \end{cases}$$

Then it is easy to see that

$$\dim \mathcal{S}_{n,n-2} = 2.$$

When $0 \leq k \leq n-3$, the solutions of (3) in $\mathcal{S}_{n,k}$ are given by (14) subject to the conditions

$$\begin{cases} \sigma' + \tau' = 0, \\ \alpha = \beta = \beta' = \gamma = \gamma' = 0. \end{cases}$$

Thus

$$\dim \mathcal{S}_{n,k} = 1.$$

Therefore we have

$$\dim \mathcal{M}_{n,k}(0) = \begin{cases} n^2 - 4n + 6 & \text{if } n-2 \leq k \leq n, \\ n^2 - 2n - 2k + 1 & \text{if } 0 \leq k \leq n-3. \end{cases}$$

□

Lemma 3.6. *If $2 \mid n$, then*

$$\dim \mathcal{M}_{n,k}(0) = \begin{cases} n^2 - 4n + 7 & \text{if } n-2 \leq k \leq n, \\ n^2 - 4n + 8 & \text{if } k = n-3, \\ n^2 - 2n - 2k + 1 & \text{if } 0 \leq k \leq n-4. \end{cases}$$

Proof of Lemma 3.6. By (7) we have

$$(\Delta^2 c)(j) = (\Delta^2 d)(j) = \begin{cases} \alpha_0 & \text{if } j \equiv 0 \pmod 2, \\ \alpha_1 & \text{if } j \equiv 1 \pmod 2, \end{cases}$$

where $\alpha_0, \alpha_1 \in F$. Thus

$$\begin{aligned} (\Delta c)(2j) &= (\Delta^2 c)(0) + \cdots + (\Delta^2 c)(2j-1) + \beta \qquad (\beta = (\Delta c)(0)) \\ &= \alpha_0 + \alpha_1 + \cdots + \alpha_0 + \alpha_1 + \beta \\ &= j\alpha_0 + j\alpha_1 + \beta, \end{aligned}$$

and

$$(\Delta c)(2j+1) = (j+1)\alpha_0 + j\alpha_1 + \beta.$$

Consequently,

$$\begin{aligned} c(2j) = {} & (\Delta c)(0) + \cdots + (\Delta c)(2j-1) + \gamma \qquad (\gamma = c(0)) \\ = {} & \quad 0\alpha_0 + 0\alpha_1 + \beta \\ & + 1\alpha_0 + 0\alpha_1 + \beta \\ & + 1\alpha_0 + 1\alpha_1 + \beta \\ & + 2\alpha_0 + 1\alpha_1 + \beta \\ & \quad \vdots \\ & + (j-1)\alpha_0 + (j-1)\alpha_1 + \beta \\ & + j\alpha_0 + (j-1)\alpha_1 + \beta + \gamma \\ = {} & j^2\alpha_0 + j(j-1)\alpha_1 + 2j\beta + \gamma, \end{aligned} \tag{17}$$

and

$$c(2j+1) = j(j+1)\alpha_0 + j^2\alpha_1 + (2j+1)\beta + \gamma. \tag{18}$$

In the same way,

$$\begin{cases} d(2j) = j^2\alpha_0 + j(j-1)\alpha_1 + 2j\beta' + \gamma', \\ d(2j+1) = j(j+1)\alpha_0 + j^2\alpha_1 + (2j+1)\beta' + \gamma'. \end{cases} \tag{19}$$

From (10), (11), (17) – (19), we have

$$\begin{aligned}
&a(2i) = f(2i,0) + \sigma \\
= &-2(\alpha_0+\alpha_1)i^2 + (-\alpha_0 - 3\alpha_1 + 2\beta + 2\beta')i - \alpha_1 + \beta - \gamma - \gamma' + \sigma,
\end{aligned}$$

$$\begin{aligned}
&a(2i+1) = f(2i+1,0) + \sigma \\
= &-2(\alpha_0+\alpha_1)i^2 + (-3\alpha_0 - 5\alpha_1 + 2\beta + 2\beta')i - \alpha_0 - 3\alpha_1 + 2\beta + \beta' - \gamma - \gamma' + \sigma,
\end{aligned}$$

$$\begin{aligned}
&b(2i) = f(0,2i) + \tau \\
= &-2(\alpha_0+\alpha_1)i^2 + (-\alpha_0 - \alpha_1 + 2\beta - 2\beta')i - \alpha_1 + \beta - \gamma - \gamma' + \tau,
\end{aligned}$$

$$\begin{aligned}
&b(2i+1) = f(0,2i+1) + \tau \\
= &-2(\alpha_0+\alpha_1)i^2 + (-3\alpha_0 - 3\alpha_1 + 2\beta - 2\beta')i - \alpha_0 - 2\alpha_1 + 2\beta - \beta' - \gamma - \gamma' + \tau,
\end{aligned}$$

where

$$\sigma + \tau = -f(0,0) = \alpha_1 - \beta + \gamma + \gamma'.$$

It is important to note that in order for the functions a, b, c, d obtained above to be well defined on $\mathbb{Z}_n$, it is necessary and sufficient that

$$\begin{cases} \frac{n}{2}\Big(\Big(\frac{n}{2}+1\Big)\alpha_0 + \frac{n}{2}\alpha_1\Big) = 0, \\ \frac{n(n+1)}{2}(\alpha_0+\alpha_1) = 0. \end{cases} \tag{20}$$

Therefore the solutions of (3) are given by

$$\begin{cases}
a(2i) = -2(\alpha_0+\alpha_1)i^2 + (-\alpha_0 - 3\alpha_1 + 2\beta + 2\beta')i + \sigma', \\
a(2i+1) = -2(\alpha_0+\alpha_1)i^2 + (-3\alpha_0 - 5\alpha_1 + 2\beta + 2\beta')i - \alpha_0 - 2\alpha_1 + \beta + \beta' + \sigma', \\
b(2i) = -2(\alpha_0+\alpha_1)i^2 + (-\alpha_0 - \alpha_1 + 2\beta - 2\beta')i + \tau', \\
b(2i+1) = -2(\alpha_0+\alpha_1)i^2 + (-3\alpha_0 - 3\alpha_1 + 2\beta - 2\beta')i - \alpha_0 - \alpha_1 + \beta - \beta' + \tau', \\
c(2i) = i^2\alpha_0 + i(i-1)\alpha_1 + 2i\beta + \gamma, \\
c(2i+1) = i(i+1)\alpha_0 + i^2\alpha_1 + (2i+1)\beta + \gamma, \\
d(2i) = i^2\alpha_0 + i(i-1)\alpha_1 + 2i\beta' + \gamma', \\
d(2i+1) = i(i+1)\alpha_0 + i^2\alpha_1 + (2i+1)\beta' + \gamma',
\end{cases} \tag{21}$$

where $\alpha_0, \alpha_1, \beta, \beta', \gamma, \gamma', \sigma', \tau' \in F$, $\sigma' + \tau' = -\alpha_1 + \beta - \gamma - \gamma'$, and α_0, α_1 satisfy (20).

Case 1. Assume $\operatorname{char} F \nmid \frac{n}{2}$. Then $\operatorname{char} F = 2$. From (20) we have $\alpha_0 = \alpha_1 = 0$. Thus (21) becomes

$$\begin{cases} a(2i) = \sigma', \\ a(2i+1) = \beta + \beta' + \sigma', \\ b(2i) = \tau', \\ b(2i+1) = \beta - \beta' + \tau', \\ c(2i) = \gamma, \\ c(2i+1) = \beta + \gamma, \\ d(2i) = \gamma', \\ d(2i+1) = \beta' + \gamma', \end{cases} \qquad i \in \mathbb{Z}_n, \tag{22}$$

where $\beta, \beta', \gamma, \gamma', \sigma', \tau' \in F$ and $\sigma' + \tau' = \beta - \gamma - \gamma'$.

The F-map

$$\begin{array}{rccc} \psi : & \left\{ \begin{array}{l} (\beta, \beta', \gamma, \gamma', \sigma', \tau') \in F^6 \\ : \sigma' + \tau' = \beta - \gamma - \gamma' \end{array} \right\} & \longrightarrow & \mathcal{S}_n \\ & (\beta, \ldots, \tau') & \longmapsto & \text{the solution in (22)} \end{array}$$

is onto with $\ker \psi = 0$. Thus

$$\dim \mathcal{S}_n = 5.$$

When $k = n-1$, the solutions of (3) in $\mathcal{S}_{n,k} = \mathcal{S}_{n,n-1}$ are given by (22) subject to the conditions $\sigma' + \tau' = \beta - \gamma - \gamma'$, $\beta + \gamma = 0$, $\beta' + \gamma' = 0$. Thus

$$\dim \mathcal{S}_{n,n-1} = 3.$$

When $0 \le k \le n-2$, the solutions of (3) in $\mathcal{S}_{n,k}$ are given by (22) subject to the conditions $\sigma' + \tau' = 0$, $\beta = \beta' = \gamma = \gamma' = 0$. Thus

$$\dim \mathcal{S}_{n,k} = 1.$$

It follows from (5) that

$$\dim \mathcal{M}_{n,k}(0) = \begin{cases} n^2 - 4n + 5 & \text{if } k = n \text{ or } n-1, \\ n^2 - 2n - 2k + 1 & \text{if } 0 \le k \le n-2. \end{cases}$$

Case 2. Assume $\operatorname{char} F \mid \frac{n}{2}$. Note that (20) is automatically satisfied. The F-map

$$\begin{array}{rccc} \Phi : & \left\{ \begin{array}{l} (\alpha_0, \alpha_1, \beta, \beta', \gamma, \gamma', \sigma', \tau') \in F^8 \\ : \sigma' + \tau' = -\alpha_1 + \beta - \gamma - \gamma' \end{array} \right\} & \longrightarrow & \mathcal{S}_n \\ & (\alpha_0, \ldots, \tau') & \longmapsto & \text{the solution in (21)} \end{array}$$

is onto with $\ker \Phi = 0$. So
$$\dim \mathcal{S}_n = 7.$$

When $k = n-1$, the solutions of (3) in $\mathcal{S}_{n,k} = \mathcal{S}_{n,n-1}$ are given by (21) subject to the conditions

$$\begin{cases} \sigma' + \tau' = -\alpha_1 + \beta - \gamma - \gamma', \\ \alpha_1 - \beta + \gamma = 0, \\ \alpha_1 - \beta' + \gamma' = 0. \end{cases}$$

Therefore
$$\dim \mathcal{S}_{n,n-1} = 5.$$

When $k = n-2$, the solutions of (3) in $\mathcal{S}_{n,k} = \mathcal{S}_{n,n-2}$ are given by (21) subject to the conditions

$$\begin{cases} \sigma' + \tau' = -\alpha_1 + \beta - \gamma - \gamma', \\ \alpha_1 - \beta + \gamma = 0, \\ \alpha_1 - \beta' + \gamma' = 0, \\ \alpha_0 + 2\alpha_1 - 2\beta + \gamma = 0, \\ \alpha_0 + 2\alpha_1 - 2\beta' + \gamma' = 0. \end{cases} \tag{23}$$

The system (23) is equivalent to

$$\begin{cases} \sigma' + \tau' = -\alpha_1 + \beta - \gamma - \gamma', \\ \alpha_0 = \gamma = \gamma', \\ \alpha_1 = \beta - \gamma, \\ \beta = \beta'. \end{cases}$$

Thus it is easy to see that
$$\dim \mathcal{S}_{n,n-2} = 3.$$

When $k = n-3$, the solutions of (3) in $\mathcal{S}_{n,k} = \mathcal{S}_{n,n-3}$ are given by (21) subject to the conditions

$$\begin{cases} \sigma' + \tau' = -\alpha_1 + \beta - \gamma - \gamma', \\ \alpha_0 = \beta = \beta' = \gamma = \gamma', \\ \alpha_1 = 0. \end{cases}$$

Hence
$$\dim \mathcal{S}_{n,n-3} = 2.$$

When $0 \le k \le n-4$, the solutions of (3) in $\mathcal{S}_{n,k}$ are given by (21) subject to the conditions

$$\begin{cases} \sigma' + \tau' = -\alpha_1 + \beta - \gamma - \gamma', \\ \alpha_0 = \alpha_1 = \beta = \beta' = \gamma = \gamma' = 0. \end{cases}$$

Thus

$$\dim \mathcal{S}_{n,k} = 1.$$

Now (5) gives

$$\dim \mathcal{M}_{n,k}(0) = \begin{cases} n^2 - 4n + 7 & \text{if } n-2 \le k \le n, \\ n^2 - 4n + 8 & \text{if } k = n-3, \\ n^2 - 2n - 2k + 1 & \text{if } 0 \le k \le n-4. \end{cases}$$

□

Combining Lemmas 3.5 and 3.6 yields Theorem 2.2.

4. Sufficient conditions for $\mathcal{M}_{n,n}(1)$ to be nonempty

In [4], Small proved that

$$\mathcal{M}_{n,1}(1) \begin{cases} = \emptyset & \text{if } \operatorname{char} F = n = 2 \text{ or } \operatorname{char} F = n = 3, \\ \neq \emptyset & \text{otherwise.} \end{cases}$$

It follows that $\mathcal{M}_{n,n}(1) = \emptyset$ if $\operatorname{char} F = n = 2$ or $\operatorname{char} F = n = 3$. In this section, we collect some sufficient conditions for $\mathcal{M}_{n,n}(1)$ to be nonempty.

Fact 4.1. *If* $\operatorname{char} F \nmid n$*, then* $\mathcal{M}_{n,n}(1) \neq \emptyset$.

In fact,

$$\frac{1}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \in \mathcal{M}_{n,n}(1).$$

Fact 4.2. *If* $2 \nmid n$ *and* $3 \nmid n$*, then* $\mathcal{M}_{n,n}(1) \neq \emptyset$.

To see this fact, let $\sigma(i) = 2i$, $i \in \mathbb{Z}_n$. Then σ, $\sigma - \mathrm{id}$, $\sigma + \mathrm{id}$ are all permutations of $\mathbb{Z}_n$. It is easy to see that the permutation matrix $[a_{i,j}]$ of σ belongs to $\mathcal{M}_{n,n}(1)$, where

$$a_{ij} = \begin{cases} 1 & \text{if } j = 2i, \\ 0 & \text{otherwise.} \end{cases}$$

Fact 4.3. *If* $A \in \mathcal{M}_{m,m}(\alpha)$, $B \in \mathcal{M}_{n,n}(\beta)$*, where* m, n *are positive integers and* $\alpha, \beta \in F$*, then* $A \otimes B \in \mathcal{M}_{mn,mn}(\alpha\beta)$*. In particular, if* $\mathcal{M}_{m,m}(1) \neq \emptyset$ *and* $\mathcal{M}_{n,n}(1) \neq \emptyset$*, then* $\mathcal{M}_{mn,mn}(1) \neq \emptyset$.

We leave the proof of Fact 4.3 to the reader.

By Fact 4.1 and 4.2, $\mathcal{M}_{n,n}(1)$ can possibly be empty only when $\operatorname{char} F = 2$ or 3 and $\operatorname{char} F \mid n$.

Example 4.4. Assume $\operatorname{char} F = 2$. We have $\mathcal{M}_{4,4}(1) \neq \emptyset$ since

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_{4,4}(1).$$

Interested readers may compare this example with Example (a) in [4, §2]. The matrix there belongs to $\mathcal{M}_{4,1}(1)$ but not to $\mathcal{M}_{4,2}(1)$.

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