On m-ary Overpartitions

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Abstract

Presently there is a lot of activity in the study of overpartitions, objects that were discussed by MacMahon, and which have recently proven useful in several combinatorial studies of basic hypergeometric series. In this paper we study some similar objects, which we name m-ary overpartitions. We consider divisibility properties of the number of m-ary overpartitions of a natural number, and we prove a theorem which is a lifting to general m of the well-known Churchhouse congruences for the binary partition function.

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1 Introduction

Presently there is a lot of activity in the study of the objects named *overpartitions* by Corteel and Lovejoy [2]. According to Corteel and Lovejoy, these objects were discussed by MacMahon and have recently proven useful in several combinatorial studies of basic hypergeometric series. In this paper we study some similar objects, which we name m-ary overpartitions.

Let $m \geq 2$ be an integer. An m-ary partition of a natural number n is a non-increasing sequence of non-negative integral powers of m whose sum is n. An m-ary overpartition of n is a non-increasing sequence of non-negative integral powers of m whose sum is n, and where the first occurrence of a power of m may be overlined. We denote the number of m-ary overpartitions of n by $\overline{b}_m(n)$. The overlined parts form an m-ary partition into distinct parts, and the non-overlined parts form an ordinary m-ary partition. Thus, putting $\overline{b}_m(0) = 1$, we have the generating function

$$F_m(q) = \sum_{n=0}^{\infty} \overline{b}_m(n) q^n = \prod_{i=0}^{\infty} \frac{1+q^{m^i}}{1-q^{m^i}}.$$

For example, for $m = 2$, we find

$$\sum_{n=0}^{\infty} \overline{b}_2(n) q^n = 1 + 2q + 4q^2 + 6q^3 + 10q^4 + 14q^5 + \dots,$$

where the 10 binary overpartitions of 4 are

$$1+1+1+1,\ \overline{1}+1+1+1,\ 2+1+1,\ 2+\overline{1}+1,$$

$$\overline{2}+1+1,\ \overline{2}+\overline{1}+1,\ 2+2,\ \overline{2}+2,\ 4,\ \overline{4}.$$

The main object of this paper is to prove the following theorem.

Theorem 1 *For each integer $r \geq 1$, we have*

$$\overline{b}_m(m^{r+1}n) - \overline{b}_m(m^{r-1}n) \equiv 0 \pmod{4m^{\lfloor 3r/2 \rfloor}/c^{\lfloor (r-1)/2 \rfloor}},$$

where $c = \gcd(3, m)$.

Putting $m = 2$ in Theorem 1, we get

$$\overline{b}_2(2^{r+1}n) - \overline{b}_2(2^{r-1}n) \equiv 0 \pmod{2^{\lfloor 3r/2 \rfloor + 2}} \quad \text{for } r \geq 1. \tag{1}$$

Writing $b_2(n)$ for the number of binary partitions of n, we have, as noted in the next section, that $\overline{b}_2(n) = b_2(2n)$. Thus (1) can be written as

$$b_2(2^{r+2}n) - b_2(2^r n) \equiv 0 \pmod{2^{\lfloor 3r/2 \rfloor + 2}} \quad \text{for } r \geq 1. \tag{2}$$

This result was conjectured by Churchhouse [1]. A number of proofs of (2) have been given by several authors; cf. [4]. Families of congruences also appear in the literature for the m-ary partition function which are valid for any $m \geq 2$; cf. [3]. But as far as we know, none of these m-ary results give the Churchhouse congruences when $m = 2$. So, Theorem 1 seems to be the first known lifting to general m of the Churchhouse congruences for the binary partition function.

We prove Theorem 1 by adapting the technique used in [3]. In Section 2 below we introduce some tools and prove three lemmata. In Section 3 we complete the proof of Theorem 1. Finally, in Section 4 we state a theorem which generalizes Theorem 1 and sketch its proof.

2 Auxiliaries

Although many of the objects below depend on m or q (or both), such dependence will sometimes be suppressed by the chosen notation.

The power series in this paper will be elements of $\mathbb{Z}[[q]]$, the ring of formal power series in q with coefficients in $\mathbb{Z}$. We define a $\mathbb{Z}$-linear operator

$$U : \mathbb{Z}[[q]] \longrightarrow \mathbb{Z}[[q]]$$

by

$$U \sum_n a(n)q^n = \sum_n a(mn)q^n.$$

Notice that if $f(q), g(q) \in \mathbb{Z}[[q]]$, then

$$U(f(q)g(q^m)) = (Uf(q))g(q). \tag{3}$$

Moreover, if $f(q) = \sum_n a(n)q^n \in \mathbb{Z}[[q]]$, and M is a positive integer, then we have

$$f(q) \equiv 0 \pmod{M} \qquad (\text{in } \mathbb{Z}[[q]])$$

if and only if, for all n,

$$a(n) \equiv 0 \pmod{M} \qquad (\text{in } \mathbb{Z}).$$

At this point, a note is in order on the relationship between the binary partition function $b_2(n)$ and the binary overpartition function $\overline{b}_2(n)$. Since each natural number has a unique representation as a sum of distinct non-negative powers of 2, we have

$$\prod_{i=0}^{\infty}(1+q^{2^i}) = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q},$$

so that

$$\sum_{n=0}^{\infty} \overline{b}_2(n)q^n = \frac{1}{1-q}\prod_{i=0}^{\infty}\frac{1}{1-q^{2^i}}. \tag{4}$$

For the binary partition function $b_2(n)$, we have

$$\sum_{n=0}^{\infty} b_2(n)q^n = \prod_{i=0}^{\infty}\frac{1}{1-q^{2^i}} = \frac{1}{1-q}\prod_{i=0}^{\infty}\frac{1}{1-q^{2^{i+1}}}.$$

Applying the U-operator with $m=2$, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} b_2(2n)q^n &= U\left(\frac{1}{1-q}\prod_{i=0}^{\infty}\frac{1}{1-q^{2^{i+1}}}\right) \\
&= \left(U\frac{1}{1-q}\right)\prod_{i=0}^{\infty}\frac{1}{1-q^{2^i}} && \text{by (3)} \\
&= \frac{1}{1-q}\prod_{i=0}^{\infty}\frac{1}{1-q^{2^i}} \\
&= \sum_{n=0}^{\infty} \overline{b}_2(n)q^n && \text{by (4),}
\end{aligned}$$

so that

$$\overline{b}_2(n) = b_2(2n),$$

as mentioned in the Introduction.

Alternatively, it is rather easy to construct a bijection between the set of binary overpartitions of n and the set of binary partitions of $2n$: Consider a binary overpartition of n. Multiply each part by 2. Write the overlined parts as sums of 1s. Then we have a binary partition of $2n$. On the other hand, let a binary partition of $2n$ be given. The number of 1s is even, and the sum of 1s can in a unique way be written as a sum of distinct positive powers of 2. Overline these powers of 2. Divide all parts by 2. Then we have a binary overpartition of n.

For example, starting with a binary overpartition of 11, we get

$$4 + \overline{2} + 2 + \overline{1} + 1 + 1 \rightarrow 8 + \overline{4} + 4 + \overline{2} + 2 + 2 \rightarrow$$

$$8 + 1 + 1 + 1 + 1 + 4 + 1 + 1 + 2 + 2 = 8 + 4 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1,$$

which is a binary partition of 22. Conversely, starting with this binary partition of 22, we find

$$8 + 4 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 \rightarrow 8 + 4 + 2 + 2 + \overline{4} + \overline{2}$$

$$\rightarrow 4 + 2 + 1 + 1 + \overline{2} + \overline{1} = 4 + \overline{2} + 2 + \overline{1} + 1 + 1,$$

which is the original binary overpartition of 11.

We shall use the following result for binomial coefficients.

Lemma 1 *For each positive integer r there exist unique integers $\alpha_r(i)$, such that for all n,*

$$\binom{mn + r - 1}{r} = \sum_{i=1}^{r} \alpha_r(i) \binom{n + i - 1}{i}. \tag{5}$$

Proof. See the proof of [3, Lemma 1]. ∎

Comparing the coefficients of n^r in (5), we get

$$\alpha_r(r) = m^r,$$

and comparing the coefficients of m^{r-1}, we get

$$\alpha_r(r-1) = -\tfrac{1}{2}(r-1)(m-1)m^{r-1},$$

so that

$$\alpha_{r-1}(r-1) - 2\alpha_r(r-1) = (rm - m - r + 2)m^{r-1}. \tag{6}$$

We also note that by setting $n = -j$ in (5), we get

$$(-1)^j \alpha_r(j) = (-1)^r \binom{mj}{r} - \sum_{i=1}^{j-1} (-1)^i \binom{j}{i} \alpha_r(i), \quad j = 1, 2, \ldots, r.$$

Next, we put

$$h_i = h_i(q) = \frac{q}{(1-q)^{i+1}} \qquad \text{for } i \geq 0.$$

Then

$$h_i = \sum_{n=1}^{\infty} \binom{n+i-1}{i} q^n, \tag{7}$$

so that

$$Uh_r = \sum_{n=1}^{\infty} \binom{mn+r-1}{r} q^n.$$

It follows from Lemma 1 and (7) that

$$Uh_r = \sum_{i=1}^{r} \alpha_r(i) h_i \qquad \text{for } r \geq 1. \tag{8}$$

In particular,

$$\begin{aligned} Uh_1 &= mh_1, && (9)\\ Uh_2 &= m^2 h_2 - \tfrac{1}{2}(m-1)mh_1, && (10)\\ Uh_3 &= m^3 h_3 - (m-1)m^2 h_2 + \tfrac{1}{6}(m-2)(m-1)mh_1. \end{aligned}$$

Let

$$M_1 = 4mh_1 \qquad \text{and} \quad M_{i+1} = U\left(\frac{1+q}{1-q}M_i\right) \quad \text{for } i \geq 1.$$

Then

$$\begin{aligned} M_2 &= U\left(\left(\frac{2}{1-q} - 1\right) 4mh_1\right) \\ &= 4m(2Uh_2 - Uh_1) \\ &= 4m(2m^2h_2 - m^2h_1) \quad \text{by (9) and (10)} \\ &= 2^3m^3h_2 - m^2M_1. \end{aligned}$$

Similarly, we find

$$M_3 = 2^4m^6h_3 - 2m^3M_2 - \tfrac{1}{3}(2m-1)(2m+1)m^3M_1.$$

Lemma 2 *For each positive integer r there exist integers $\mu_r(i)$ such that*

$$M_r = 2^{r+1}m^{r(r+1)/2}h_r - \sum_{i=1}^{r-1}\mu_r(i)M_i, \tag{11}$$

where

$$3\mu_r(i) \equiv 0 \pmod{m^{\lfloor (3(r-i)+1)/2 \rfloor}} \tag{12}$$

for $i = 1, 2, \ldots, r-1$.

Note. In the following we set $\mu_r(r) = 1$ and $\mu_r(0) = 0$ for $r \geq 1$. Notice that these values of μ satisfy (12).

Proof. We use induction on r. The lemma is true for $r = 1$. Suppose that for some $r > 1$, we have

$$M_j = 2^{j+1}m^{j(j+1)/2}h_j - \sum_{i=1}^{j-1}\mu_j(i)M_i \quad \text{for } j = 1, 2, \ldots, r-1, \tag{13}$$

where all the $\mu_j(i)$ are integers satisfying

$$3\mu_j(i) \equiv 0 \pmod{m^{\lfloor (3(j-i)+1)/2 \rfloor}}, \quad i = 1, 2, \ldots, j-1.$$

Then, using (13) with $j = r-1$, we get

$$\begin{aligned} M_r &= U\left(\frac{1+q}{1-q}M_{r-1}\right) \\ &= 2^r m^{r(r-1)/2} U\left(\left(\frac{2}{1-q}-1\right)h_{r-1}\right) - \sum_{i=1}^{r-2}\mu_{r-1}(i)U\left(\frac{1+q}{1-q}M_i\right) \\ &= 2^{r+1}m^{r(r-1)/2}Uh_r - 2^r m^{r(r-1)/2}Uh_{r-1} - \sum_{i=1}^{r-2}\mu_{r-1}(i)M_{i+1}. \end{aligned}$$

By (8), we further get

$$\begin{aligned} M_r &= 2^{r+1}m^{r(r-1)/2}\sum_{i=1}^{r}\alpha_r(i)h_i - 2^r m^{r(r-1)/2}\sum_{i=1}^{r-1}\alpha_{r-1}(i)h_i \\ &\quad - \sum_{i=1}^{r-2}\mu_{r-1}(i)M_{i+1} \\ &= 2^{r+1}m^{r(r+1)/2}h_r - \sum_{i=2}^{r-1}\mu_{r-1}(i-1)M_i \\ &\quad - \sum_{j=1}^{r-1}2^r m^{r(r-1)/2}(\alpha_{r-1}(j)-2\alpha_r(j))h_j. \end{aligned}$$

Moreover, by (13),

$$\begin{aligned} M_r &= 2^{r+1}m^{r(r+1)/2}h_r - \sum_{i=2}^{r-1}\mu_{r-1}(i-1)M_i \\ &\quad - \sum_{j=1}^{r-1}2^{r-1-j}m^{r(r-1)/2-j(j+1)/2}(\alpha_{r-1}(j)-2\alpha_r(j))\sum_{i=1}^{j}\mu_j(i)M_i \\ &= 2^{r+1}m^{r(r+1)/2}h_r - \sum_{i=2}^{r-1}\mu_{r-1}(i-1)M_i \\ &\quad - \sum_{i=1}^{r-1}\sum_{j=i}^{r-1}2^{r-1-j}m^{r(r-1)/2-j(j+1)/2}(\alpha_{r-1}(j)-2\alpha_r(j))\mu_j(i)M_i. \end{aligned}$$

Thus (11) holds with

$$(14)\quad \mu_r(i) = \mu_{r-1}(i-1)$$

$$+\sum_{j=i}^{r-1} 2^{r-1-j} m^{r(r-1)/2-j(j+1)/2}(\alpha_{r-1}(j)-2\alpha_r(j))\mu_j(i),$$

so that $\mu_r(i) \in \mathbb{Z}$. We continue to show that (12) holds. Since (12) is true for $r = 1, 2, 3$, we can assume that $r > 3$. With exponents of m in mind, we have

$$\frac{1}{2}r(r-1) - \frac{1}{2}j(j+1) + \left\lfloor \frac{3(j-i)+1}{2} \right\rfloor \geq \left\lfloor \frac{3(r-i)+1}{2} \right\rfloor$$

for $j \leq r-2$. Thus we get by (14), (6), and the induction hypothesis,

$$\begin{aligned} 3\mu_r(i) &\equiv 3\mu_{r-1}(i-1) + (\alpha_{r-1}(r-1) - 2\alpha_r(r-1)) \cdot 3\mu_{r-1}(i) \\ &\equiv (rm - m - r + 2)m^{r-1} \cdot 3\mu_{r-1}(i) \pmod{m^{\lfloor (3(r-i)+1)/2 \rfloor}}. \end{aligned}$$

Now, looking at exponents of m, we have

$$r - 1 + \left\lfloor \frac{3(r-1-i)+1}{2} \right\rfloor > \left\lfloor \frac{3(r-i)+1}{2} \right\rfloor,$$

and (12) follows. ■

We notice that by putting $i = r-1$ in (14), we get

$$\mu_r(r-1) = \mu_{r-1}(r-2) + \alpha_{r-1}(r-1) - 2\alpha_r(r-1),$$

so that, by (6),

$$\mu_r(r-1) = \mu_{r-1}(r-2) + (rm - m - r + 2)m^{r-1} \quad \text{for } r > 1.$$

Since $\mu_1(0) = 0$, induction on r gives

(15) $$\mu_r(r-1) = (r-1)m^r \qquad \text{for } r \geq 1.$$

Lemma 3 *For $r \geq 1$, we have*

(16) $$3^{\lfloor (r-1)/2 \rfloor} M_r \equiv 0 \pmod{4m^{\lfloor 3r/2 \rfloor}}.$$

Proof. We use induction on r. The lemma is true for $r = 1$. Suppose that for some $r > 1$, we have

$$3^{\lfloor (i-1)/2 \rfloor} M_i \equiv 0 \pmod{4m^{\lfloor 3i/2 \rfloor}}$$

for $i = 1, 2, \ldots, r-1$. By Lemma 2 and the induction hypothesis, we find

$$\begin{aligned} 3^{\lfloor (r-1)/2 \rfloor} M_r &= 3^{\lfloor (r-1)/2 \rfloor} 2^{r+1} m^{r(r+1)/2} h_r - \mu_r(r-1) 3^{\lfloor (r-1)/2 \rfloor} M_{r-1} \\ &\quad - \sum_{i=1}^{r-2} 3^{\lfloor (r-1)/2 \rfloor - \lfloor (i-1)/2 \rfloor - 1} \cdot 3\mu_r(i) \cdot 3^{\lfloor (i-1)/2 \rfloor} M_i \\ &\equiv -\mu_r(r-1) \cdot 3^{\lfloor (r-1)/2 \rfloor} M_{r-1} \pmod{4m^{\lfloor 3r/2 \rfloor}}, \end{aligned}$$

and using (15), (16) follows. ■

3 Proof of Theorem 1, Concluded

Theorem 1 now follows from Lemma 3 and the following result.

Lemma 4 *For $r \geq 1$, we have*

$$\sum_{n=1}^{\infty} (\overline{b}_m(m^{r+1} n) - \overline{b}_m(m^{r-1} n)) q^n = M_r F_m(q).$$

Proof. We use induction on r. We have

$$F_m(q) = \sum_{n=0}^{\infty} \overline{b}_m(n) q^n = \prod_{i=0}^{\infty} \frac{1 + q^{m^i}}{1 - q^{m^i}} = \frac{1+q}{1-q} F_m(q^m),$$

so that

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{b}_m(mn) q^n &= \left(U \frac{1+q}{1-q} \right) F_m(q) \\ &= \frac{1+q}{1-q} F_m(q) \\ &= \left(\frac{1+q}{1-q} \right)^2 F_m(q^m); \end{aligned}$$

that is

$$\sum_{n=0}^{\infty} \overline{b}_m(mn)q^n = (4h_1+1)F_m(q^m),$$

and it follows that

$$\sum_{n=1}^{\infty} (\overline{b}_m(m^2n) - \overline{b}_m(n))q^n = M_1 F_m(q),$$

so the lemma is true for $r = 1$.

Suppose that for some $r \geq 2$, we have

$$\sum_{n=1}^{\infty} (\overline{b}_m(m^r n) - \overline{b}_m(m^{r-2}n))q^n = M_{r-1} F_m(q).$$

Then

$$\sum_{n=1}^{\infty} (\overline{b}_m(m^r n) - \overline{b}_m(m^{r-2}n))q^n = \frac{1+q}{1-q} M_{r-1} F_m(q^m),$$

so that

$$\begin{aligned} \sum_{n=1}^{\infty} (\overline{b}_m(m^{r+1}n) - \overline{b}_m(m^{r-1}n))q^n &= \left(U \left(\frac{1+q}{1-q} M_{r-1} \right) \right) F_m(q) \\ &= M_r F_m(q), \end{aligned}$$

and the proof is complete. ■

4 Restricted m-ary Overpartitions

We close by noting that we can actually prove a result which is stronger than Theorem 1. For a positive integer k, let $\overline{b}_{m,k}(n)$ denote the number of m-ary overpartitions of n, where the largest part is at most m^{k-1}. For this restricted m-ary overpartition function we have the following result.

Theorem 2 *Let* $s = \min(r, k-1) \geq 1$. *Then we have*

$$\overline{b}_{m,k}(m^{r+1}n) - \overline{b}_{m,k-2}(m^{r-1}n) \equiv 0 \pmod{4m^{r+\lfloor s/2 \rfloor}/c^{\lfloor (s-1)/2 \rfloor}},$$

where $c = \gcd(3, m)$.

For a given n, we have $\overline{b}_m(n) = \overline{b}_{m,k}(n)$ for a sufficiently large value of k. Therefore, Theorem 2 implies Theorem 1.

We now sketch a proof of Theorem 2. We have the generating function

$$F_{m,k}(q) = \sum_{n=0}^{\infty} \overline{b}_{m,k}(n) q^n = \prod_{i=0}^{k-1} \frac{1+q^{m^i}}{1-q^{m^i}}.$$

Putting $F_{m,0}(q) = 1$, minor modifications in the proof of Lemma 4 give the following lemma.

Lemma 5 *For $1 \le r \le k-1$, we have*

$$\sum_{n=1}^{\infty} (\overline{b}_{m,k}(m^{r+1}n) - \overline{b}_{m,k-2}(m^{r-1}n)) q^n = M_r F_{m,k-1-r}(q).$$

Now, by Lemma 3 and Lemma 5, Theorem 2 holds for $r \le k-1$.

For the remaining case $r \ge k$, we need one more lemma.

Lemma 6 *For $v \ge 1$ and $t \ge 0$, there exist integers $\lambda_{v,t}(i)$ such that*

$$U^t M_v = \sum_{i=1}^{v} \lambda_{v,t}(i) M_i,$$

where

$$3^{\lfloor (v-i)/2 \rfloor} \lambda_{v,t}(i) \equiv 0 \pmod{m^{\lfloor (3(v-i)+1)/2 \rfloor + t}}.$$

This lemma is proven by induction on t. However, for the induction step we need the special case $t = 1$ of the lemma, and this special case is proven by induction on v.

By Lemma 3 and Lemma 6, we find that

$$3^{\lfloor (v-1)/2 \rfloor} U^t M_v \equiv 0 \pmod{4m^{\lfloor 3v/2 \rfloor + t}} \tag{17}$$

for $v \ge 1$ and $t \ge 0$. Applying the operator U^t to the identity of Lemma 5 with $r = k-1 \ge 1$, we get, by (17), that Theorem 2 also holds if $r = k-1+t \ge k$.

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