Arithmetic properties of overpartitions into odd parts

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Abstract

In this article, we consider various arithmetic properties of the function $\overline{p_o}(n)$ which denotes the number of overpartitions of n using only odd parts. This function has arisen in a number of recent papers, but in contexts which are very different from overpartitions. We prove a number of arithmetic results including several Ramanujan-like congruences satisfied by $\overline{p_o}(n)$ and some easily-stated characterizations of $\overline{p_o}(n)$ modulo small powers of two. For example, it is proven that, for $n \geq 1$, $\overline{p_o}(n) \equiv 0 \pmod{4}$ if and only if n is neither a square nor twice a square.

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1 Introduction

Throughout this work, we let $\overline{p}(n)$ be the number of overpartitions of the integer n and $\overline{p_o}(n)$ be the number of overpartitions of n in which only odd parts are used. Here an **overpartition** of the nonnegative integer n is a nonincreasing sequence of natural numbers whose sum is n where the first occurrence of parts of each size may be overlined. For example, the overpartitions of the integer 3 are

 $3,\ \overline{3},\ 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1,\ \overline{1}+1+1.$

Thus, from this example, we see that $\overline{p}(3) = 8$ and $\overline{p_o}(3) = 4$.

The function $\overline{p}(n)$ has been considered recently by a number of mathematicians including Corteel, Lovejoy, Mahlburg, Yee, and the authors. See [4, 5] and [8, 9, 10, 11, 12]. In [8] and [12], several Ramanujan-like congruences modulo small powers of two were proven for $\overline{p}(n)$. Indeed, all of these (and many more) congruences modulo small powers of two follow from a functional equation satisfied by the generating function for $\overline{p}(n)$. Our goal in this note is to focus on similar results which hold for $\overline{p_o}(n)$. Interestingly enough, the generating function for $\overline{p_o}(n)$, which we will denote by $\overline{P_o}(q)$, has appeared quite recently in the works of Ardonne, Kedem, and Stone [2], Bessenrodt [3], Santos and Sills [13]. However, in none of these cases do the authors connect their work to overpartitions into odd parts. Moreover, they do not consider arithmetic properties satisfied by $\overline{p_o}(n)$; this was simply not their focus. But given the wealth of results now known for $\overline{p}(n)$, it is natural to consider whether $\overline{p_o}(n)$ satisfies similar properties.

In Section 2, we prove functional equations involving the generating functions for $\overline{p}(n)$ and $\overline{p_o}(n)$, respectively. We then utilize these to develop characterizations of $\overline{p_o}(n)$ modulo small powers of 2. In particular, we prove the following two theorems:

Theorem 1.1. For all $n \ge 1$,

 $\overline{p_o}(n) \equiv \begin{cases} 2 \pmod{4} & \text{if } n \text{ is a square or } n \text{ is twice a square,} \\ 0 \pmod{4} & \text{otherwise.} \end{cases}$

Theorem 1.2. Assume the prime factorization of n is given by

$$n = 2^{\alpha} \prod p_i^{\alpha_i} q_i^{\beta_i} r_i^{\gamma_i} s_i^{\delta_i}$$

where each

 $p_i \equiv 1 \pmod{8}, q_i \equiv 3 \pmod{8}, r_i \equiv 5 \pmod{8}, s_i \equiv 7 \pmod{8}.$

Then $\overline{p_o}(n) \equiv 0 \pmod{8}$ if and only if one of the following holds:

- at least one δ_i is odd
- all δ_i are even and at least one γ_i is odd
- all δ_i are even, all γ_i are even, at least one β_i is odd and

$$\prod (\alpha_i + 1)(\beta_i + 1) \equiv 0 \pmod{4}$$

• all δ_i are even, all γ_i are even, all β_i are even and

$$\prod (\alpha_i + 1) \equiv 0 \pmod{4}$$

We then state a number of corollaries of Theorems 1.1 and 1.2 which yield nice Ramanujan-like congruence results modulo powers of 2. We close Section 2 by proving a number of generating function identities for $\overline{p_o}(n)$ for specific arithmetic progressions.

In Section 3, we consider two infinite families of results culminating in the proof of the following theorem:

Theorem 1.3. For all $n \ge 0$ and all $\alpha \ge 0$,

$$\overline{p_o}(9^{\alpha}(9n+6)) \equiv 0 \pmod{12} \quad and$$
$$\overline{p_o}(9^{\alpha}(27n+9)) \equiv 0 \pmod{6}.$$

The techniques employed below are elementary and involve a number of generating function manipulations. Moreover, Jacobi's Triple Product Identity [1, Theorem 2.8] is used often, so we state it here:

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq; q^2)_{\infty} (-z^{-1}q; q^2)_{\infty} (q^2; q^2)_{\infty}$$
(1)

where $(a;q)_{\infty} := (1-a)(1-aq)(1-aq^2)\dots$. We will, at times, shorten $(q;q)_{\infty}$ by simply writing $(q)_{\infty}$.

2 Results Modulo Powers of Two

We begin with the generating function for $\overline{p}(n)$ which is given by

$$\overline{P}(q) = \sum_{n \ge 0} \overline{p}(n)q^n = \prod_{n \ge 1} \frac{1+q^n}{1-q^n} = \frac{(-q;q)_{\infty}}{(q;q)_{\infty}} = \frac{(q^2;q^2)_{\infty}}{(q;q)_{\infty}^2}.$$

Our first goal is to prove a functional equation for $\overline{P}(q)$ which involves

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2 \sum_{n \ge 1} q^{n^2}.$$

Theorem 2.1.

$$\overline{P}(q) = \phi(q)\overline{P}(q^2)^2$$

Proof. From (1), it is clear that

$$\phi(q) = (-q;q^2)_{\infty}^2 (q^2;q^2)_{\infty} = \frac{(q^2;q^2)_{\infty}^5}{(q;q)_{\infty}^2 (q^4;q^4)_{\infty}^2}$$

after straightforward manipulations. Therefore

$$\begin{split} \phi(q)\overline{P}(q^2)^2 &= \frac{(q^2;q^2)_{\infty}^5}{(q;q)_{\infty}^2(q^4;q^4)_{\infty}^2} \times \left(\frac{(q^4;q^4)_{\infty}}{(q^2;q^2)_{\infty}^2}\right)^2 \\ &= \frac{(q^2;q^2)_{\infty}}{(q;q)_{\infty}^2} \\ &= \overline{P}(q). \end{split}$$

Iteration of Theorem 2.1 yields the following theorem.

Theorem 2.2.
$$\sum_{n\geq 0} \overline{p}(n)q^n = \phi(q) \phi(q^2)^2 \phi(q^4)^4 \phi(q^8)^8 \cdots$$

We note that various congruences modulo small powers of two can easily be proved using Theorem 2.2 and we refer the reader to [8] and [12] for more information. (Note that the methods employed in [8] are quite different from those used in [12].)

We now obtain a theorem similar to Theorem 2.2 for $\overline{p_o}(n)$.

Theorem 2.3.

$$\overline{P_o}(q) = \phi(q)\overline{P}(q^2)$$

Proof. We begin with the generating function for $\overline{p_o}(n)$ which is

$$\overline{P_o}(q) = \sum_{n \ge 0} \overline{p_o}(n) q^n = \prod_{n \ge 1} \frac{1 + q^{2n-1}}{1 - q^{2n-1}} = \frac{(-q; q^2)_\infty}{(q; q^2)_\infty} = \frac{(q^2; q^2)_\infty^3}{(q; q)_\infty^2 (q^4; q^4)_\infty}$$

after some manipulations. Therefore

$$\begin{split} \phi(q)\overline{P}(q^2) &= \frac{(q^2;q^2)_{\infty}^5}{(q;q)_{\infty}^2(q^4;q^4)_{\infty}^2} \times \frac{(q^4;q^4)_{\infty}}{(q^2;q^2)_{\infty}^2} \\ &= \frac{(q^2;q^2)_{\infty}^3}{(q;q)_{\infty}^2(q^4;q^4)_{\infty}} \\ &= \overline{P_o}(q). \end{split}$$

We can combine Theorem 2.3 with Theorem 2.1 to obtain the following:

Theorem 2.4. $\sum_{n\geq 0} \overline{p_o}(n)q^n = \phi(q) \phi(q^2) \phi(q^4)^2 \phi(q^8)^4 \cdots$

Theorem 2.4 can be utilized to obtain characterizations of $\overline{p_o}(n)$ modulo small powers of two in a very straightforward way. We do this below for the moduli 4 and 8.

For completeness' sake, we note that for all n > 0, $\overline{p_o}(n) \equiv 0 \pmod{2}$. This follows since, modulo 2,

$$\sum_{n \ge 0} \overline{p_o}(n) q^n = \prod_{n \ge 1} \frac{1 + q^{2n-1}}{1 - q^{2n-1}} \equiv \prod_{n \ge 1} \frac{1 - q^{2n-1}}{1 - q^{2n-1}} = 1.$$

We turn our attention to a characterization of $\overline{p_o}(n)$ modulo 4 by proving Theorem 1.1 mentioned above.

Proof. (of Theorem 1.1) Note that $(\phi(q))^{2^k} \equiv 0 \pmod{4}$ for $k \geq 1$. This is clear when one writes

$$\phi(q)^{2^k} = \left(1 + 2\sum_{n\geq 1} q^{n^2}\right)^{2^k}$$

and then expands via the binomial theorem. Therefore we have

$$\sum_{n\geq 0} \overline{p_o}(n)q^n = \phi(q) \phi(q^2) \phi(q^4)^2 \phi(q^8)^4 \cdots$$

$$\equiv \phi(q)\phi(q^2) \pmod{4}$$

$$= (1+2\sum_{n\geq 1} q^{n^2})(1+2\sum_{n\geq 1} q^{2n^2})$$

$$\equiv 1+2\sum_{n\geq 1} q^{n^2}+2\sum_{n\geq 1} q^{2n^2} \pmod{4}.$$

The result follows.

From Theorem 1.1, we see that numerous Ramanujan-like congruences modulo 4 are satisfied by $\overline{p_o}(n)$. All that is necessary is to find arithmetic progressions which fail to contain squares and doubles of squares. For example, we know that, for all $n \ge 0$, $\overline{p_o}(2^j n + (2r+1)) \equiv 0 \pmod{4}$ for all $j \ge 2$ and $1 \le r \le 2^{j-1} - 1$ where r is not a multiple of 4. Many additional examples could be stated.

We next consider a characterization of $\overline{p_o}(n)$ modulo 8 by proving Theorem 1.2. But before we prove Theorem 1.2, we state one important lemma.

Lemma 2.5. Let $r_{\{\Box+\Box\}}(n)$ be the number of ways to represent the integer n as the sum of two (possibly equal) squares and let $r_{\{\Box+2\Box\}}(n)$ be the number of ways to represent the integer n as the sum of a square and twice a square. Also, assume the prime factorization of n is given by

$$n=2^{\alpha}\prod p_{i}^{\alpha_{i}}q_{i}^{\beta_{i}}r_{i}^{\gamma_{i}}s_{i}^{\delta_{i}}$$

 $where \ each$

$$p_i \equiv 1 \pmod{8}, \ q_i \equiv 3 \pmod{8}, \ r_i \equiv 5 \pmod{8}, \ s_i \equiv 7 \pmod{8}$$

Then

$$r_{\{\Box+\Box\}}(n) = \begin{cases} 0 & \text{if any } \beta_i \text{ or } \delta_i \text{ is odd} \\ 4\prod(\alpha_i+1)(\gamma_i+1) & \text{if all } \beta_i, \ \delta_i \text{ are even,} \end{cases}$$

and

$$r_{\{\Box+2\Box\}}(n) = \begin{cases} 0 & \text{if any } \gamma_i \text{ or } \delta_i \text{ is odd} \\ 2\prod(\alpha_i+1)(\beta_i+1) & \text{if all } \gamma_i, \ \delta_i \text{ are even.} \end{cases}$$

Proof. These results follow from results of Jacobi and Dirichlet. For proofs of these see [6].

With Lemma 2.5 in hand, we can now prove Theorem 1.2.

Proof. (of Theorem 1.2) We consider the following set of equalities and congruences:

$$\begin{split} \sum_{n\geq 0} \overline{p_o}(n) q^n &= \phi(q) \phi(q^2) \phi(q^4)^2 \phi(q^8)^4 \cdots \\ &\equiv \phi(q) \phi(q^2) \phi(q^4)^2 \pmod{8} \\ &\equiv \phi(q) \left(\phi(q^2) + \phi(q^4)^2 - 1 \right) \pmod{8} \\ &= \phi(q) \phi(q^2) + \phi(q) \phi(q^4)^2 - \phi(q) \\ &\equiv \phi(q) \phi(q^2) + \left(\phi(q) + \phi(q^4)^2 - 1 \right) - \phi(q) \pmod{8} \\ &= \phi(q) \phi(q^2) + \phi(q^4)^2 - 1 \\ &= \sum_{n\geq 0} r_{\{\Box+2\Box\}}(n) q^n + \sum_{n\geq 0} r_{\{\Box+\Box\}}(n) q^{4n} - 1. \end{split}$$

It follows that, modulo 8,

$$\overline{p_o}(n) \equiv \begin{cases} r_{\{\Box+2\Box\}}(n) & \text{if } 4 \nmid n \\ r_{\{\Box+2\Box\}}(n) + r_{\{\Box+\Box\}}(n) & \text{if } 4|n. \end{cases}$$

It is now an easy matter to use Lemma 2.5 to complete the proof of the theorem.

Indeed, we can do better. We can provide a full characterization of $\overline{p_o}(n)$ modulo 8 as follows:

Theorem 2.6. For all $n \ge 1$, $\overline{p_o}(n) \equiv 0 \pmod{8}$ if and only if one of the following holds:

- at least one δ_i is odd,
- all δ_i are even and at least one γ_i is odd,
- all δ_i are even, all γ_i are even, at least one β_i is odd and

$$\prod (\alpha_i + 1)(\beta_i + 1) \equiv 0 \pmod{4},$$

• all δ_i are even, all γ_i are even, all β_i are even and

$$\prod (\alpha_i + 1) \equiv 0 \pmod{4};$$

 $\overline{p_o}(n) \equiv 4 \pmod{8}$ if and only if one of the following holds:

• all δ_i are even, all γ_i are even, any β_i is odd and

$$\prod (\alpha_i + 1)(\beta_i + 1) \equiv 2 \pmod{4},$$

• all δ_i are even, all γ_i are even, all β_i are even and

$$\prod (\alpha_i + 1) \equiv 2 \pmod{4};$$

 $\overline{p_o}(n) \equiv 2 \pmod{8}$ if and only if one of the following holds:

• n is an odd square or twice an odd square with

$$\sum (\alpha_i + \beta_i) \equiv 0 \pmod{4},$$

• n is an even square or twice an even square with

$$\sum (\alpha_i + \beta_i) \equiv 2 \pmod{4};$$

 $\overline{p_o}(n) \equiv 6 \pmod{8}$ if and only if one of the following holds:

 \bullet n is an odd square or twice an odd square with

$$\sum (\alpha_i + \beta_i) \equiv 2 \pmod{4},$$

• n is an even square or twice an even square with

$$\sum (\alpha_i + \beta_i) \equiv 0 \pmod{4}.$$

At this point, it is worthwhile to highlight a number of corollaries to Theorems 1.2 and 2.6, as numerous Ramanujan-like congruences can readily be seen from the theorems.

Corollary 2.7. For all $n \ge 0$,

$$\overline{p_o}(8n+5) \equiv 0 \pmod{8} \quad and$$
$$\overline{p_o}(8n+7) \equiv 0 \pmod{8}.$$

Proof. In the case of numbers of the form 8n + 5 or 8n + 7, either at least one δ_i or one γ_i is odd (using the notation employed in Theorem 1.2 and Theorem 2.6). This implies the result.

Corollary 2.8. For all $n \ge 0$,

$$\overline{p_o}(18n+15) \equiv 0 \pmod{8} \quad and$$
$$\overline{p_o}(36n+21) \equiv 0 \pmod{8}.$$

Proof. Note that 18n + 15 = 3(6n + 5), $3 \nmid 6n + 5$ and 6n + 5 is not a square. It follows that either at least one δ_i , γ_i or α_i is odd, or at least two of the β_i are odd. In any case the result follows. Similarly, we have 36n + 21 = 3(12n + 7), $3 \nmid 12n + 7$ and 12n + 7 cannot be a square. The proof proceeds as above.

We next consider a corollary which produces infinitely many congruences modulo 8 for an infinite subset of primes.

Corollary 2.9. Let p be a prime such that $p \equiv 5 \pmod{8}$ or $p \equiv 7 \pmod{8}$ and let $r \in \{1, \dots, p-1\}$. Then, for all $n \ge 0$, $\overline{p_0}(p^2n + pr) \equiv 0 \pmod{8}$.

Proof. Note that $p^2n + pr = p(pn + r)$ and $p \nmid pn + r$, so at least one δ_i or γ_i is odd. The result follows.

Corollary 2.10. For all $n \ge 0$,

$$\overline{p_o}(2n) \equiv \begin{cases} -\overline{p_o}(n) \pmod{8} & \text{if } n \text{ is twice an odd square,} \\ \overline{p_o}(n) \pmod{8} & \text{otherwise.} \end{cases}$$

In particular, if $\overline{p_o}(n) \equiv 0 \pmod{8}$, then $\overline{p_o}(2n) \equiv 0 \pmod{8}$.

Thus, for example, for all $n \ge 0$ and all $\alpha \ge 0$, we have

$$\overline{p_o}(2^{\alpha}(8n+5)) \equiv 0 \pmod{8} \text{ and}$$
$$\overline{p_o}(2^{\alpha}(8n+7)) \equiv 0 \pmod{8}.$$

Because the analysis of characterizations of $\overline{p_o}(n)$ becomes difficult modulo higher powers of 2, we now consider a different strategy. Using elementary generating function manipulation techniques, we can prove a number of generating function identities for $\overline{p_o}(n)$ for certain arithmetic progressions. We first provide a lemma which will prove useful within the intermediate steps of the proof.

First, we define four functions of q, one of which has already been used above:

$$\begin{split} \phi(q) &= \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n \ge 1} (1+q^{2n-1})^2 (1-q^{2n}) = \frac{(q^2)_{\infty}^5}{(q)_{\infty}^2 (q^4)_{\infty}^2}, \\ \psi(q) &= \sum_{n=-\infty}^{\infty} q^{2n^2-n} = \prod_{n \ge 1} (1+q^{4n-3})(1+q^{4n-1})(1-q^{4n}) = \frac{(q^2)_{\infty}^2}{(q)_{\infty}}, \\ D(q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n \ge 1} (1-q^{2n-1})^2 (1-q^{2n}) = \prod_{n \ge 1} \frac{1-q^n}{1+q^n} = \frac{(q)_{\infty}^2}{(q^2)_{\infty}}, \text{ and} \\ Y(q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n} = \prod_{n \ge 1} (1-q^{6n-5})(1-q^{6n-1})(1-q^{6n}) = \frac{(q)_{\infty} (q^6)_{\infty}^2}{(q^2)_{\infty} (q^3)_{\infty}} \end{split}$$

Next, we state without proof a lemma which contains a number of elementary results relating these four functions. The proofs of each of these identities are relatively straightforward.

Lemma 2.11.

$$\begin{split} \phi(q) &= \phi(q^4) + 2q\psi(q^8), \\ \phi(q)^2 &= \phi(q^2)^2 + 4q\psi(q^4)^2, \\ \frac{1}{D(q)} &= \prod_{n \ge 1} \frac{1+q^n}{1-q^n} = \frac{(q^2)_\infty}{(q)_\infty^2} = \frac{\phi(q)}{D(q^2)^2}. \end{split}$$

With this lemma in hand, we can now proceed to prove a number of generating function identities for particular arithmetic progressions. We combine these in the statement of one theorem.

Theorem 2.12.

$$\sum_{n\geq 0} \overline{p_o}(2n+1)q^n = 2\frac{(q^2)_{\infty}(q^8)_{\infty}^2}{(q)_{\infty}^2(q^4)_{\infty}},$$

$$\sum_{n\geq 0} \overline{p_o}(4n+3)q^n = 4\frac{(q^2)_{\infty}(q^4)_{\infty}(q^8)_{\infty}^2}{(q)_{\infty}^4},$$

$$\sum_{n\geq 0} \overline{p_o}(8n+7)q^n = 16\frac{(q^2)_{\infty}^3(q^4)_{\infty}^6}{(q)_{\infty}^9}.$$

Proof. The proof follows from generating function manipulations that rely heavily on the lemmas mentioned above. First, note that

$$\sum_{n \ge 0} \overline{p_o}(n) q^n = \prod_{n \ge 1} \frac{1 + q^{2n-1}}{1 - q^{2n-1}}$$

$$= \prod_{n \ge 1} \frac{1 + q^n}{1 - q^n} / \prod_{n \ge 1} \frac{1 + q^{2n}}{1 - q^{2n}}$$

$$= \frac{D(q^2)}{D(q)}$$

$$= \frac{\phi(q)}{D(q^2)} \text{ using Lemma 2.11}$$

$$= \frac{\phi(q^4) + 2q\psi(q^8)}{D(q^2)} \text{ using Lemma 2.11.}$$

It follows that

$$\sum_{n\geq 0} \overline{p_o}(2n+1)q^n = 2\frac{\psi(q^4)}{D(q)} = 2\frac{(q^2)_{\infty}(q^8)_{\infty}^2}{(q)_{\infty}^2(q^4)_{\infty}}.$$

This proves the first identity in the theorem. Next, we can use Lemma 2.11 once again to write the generating function for $\overline{p_o}(2n+1)$ in a different way to obtain the generating function result for $\overline{p_o}(4n+3)$. Doing so yields

$$\sum_{n\geq 0} \overline{p_o}(2n+1)q^n = 2\frac{\psi(q^4)\phi(q)}{D(q^2)^2} = 2\frac{\psi(q^4)(\phi(q^4) + 2q\psi(q^8))}{D(q^2)^2}.$$

Then we see that

$$\sum_{n\geq 0} \overline{p_o}(4n+3)q^n = 4\frac{\psi(q^2)\psi(q^4)}{D(q)^2} = 4\frac{(q^4)_\infty^2}{(q^2)_\infty}\frac{(q^8)_\infty^2}{(q^4)_\infty}\frac{(q^2)_\infty^2}{(q)_\infty^4} = 4\frac{(q^2)_\infty(q^4)_\infty(q^8)_\infty^2}{(q)_\infty^4}.$$

We can apply this principle once again to obtain the result for $\overline{p_o}(8n+7)$. Notice that

$$\sum_{n\geq 0} \overline{p_o}(4n+3)q^n = 4\frac{\psi(q^2)\psi(q^4)\phi(q)^2}{D(q^2)^4} = 4\frac{\psi(q^2)\psi(q^4)(\phi(q^2)^2 + 4q\psi(q^4)^2)}{D(q^2)^4}.$$

Thus,

$$\sum_{n \ge 0} \overline{p_o}(8n+7)q^n = 16 \frac{\psi(q)\psi(q^2)^3}{D(q)^4} = 16 \frac{(q^2)_\infty^2}{(q)_\infty} \frac{(q^4)_\infty^6}{(q^2)_\infty^3} \frac{(q^2)_\infty^4}{(q)_\infty^8} = 16 \frac{(q^2)_\infty^3(q^4)_\infty^6}{(q)_\infty^9}.$$

This completes the proof.

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One obvious and immediate corollary of Theorem 2.12 follows:

Corollary 2.13. For all $n \ge 0$,

$$\overline{p_o}(8n+7) \equiv 0 \pmod{16}.$$

3 Results Modulo Multiples of Three

We now consider a number of results modulo multiples of three. Before proving the main results of the section, we state a few important lemmas. The first lemma, a series of three short results, is stated without proof, while outlines of the proofs of the other lemmas are given.

Lemma 3.1.

$$\begin{split} D(q) &= D(q^9) - 2qY(q^3), \\ D(q)D(\omega q)D(\omega^2 q) &= \frac{D(q^3)^4}{D(q^9)} \quad where \quad \omega = e^{2\pi i/3}, \quad and \\ D(q^3)^3 - 8qY(q)^3 &= \frac{D(q)^4}{D(q^3)}. \end{split}$$

Lemma 3.2.

$$\frac{(q^3)_{\infty}^3}{(q)_{\infty}} - q\frac{(q^{12})_{\infty}^3}{(q^4)_{\infty}} = \frac{(q^4)_{\infty}^3(q^6)_{\infty}^2}{(q^2)_{\infty}^2(q^{12})_{\infty}}.$$

Proof. Let

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2 + mn + n^2 + m + n}$$

In [7], it was shown that

$$c(q) = 3 \frac{(q^3)_{\infty}^3}{(q)_{\infty}}$$
 and $c(q) - qc(q^4) = 3 \frac{(q^4)_{\infty}^3 (q^6)_{\infty}^2}{(q^2)_{\infty}^2 (q^{12})_{\infty}}$

The result follows.

Lemma 3.3.

$$\frac{(q^4)_{\infty}^3}{(q^{12})_{\infty}} - 3q \frac{(q^2)_{\infty}^2 (q^{12})_{\infty}^3}{(q^4)_{\infty} (q^6)_{\infty}^2} = \frac{(q)_{\infty}^3}{(q^3)_{\infty}}.$$

Proof. Let

$$b(q) = \sum_{m,m=-\infty}^{\infty} \omega^{m-n} q^{m^2 + mn + n^2}$$

where $\omega = e^{2\pi i/3}$. In [7], it was shown that

$$b(q) = \frac{(q)_{\infty}^3}{(q^3)_{\infty}}$$
 and $b(q) = b(q^4) - 3q \frac{(q^2)_{\infty}^2 (q^{12})_{\infty}^3}{(q^4)_{\infty} (q^6)_{\infty}^2}$

The result follows.

Lemma 3.4.

$$\frac{(q^3)_{\infty}^3}{(q)_{\infty}} - 4q \frac{(q^{12})_{\infty}^3}{(q^4)_{\infty}} = \frac{(q)_{\infty}^3 (q^6)_{\infty}^2}{(q^2)_{\infty}^2 (q^3)_{\infty}}.$$

 $\it Proof.$ From Lemmas 3.2 and 3.3 we have

$$\begin{aligned} \frac{(q^3)_{\infty}^3}{(q)_{\infty}} - 4q \frac{(q^{12})_{\infty}^3}{(q^4)_{\infty}} &= \left(\frac{(q^3)_{\infty}^3}{(q)_{\infty}} - q \frac{(q^{12})_{\infty}^3}{(q^4)_{\infty}}\right) - 3q \frac{(q^{12})_{\infty}^3}{(q^4)_{\infty}} \\ &= \frac{(q^4)_{\infty}^3 (q^6)_{\infty}^2}{(q^2)_{\infty}^2 (q^{12})_{\infty}} - 3q \frac{(q^{12})_{\infty}^3}{(q^4)_{\infty}} \\ &= \frac{(q^6)_{\infty}^2}{(q^2)_{\infty}^2} \left(\frac{(q^4)_{\infty}^3}{(q^{12})_{\infty}} - 3q \frac{(q^2)_{\infty}^2 (q^{12})_{\infty}^3}{(q^4)_{\infty} (q^6)_{\infty}^2}\right) \\ &= \frac{(q)_{\infty}^3 (q^6)_{\infty}^2}{(q^2)_{\infty}^2 (q^3)_{\infty}}.\end{aligned}$$

We now state and prove a theorem in the same vein as Theorem 2.12.

Theorem 3.5.

$$\sum_{n\geq 0} \overline{p_o}(3n)q^n = \frac{(q^2)^2_{\infty}(q^3)^2_{\infty}(q^6)_{\infty}}{(q)^4_{\infty}(q^{12})_{\infty}} \quad and$$
$$\sum_{n\geq 0} \overline{p_o}(9n+6)q^n = 12\frac{(q^2)^7_{\infty}(q^3)^6_{\infty}}{(q)^{12}_{\infty}(q^4)_{\infty}}.$$

Proof.

It follows that

$$\begin{split} \sum_{n\geq 0} \overline{p_o}(3n)q^n &= \frac{D(q^3)}{D(q)^4} \left(D(q^6) D(q^3)^2 - 4q D(q^3) Y(q) Y(q^2) \right) \\ &= \frac{(q^2)_{\infty}^4 (q^3)_{\infty}^2}{(q)_{\infty}^8 (q^6)_{\infty}} \left(\frac{(q^3)_{\infty}^4}{(q^{12})_{\infty}} - 4q \frac{(q)_{\infty} (q^3)_{\infty} (q^{12})_{\infty}^2}{(q^4)_{\infty}} \right) \\ &= \frac{(q^2)_{\infty}^4 (q^3)_{\infty}^3}{(q)_{\infty}^7 (q^6)_{\infty} (q^{12})_{\infty}} \left(\frac{(q^3)_{\infty}^3}{(q)_{\infty}} - 4q \frac{(q^{12})_{\infty}^3}{(q^4)_{\infty}} \right) \\ &= \frac{(q^2)_{\infty}^2 (q^3)_{\infty}^2 (q^6)_{\infty}}{(q)_{\infty}^4 (q^{12})_{\infty}} \quad \text{using Lemma 3.4.} \end{split}$$

Now we wish to dissect the generating function for $\overline{p_o}(3n)$ in order to obtain the generating function

for $\overline{p_o}(9n+6)$ and complete our proof. We do so by the following manipulations:

$$\begin{split} \sum_{n\geq 0} \overline{p_o}(3n)q^n &= \frac{(q^2)^2_{\infty}(q^3)^2_{\infty}(q^6)_{\infty}}{(q)^4_{\infty}(q^{12})_{\infty}} \\ &= \frac{D(q^3)D(q^6)}{D(q)^2} \\ &= D(q^3)D(q^6)\left(\frac{D(\omega q)D(\omega^2 q)}{D(q^3)^4/D(q^9)}\right)^2 \\ &= \frac{D(q^6)D(q^9)^2}{D(q^3)^7}\left(D(q^9)^2 + 2qD(q^9)Y(q^3) + 4q^2Y(q^3)^2\right)^2. \end{split}$$

It follows that

$$\begin{split} \sum_{n\geq 0} \overline{p_o}(9n+6)q^n &= \frac{D(q^2)D(q^3)^2}{D(q)^7} \Big(12D(q^3)^2 Y(q)^2\Big) \\ &= 12 \frac{D(q^2)D(q^3)^4 Y(q)^2}{D(q)^7} \\ &= 12 \frac{(q^2)_{\infty}^7 (q^3)_{\infty}^6}{(q)_{\infty}^{12} (q^4)_{\infty}}. \end{split}$$

Two corollaries of Theorem 3.5 are worthy of note here.

Corollary 3.6.

$$\overline{p_o}(9n+6) \equiv \begin{cases} 12 \pmod{24} & \text{if } n = 6k^2 \pm 4k \text{ for some } k \ge 0\\ 0 \pmod{24} & \text{otherwise.} \end{cases}$$

Proof. Notice that

$$\sum_{n\geq 0} \frac{\overline{p_o}(9n+6)}{12} q^n = \frac{(q^2)_\infty^7 (q^3)_\infty^6}{(q)_\infty^{12} (q^4)_\infty}$$
$$\equiv \frac{(q^2)_\infty (q^{12})_\infty^2}{(q^4)_\infty (q^6)_\infty} \pmod{2}$$
$$= Y(q^2)$$
$$\equiv 1 + \sum_{k\geq 1} (q^{6k^2 - 4k} + q^{6k^2 + 4k}) \pmod{2}.$$

The result follows.

Corollary 3.7. For all $n \ge 0$,

$$\overline{p_o}(72n+15) \equiv 0 \pmod{48}.$$

Proof. This follows from Corollary 2.13 and Corollary 3.6.

We close this article by proving the two infinite families of congruences stated in Theorem 1.3. These follow as corollaries of the following theorem:

Theorem 3.8. For all $n \ge 0$,

$$\overline{p_o}(27n+9) \equiv 0 \pmod{6} \quad and$$
$$\overline{p_o}(27n) \equiv \overline{p_o}(3n) \pmod{12}.$$

Proof. Using what has already been proven above regarding the generating function for $\overline{p_o}(3n)$, we have

$$\begin{split} \sum_{n\geq 0} \overline{p_0}(9n)q^n &= \frac{D(q^2)D(q^3)^2}{D(q)^7} \left(D(q^3)^4 + 16qD(q^3)Y(q)^3 \right) \\ &= \frac{D(q^2)D(q^3)^3}{D(q)^7} \left(D(q^3)^3 + 16qY(q)^3 \right) \\ &\equiv \frac{D(q^2)D(q^3)^3}{D(q)^7} \left(D(q^3)^3 - 8qY(q)^3 \right) \pmod{24} \\ &= \frac{D(q^2)D(q^3)^2}{D(q)^7} \left(\frac{D(q)^4}{D(q^3)} \right) \\ &= \frac{D(q^2)D(q^3)^2}{D(q)^7} \left(\frac{D(\omega q)D(\omega^2 q)}{D(q^3)^4} \right)^3 \\ &= D(q^2)D(q^3)^2 \left(\frac{D(\omega q)D(\omega^2 q)}{D(q^3)^4} \right)^3 \\ &= D(q^2)D(q^3)^2 \left(D(q^9)^2 + 2qD(q^9)Y(q^3) + 4q^2Y(q^3)^2 \right)^3 \\ &= \frac{D(q^9)^3}{D(q^3)^{10}}D(q^2) \left(D(q^9)^6 + 6qD(q^9)^5Y(q^3) + 24q^2D(q^9)^4Y(q^3)^2 \\ &\quad +56q^3D(q^9)^3Y(q^3)^3 + 96q^4D(q^9)^2Y(q^3)^4 + 96q^5D(q^9)Y(q^3)^5 + 64q^6Y(q^3)^6 \right) \\ &= \frac{D(q^9)^3}{D(q^3)^{10}}D(q^2) \left(D(q^9)^6 - 16q^3D(q^9)^3Y(q^3)^3 + 64q^6Y(q^3)^6 + 6qD(q^9)^5Y(q^3) \right) \\ &= \frac{D(q^9)^3}{D(q^3)^{10}}D(q^2) \left(\left(D(q^9)^3 - 8q^3Y(q^3)^3 \right)^2 + 6qD(q^9)^5Y(q^3) \right) \\ &= \frac{D(q^9)^3}{D(q^3)^{10}}D(q^2) \left(\left(\frac{D(q^3)^4}{D(q^9)} \right)^2 + 6qD(q^9)^5Y(q^3) \right) \\ &= \frac{D(q^9)}{D(q^3)^{10}}D(q^2) \left(D(q^9)^8 + 6qD(q^9)^5Y(q^3) \right) \\ &= \frac{D(q^9)}{D(q^3)^{10}}D(q^2) \left(D(q^3)^8 + 6qD(q^9)^5Y(q^3) \right) \\ &= \frac{D(q^9)}{D(q^3)^{10}}D(q^2) \left(D(q^3)^8 + 6qD(q^9)^7Y(q^3) \right) . \end{split}$$

It follows that, modulo 24,

$$\sum_{n\geq 0} \overline{p_o}(27n+9)q^n \equiv 6 \frac{D(q^3)^8 D(q^6) Y(q)}{D(q)^{10}} = 6 \frac{(q^2)_{\infty}^9 (q^3)_{\infty}^{15}}{(q)_{\infty}^{19} (q^6)_{\infty}^4 (q^{12})_{\infty}},$$

and, modulo 12,

$$\sum_{n\geq 0} \overline{p_o}(27n)q^n \equiv \frac{D(q^3)D(q^6)}{D(q)^2} = \sum_{n\geq 0} \overline{p_o}(3n)q^n.$$

This completes the proof.

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