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CONGRUENCES INVOLVING GENERALIZED FROBENIUS PARTITIONS

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ABSTRACT. The goal of this paper is to discuss congruences involving the function $\overline{c\phi_m}(n)$, which denotes the number of generalized Frobenius partitions of n with m colors whose order is m under cyclic permutation of the m colors.

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1. INTRODUCTION.

In 1984, George Andrews [1] introduced the idea of generalized Frobenius partitions, or F–partitions, and discussed many of the properties associated with them. In particular, he studied the function $c\phi_m(n)$, the number of F–partitions of n with m colors. One of the results that Andrews obtained was the following: If m is prime then

$$c\phi_m(n) \equiv 0 \pmod{m^2} \tag{1.1}$$

for all $n \geq 1$ not divisible by m.

More recently, Louis Kolitsch [2,3] has considered the function $\overline{c\phi_m}(n)$, which denotes the number of F–partitions of n with m colors whose order is m under cyclic permutation of the m colors. Kolitsch has proven that, for $m \geq 2$ and for all $n \geq 1$,

$$\overline{c\phi_m}(n) \equiv 0 \pmod{m^2}. \tag{1.2}$$

2. MAIN RESULT.

We now want to prove the following congruence related to (1.2).

THEOREM 1: For $m = 5$, 7, and 11, and for all $n \geq 1$,

$$\overline{c\phi_m}(mn) \equiv 0 \pmod{m^3}. \tag{2.1}$$

Proof: In [3], Kolitsch proved that, for all $n \geq 1$,

$$\begin{aligned}\overline{c\phi_5}(n) &= 5\,p(5n-1),\\ \overline{c\phi_7}(n) &= 7\,p(7n-2), \text{ and}\\ \overline{c\phi_{11}}(n) &= 11\,p(11n-5)\end{aligned}$$

where $p(n)$ is the ordinary partition function. Now we note that

$$\begin{aligned}\overline{c\phi_5}(5n) &= 5\,p(25n-1),\\ \overline{c\phi_7}(7n) &= 7\,p(49n-2), \text{ and}\\ \overline{c\phi_{11}}(11n) &= 11\,p(121n-5).\end{aligned}$$

Moreover, several authors have shown that

$$\begin{aligned}p(25n-1) &\equiv 0 \pmod{5^2},\\ p(49n-2) &\equiv 0 \pmod{7^2}, \text{ and}\\ p(121n-5) &\equiv 0 \pmod{11^2}.\end{aligned}$$

(See Andrews [4] for an excellent discussion of these congruences first noticed by Srinivasa Ramanujan.) Hence, we see that

$$\begin{aligned}\overline{c\phi_5}(5n) &\equiv 0 \pmod{5^3},\\ \overline{c\phi_7}(7n) &\equiv 0 \pmod{7^3}, \text{ and}\\ \overline{c\phi_{11}}(11n) &\equiv 0 \pmod{11^3}.\end{aligned}$$

This is the desired result. ■

3. FINAL REMARKS.

Now it would appear that congruences like (2.1) above hold for other values of m as well. This author has considered congruences of the form above for $m = 2$ and 3. Values involving $\overline{c\phi_2}(2n)$ and $\overline{c\phi_3}(3n)$ have been found for several values of n, which were easily computed using the generating functions for $c\phi_2(n)$ and $c\phi_3(n)$ developed in [1] and the fact that

$$\overline{c\phi_m}(mn) = c\phi_m(mn) - p(n)$$

for prime m. Given these, it appears that the following congruences hold:

Conjecture: For all $n \geq 1$,

$$\begin{aligned}\overline{c\phi_2}(2n) &\equiv 0 \pmod{2^3} \text{ and}\\ \overline{c\phi_3}(3n) &\equiv 0 \pmod{3^3}.\end{aligned}$$

It may be possible that such a congruence holds for each prime m, although this author has not pursued this.

VALUES OF $\overline{c\phi_2}(2n)$ AND $\overline{c\phi_3}(3n)$

n	$\overline{c\phi_2}(2n)$	$\overline{c\phi_3}(3n)$
1	8	81
2	40	1053
3	144	8424
4	440	50625
5	1208	2 52720
6	3048	10 99332
7	7224	43 01667
8	16264	154 51722
9	35080	517 12830
10	72968	1629 97272
11	1 47088	4879 27557
12	2 88424	13962 16926
13	5 51936	38393 79507
14	10 33360	1 01892 78765
15	18 96912	2 61910 56294
16	34 20296	6 54024 40254
17	60 66968	15 90662 95911
18	106 01000	37 76248 81413
19	182 68120	87 67386 65745
20	310 78000	199 40269 12767
21	522 41184	444 91894 14618
22	868 39912	975 17946 80439
23	1428 50088	2102 06052 45324
24	2326 87400	4460 80757 32350
25	3755 31240	9328 13551 33110
26	6007 94432	19237 81237 93026
27	9532 73544	39158 71787 90619
28	15007 49624	78725 59131 93255
29	23451 43040	1 56420 88838 88750
30	36387 99072	3 07339 60189 72779
31	56081 45688	5 97475 96846 87374
32	85878 93472	11 49781 94682 00462
33	1 30702 49344	21 91302 74194 34670
34	1 97754 21160	41 37759 78755 87103
35	2 97521 92096	77 44175 44231 50981
36	4 45208 02024	143 71142 02610 68
37	6 62751 31408	264 52213 45204 06248
38	9 81677 05768	483 08784 10303 77438
39	14 47099 70880	875 61540 95101 83201
40	21 23324 59688	1575 59882 41835 00991

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